THE UNIT TANGENT SPHERE BUNDLE WHOSE CHARACTERISTIC JACOBI OPERATOR IS PSEUDO-PARALLEL

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Abstract. We study the characteristic Jacobi operator \( \ell = \bar{R}(\cdot, \xi) \xi \) (along the Reeb flow \( \xi \)) on the unit tangent sphere bundle \( T_1M \) over a Riemannian manifold \( (M^n, g) \). We prove that if \( \ell \) is pseudo-parallel, i.e., \( \bar{R} \cdot \ell = LQ(\bar{g}, \ell) \), by a non-positive function \( L \), then \( M \) is locally flat. Moreover, when \( L \) is a constant and \( n \neq 16 \), \( M \) is of constant curvature 0 or 1.

1. Introduction

It is intriguing to study the interplay between Riemannian manifolds and their unit tangent sphere bundles. In particular, we are interested in the standard contact metric structure \( (\eta, \bar{g}, \phi, \xi) \) of a unit tangent sphere bundle \( T_1M \) over a given Riemannian manifold \( (M, g) \). It is remarkable that the characteristic vector field \( \xi \) on \( T_1M \) contains a crucial information about \( M \). In fact, all the geodesics in \( M \) are controlled by the geodesic flow on \( T_1M \) which is precisely given by \( \xi \). Apart from the defining structure tensors \( \eta, \bar{g}, \phi \) and \( \xi \), the so-called characteristic Jacobi operator \( \ell = \bar{R}(\cdot, \xi) \xi \) plays a fundamental role in contact Riemannian geometry, especially in the unit tangent sphere bundle (cf. [2]). Here, \( \bar{R} \) denotes the Riemannian curvature tensor determined by \( \bar{g} \). In Section 3, we prove that the characteristic Jacobi operator \( \ell \) vanishes if and only if \( M \) is locally flat (Proposition 2).

On the other hand, for a Riemannian manifold \( (\bar{M}, \bar{g}) \) a tensor field \( F \) of type \((1,3)\);

\[ F : \mathfrak{X}(\bar{M}) \times \mathfrak{X}(\bar{M}) \times \mathfrak{X}(\bar{M}) \to \mathfrak{X}(\bar{M}) \]

is said to be curvature-like provided that \( F \) has the symmetric properties of \( \bar{R} \). Here \( \mathfrak{X}(\bar{M}) \) is the Lie algebra of all vector fields on \( \bar{M} \). For example,
\( (\bar{X} \wedge \bar{Y}) \bar{Z} = \bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{Z}, \bar{X})\bar{Y}, \ \bar{X}, \bar{Y}, \bar{Z} \in \mathfrak{X}(\bar{M}), \) defines a curvature-like tensor field on \( \bar{M}. \) Note that a Riemannian manifold \((\bar{M}, \bar{g})\) of constant curvature \( c \) satisfies the formula \( \bar{R}(\bar{X}, \bar{Y}) = c(\bar{X} \wedge \bar{Y}). \)

As is well-known, a curvature-like tensor field \( F \) acts on the algebra \( \mathcal{T}^1_1(\bar{M}) \) of all tensor fields on \( \bar{M} \) of type \((1, s)\) as a derivation (cf. [5]). Then \( P \) is said to be semi-parallel if \( \bar{R} \cdot P = 0, \) where \( \cdot \) means that \( \bar{R} \) acts as a derivation on \( P. \) Pseudo-parallelism is defined as the natural generalization. Namely, \( P \) is said to be pseudo-parallel if \( \bar{R} \cdot P = L \mathcal{Q}(\bar{g}, P) \) for some function \( L, \) where \( \mathcal{Q}(\bar{g}, P) \) is defined by

\[
\mathcal{Q}(\bar{g}, P)(X_1, \ldots, X_s; Y, X) = (X \wedge Y)P(X_1, \ldots, X_s) - \sum_{j=1}^s P(X_1, \ldots, (X \wedge Y)X_j, \ldots, X_s).
\]

In the present paper, we study pseudo-parallelism of the characteristic Jacobi operator \( \ell \) on the unit tangent sphere bundle \( T_1 \bar{M}: \bar{R} \cdot \ell = L \mathcal{Q}(\bar{g}, \ell) \) for a function \( L \) on \( T_1 \bar{M}. \) Then we easily see that vanishing \( \ell \) implies pseudo-parallel \( \ell. \) Moreover, pseudo-parallel \( \ell \) includes the case of semi-parallel \( \ell \) \((L = 0)\). The main purpose of the present paper is to prove the following.

**Main Theorem.** Let \((\bar{M}, \bar{g})\) be an \( n \)-dimensional Riemannian manifold and \( T_1 \bar{M} \) be the unit tangent sphere bundle over \( \bar{M} \) with the standard contact metric structure \((\eta, \bar{g}, \phi, \xi). \) Suppose that the characteristic Jacobi operator \( \ell \) of \( T_1 \bar{M} \) is pseudo-parallel by a function \( L \) on \( T_1 \bar{M}. \) Then we have the following results:

(i) if \( L \leq 0, \) then \( \bar{M} \) is locally flat,

(ii) if \( L \) is constant and \( n \neq 16, \) then \( \bar{M} \) is of constant curvature 0 or 1.

Conversely, for the unit tangent sphere bundle over a space of constant curvature \( c = 0 \) or \( c = 1, \) the characteristic Jacobi operator \( \ell \) is pseudo-parallel with \( L = 0 \) or \( L = 1, \) respectively.

### 2. Preliminaries

All manifolds in the present paper are assumed to be connected and of class \( C^\infty. \) We start by collecting some fundamental material about contact metric geometry. We refer to [1] for further details. A \((2n + 1)\)-dimensional manifold \( \bar{M}^{2n+1} \) is said to be a contact manifold if it admits a global 1-form \( \eta \) such that \( \eta \wedge (d\eta)^n \neq 0 \) everywhere. Given a contact form \( \eta, \) we have a unique vector field \( \xi, \) the characteristic vector field, satisfying \( \eta(\xi) = 1 \) and \( d\eta(\xi, X) = 0 \) for any vector field \( X \) on \( \bar{M}. \) It is well-known that there exists a Riemannian metric \( \bar{g} \) on \( \bar{M} \) and a \((1, 1)\)-tensor field \( \phi \) such that

\[
(1) \quad \eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi,
\]

where \( \bar{X} \) and \( \bar{Y} \) are vector fields on \( \bar{M}. \) From (1) it follows that

\[
(2) \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi \bar{X}, \phi \bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}).
\]
A Riemannian manifold \( \bar{M} \) equipped with structure tensors \( (\eta, \bar{g}, \phi, \xi) \) satisfying (1) is said to be a contact metric manifold and is denoted by \( \bar{M} = (\bar{M}; \eta, \bar{g}, \phi, \xi) \). Given a contact metric manifold \( \bar{M} \), we define the structural operator \( h \) by
\[
h = \frac{1}{2} \xi \phi,
\]
where \( \xi \) denotes Lie differentiation. Then we may observe that \( h \) is symmetric and satisfies
\[
(3) \quad h \xi = 0 \quad \text{and} \quad h \phi = -\phi h,
\]
(4) \( \bar{\nabla}_X \xi = -\phi X - \phi h X \),
where \( \bar{\nabla} \) is the Levi-Civita connection. From (3) and (4) we see that each trajectory of \( \xi \) is a geodesic. We denote by \( \bar{R} \) the Riemannian curvature tensor defined by
\[
\bar{R}(\bar{X}, \bar{Y}) \bar{Z} = \bar{\nabla}_X (\bar{\nabla}_Y \bar{Z}) - \bar{\nabla}_Y (\bar{\nabla}_X \bar{Z}) - \bar{\nabla}_{[X,Y]} \bar{Z}
\]
for all vector fields \( \bar{X}, \bar{Y} \) and \( \bar{Z} \). Along a trajectory of \( \xi \), the Jacobi operator \( \ell = \bar{R}(\cdot, \xi) \xi \) is a symmetric \((1,1)\)-tensor field. We call it the characteristic Jacobi operator. A contact metric manifold for which \( \xi \) is Killing is called a \( K \)-contact manifold.

**Proposition 1.** For a Sasakian manifold, the characteristic Jacobi operator \( \ell \) is pseudo-parallel with \( L = 1 \).

**Proof.** Let \( \bar{M} = (\bar{M}; \eta, \bar{g}, \phi, \xi) \) be a Sasakian manifold. Then, from (5) we get
\[
(6) \quad \ell \bar{X} = \bar{X} - \eta(\bar{X}) \xi
\]
for any vector field \( \bar{X} \) on \( \bar{M} \). Using (6) we compute
\[
(\bar{R}(\bar{X}, \bar{Y}) \cdot \ell) \bar{Z} = \bar{R}(\bar{X}, \bar{Y}) \ell \bar{Z} - \ell(\bar{R}(\bar{X}, \bar{Y}) \bar{Z})
\]
\[
= \eta(\bar{X}) \bar{g}(\bar{Y}, \bar{Z}) \xi - \eta(\bar{Y}) \bar{g}(\bar{X}, \bar{Z}) \xi + \eta(\bar{X}) \eta(\bar{Z}) \bar{Y} - \eta(\bar{Y}) \eta(\bar{Z}) \bar{X},
\]
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\]
\[
L((\bar{X} \wedge \bar{Y}) \cdot \ell) \bar{Z} \\
= L\{\bar{X} \wedge \bar{Y})\ell \bar{Z} - \ell((\bar{X} \wedge \bar{Y})\bar{Z})\} \\
= L\{\eta(\bar{Y}) \bar{g}(\bar{Z}, \bar{X})\xi - \eta(\bar{Y})\bar{g}(\bar{X}, \bar{Z})\xi + \eta(\bar{X})\eta(\bar{Z})\bar{Y} - \eta(\bar{Y})\eta(\bar{Z})\bar{X}\}. 
\]

Then from (7) and (8), we can see that \( \ell \) is pseudo-parallel and \( L = 1 \). \( \square \)

3. The contact metric structure of the unit tangent sphere bundle

The basic facts and fundamental formulae about tangent bundles are well-known (cf. [6], [9], [14]). We only briefly review some notations and definitions. Let \( M = (M, g) \) be an \( n \)-dimensional Riemannian manifold and let \( TM \) denote its tangent bundle with the projection \( \pi : TM \to M, \pi(p, u) = p \). For a vector field \( X \) on \( M \), its \emph{vertical lift} \( X^v \) on \( TM \) is the vector field defined by \( X^v = \omega(X) \cdot \pi \), where \( \omega \) is a 1-form on \( M \). For the Levi Civita connection \( \nabla \) on \( M \), the \emph{horizontal lift} \( X^h \) of \( X \) is defined by \( X^h = \nabla_X \omega \). The tangent bundle \( TM \) can be endowed in a natural way with a Riemannian metric \( \bar{g} \), the so-called \emph{Sasaki metric}, depending only on the Riemannian metric \( g \) on \( M \). It is determined by

\[
\bar{g}(X^h, Y^h) = \bar{g}(X^v, Y^v) = g(X, Y) \circ \pi, \quad \bar{g}(X^h, Y^v) = 0
\]

for all vector fields \( X \) and \( Y \) on \( M \). Also, \( TM \) admits an almost complex structure tensor \( J \) defined by \( JX^h = X^v \) and \( JX^v = -X^h \). Then \( \bar{g} \) is a Hermitian metric for the almost complex structure \( J \).

The unit tangent sphere bundle \( \bar{\pi} : T_1M \to M \) is a hypersurface of \( TM \) given by \( g_p(u, u) = 1 \). Note that \( \bar{\pi} = \pi \circ i \), where \( i \) is the immersion of \( T_1M \) into \( TM \). A unit normal vector field \( N = u^v \) to \( T_1M \) is given by the vertical lift of \( u \) for \( (p, u) \). The horizontal lift of a vector is tangent to \( T_1M \), but the vertical lift of a vector is not tangent to \( T_1M \) in general. So, we define the \emph{tangential lift} of \( X \) to \( (p, u) \in T_1M \) by

\[
X^t_{(p, u)} = (X - g(X, u)u)^v.
\]

Clearly, the tangent space \( T_{(p, u)}T_1M \) is spanned by vectors of the form \( X^h \) and \( X^v \), where \( X \in T_pM \).

We now define the standard contact metric structure of the unit tangent sphere bundle \( T_1M \) over a Riemannian manifold \( (M, g) \). The metric \( g' \) on \( T_1M \) is induced from the Sasaki metric \( \bar{g} \) on \( TM \). Using the almost complex structure \( J \) on \( TM \), we define a unit vector field \( \xi' \), a 1-form \( \eta' \) and a \((1,1)\)-tensor field \( \phi' \) on \( T_1M \) by

\[
\xi' = -JN, \quad \phi' = J - \eta' \otimes N.
\]

Since \( g'(\bar{X}, \phi' \bar{Y}) = 2d\eta'(\bar{X}, \bar{Y}), (\eta', g', \phi', \xi') \) is not a contact metric structure. If we rescale this structure by

\[
\xi = 2\xi', \quad \eta = \frac{1}{2} \eta', \quad \phi = \phi', \quad \bar{g} = \frac{1}{4} g',
\]

we obtain a contact metric structure on \( T_1M \).
we get the standard contact metric structure \((\eta, \bar{g}, \phi, \xi)\). The tensors \(\xi\) and \(\phi\) are explicitly given by

\[(9) \quad \xi = 2u^h, \quad \phi X^t = -X^h + \frac{1}{2} g(X, u) \xi, \quad \phi X^t = X^t,\]

where \(X\) and \(Y\) are vector fields on \(M\).

From now on, we consider \(T_1 M = (T_1 M; \eta, \bar{g}, \phi, \xi)\) with the standard contact metric structure. Then the Levi-Civita connection \(\bar{\nabla}\) of \(T_1 M\) is described by

\[(10) \quad \bar{\nabla}_X Y^t = -g(Y, u) X^t,\]

\[\bar{\nabla}_X Y^h = \frac{1}{2} (R(u, X) Y)^h,\]

\[\bar{\nabla}_X Y^t = (\nabla X Y)^t + \frac{1}{2} (R(u, Y) X)^h,\]

\[\bar{\nabla}_X Y^h = (\nabla X Y)^h - \frac{1}{2} (R(X, Y) u)^t\]

for all vector fields \(X\) and \(Y\) on \(M\).

Also the Riemann curvature tensor \(\bar{R}\) of \(T_1 M\) is given by

\[(11) \quad \bar{R}(X^t, Y^t) Z^t = -(g(X, Z) - g(X, u)g(Z, u)) Y^t + (g(Y, Z) - g(Y, u)g(Z, u)) X^t,\]

\[\bar{R}(X^t, Y^t) Z^h = \left\{ R(X - g(X, u) Y - g(Y, u)u Z)^h \right\}
+ \frac{1}{4} \left\{ R(u, X) (R(u, Y) Z)^h \right\},\]

\[\bar{R}(X^h, Y^t) Z^t = -\frac{1}{2} \left\{ R(Y - g(Y, u) Z - g(Z, u)u X)^h \right\}
- \frac{1}{4} \left\{ R(u, Y) (R(u, Z) X)^h \right\},\]

\[\bar{R}(X^h, Y^t) Z^h = \frac{1}{2} \left\{ R(X, Y) (Y - g(Y, u) u) Z^h \right\}
- \frac{1}{4} \left\{ R(X, R(u, Y) Z) u Z^h \right\} + \frac{1}{2} \left\{ (\nabla X R)(u, Y) Z^h \right\},\]

\[\bar{R}(X^h, Y^h) Z^t = \left\{ R(X, Y) (Z - g(Z, u) u) Z^t \right\} + \frac{1}{4} \left\{ R(Y, R(u, Z) X) u - R(X, R(u, Z) Y) u^t \right\}
+ \frac{1}{2} \left\{ (\nabla X R)(u, Z) Y - (\nabla Y R)(u, Z) X^h \right\},\]

\[\bar{R}(X^h, Y^h) Z^h = (R(X, Y) Z)^h + \frac{1}{2} \left\{ R(u, R(X, Y) u) Z^h \right\}
- \frac{1}{4} \left\{ R(u, R(Y, Z) u) X - R(u, R(X, Z) u) Y^h \right\}
+ \frac{1}{2} \left\{ (\nabla Z R)(X, Y) u \right\}^t\]
for all vector fields $X$, $Y$ and $Z$ on $M$. Using the formulae (11), we get

$$
\ell X^i = (R_u^2 X)^i + 2(R_u X)^b_i,
$$
(12)

$$
\ell X^b = 4(R_u X)^b - 3(R_u^2 X)^h + 2(R_u X)^i,
$$

where $R_u = R(\cdot, u)u$, $R_u^a = (\nabla_u R)(\cdot, u)u$ and $R_u^2 = R(R(\cdot, u)u, u)u$. We can refer to [2, 3, 4] for the formulas (10) ~ (12). From (12), we have the following proposition.

**Proposition 2.** The characteristic Jacobi operator $\ell$ of $T_1M$ vanishes if and only if $M$ is locally flat.

**Proof.** Suppose that the characteristic Jacobi operator $\ell$ vanishes. Then we get from (12) $R_u^a X = 0$ and $R_u^2 X = 0$. The former implies that $(M, G)$ is a locally symmetric space ([8], [13]) and the latter does that the eigenvalues of $R_u$ are constant and equal to $0$, i.e., $(M, G)$ is a globally Osserman space (i.e., the eigenvalues of $R_u$ do not depend on the point $p$ and not on the choice of unit vector $u$ at $p$). However, a locally symmetric globally Osserman space is locally flat or locally isometric to a rank one symmetric space ([7]). Therefore, we conclude that $M$ is a space of constant curvature $0$. \[\Box\]

### 4. Proof of Main Theorem

Suppose that the characteristic Jacobi operator $\ell$ of $T_1M$ is pseudo-parallel by a function $L$ on $T_1M$. Then $T_1M$ satisfies

$$
\bar{R}(X, Y)\ell Z - \ell(\bar{R}(X, Y)Z)
$$
(13)

$$
= L(\bar{g}(Y, \ell Z)X - \bar{g}(X, \ell Z)Y - \bar{g}(\bar{Y}, \bar{Z})\ell X + \bar{g}(\bar{X}, \bar{Z})\ell Y).
$$

We put $\bar{Y} = \xi$ in (13). Then we have

$$
\bar{R}(X, \xi)\ell Z - \ell(\bar{R}(X, \xi)Z) = L(-\bar{g}(\bar{X}, \ell Z)\xi - \eta(\bar{Z})\ell X).
$$

(14)

Setting $X = X^t$, $Z = Z^t$ in (14), and applying the Riemannian metric $\bar{g}$ on $T_1M$ for $Y^h$ on both sides, then we have the following equation:

$$
-\frac{1}{2}g(R(X, R_u^2 Z)u, Y) + \frac{1}{2}g(X, u)g(R_u^4 Z, Y) + \frac{1}{4}g(R(X, u)R_u^4 Z, Y)
$$

$$
- g((\nabla_u R)(u, X)R_u^4 Z, Y) = -\frac{1}{4}Lg(X, R_u^2 Z)g(Y, u).
$$

(15)

We put $Y = u$ in (15). Then we have

$$
g(-\frac{1}{4}R_u^4 X - R_u^2 X, Z) = -\frac{1}{4}Lg(R_u^2 X, Z)
$$

for any vector fields $X$ and $Z$ on $M$, that is, it holds

$$
R_u^4 X + 4R_u^2 X = LR_u^2 X.
$$

(16)

Since $R_u$ is symmetric operator, if $L \leq 0$, from (16) we have $R_u^4 = 0$ and $R_u = 0$. Therefore, using the similar arguments in the proof of Proposition 2 we see that $M$ is locally flat. This completes the proof of (i).
Next, in order to prove the second part of Main Theorem we prepare the following lemma.

**Lemma 3.** Let \((M, g)\) be a locally symmetric space. Then the characteristic Jacobi operator \(\ell\) of \(T_1M\) is pseudo-parallel by a function \(L\) on \(T_1M\) if and only if \(M\) is of constant curvature 0 or 1.

**Proof.** If we set \(\bar{X} = X^h\), \(\bar{Z} = Z^h\) in (14), and apply the Riemannian metric \(\bar{g}\) on \(T_1M\) for \(Y^h\) on both sides, then we have the following equation:

\[(17)\]
\[
4g(R(X, u)R_uZ, Y) + 2g(R(u, R_uX)R_uZ, Y) - g(R(R_u^2Z, u)X, Y)
- g(R(X, R_uZ)u, R_uY) - 3g(R(X, u)R_u^2Z, Y) - \frac{3}{2}g(R(u, R_uX)R_u^2Z, Y)
+ \frac{3}{4}g(R(R_u^2Z, u)X, Y) - \frac{3}{4}g(R(R_u^2Z, X)u, R_uY) + g((\nabla X R)(u, R_u'Z)u, Y)
- g(((\nabla u) R)(u, R_u'Z)X, Y) - 4g(R(X, u)Z, R_uY) + 3g(R(X, u)Z, R_u^2Y)
- 2g(R(u, R_uX)Z, R_uY) + \frac{3}{2}g(R(u, R_uX)Z, R_u^2Y) + g(R(R_uZ, u)X, R_uY)
- \frac{3}{4}g(R(R_uZ, u)X, R_u^2Y) + g(R(X, Z)u, R_u^2Y) - \frac{3}{4}g(R(X, Z)u, R_u^2Y)
- g((\nabla Z R)(X, u)u, R_u^2Y)
= \frac{1}{4}L(-4g(X, R_uZ)g(Y, u) + 3g(X, R_u^2Z)g(Y, u) - 4g(R_uX, Y)g(Z, u)
+ 3g(R_u^2X, Y)g(Z, u))\]

Putting \(Y = u\) in (17), we have

\[(18)\]
\[-\frac{9}{4}R_u^4X + 6R_u^3X - 4R_u^2X - R_u'^2X = 4L(-4R_uX + 3R_u^2X)\]

We suppose that \(M\) is locally symmetric. Then from (16) and (18), we obtain

\[(19)\]
\[R_u^4X = LR_u^2X,\]

\[(20)\]
\[-9R_u^4X + 24R_u^3X - 16R_u^2X = L(-4R_uX + 3R_u^2X)\]

We assume that \(R_uX = \lambda X\) for a function \(\lambda\) on \(M\). Then from (19) and (20), we have

\[(21)\]
\[\lambda^4 = L\lambda^2,\]

\[(22)\]
\[9\lambda^4 - 24\lambda^3 + 16\lambda^2 - 4L\lambda + 3L\lambda^2 = 0.\]

From (21), we have \(\lambda = 0\) or \(L = \lambda^2\). If \(L = \lambda^2\) and \(\lambda \neq 0\), from (22), we have

\[(3\lambda - 4)(\lambda - 1) = 0.\]

Hence, \(\lambda = 0, 1\) or \(\frac{4}{3}\), and then \((M, g)\) is a globally Osserman space. But, it is also locally symmetric, and then it is locally isometric to a rank one symmetric space. However, we can easily check that \(T_1M\) of a space of constant curvature
does not satisfy pseudo-parallelism of $\ell$. Therefore, we conclude that $(M, g)$ is of constant curvature 0 or 1. By Propositions 1 and 2, the converse is easily proved. □

Now we assume that $L$ is constant. Then, from (16) and (18), we have

$$2R_a^4X - 6R_a^3X + 4R_a^2X = L(R_aX - R_a^2X).$$

If we put $R_aX = \lambda X$, we get

$$\lambda(\lambda - 1)(2\lambda^2 - 4\lambda + L) = 0.$$  

Here, we use Nikolayevsky’s results ([10, 11, 12]) on the Osserman conjecture. Then we find that $(M^n, g)$ is locally isometric to a rank one symmetric space, when $n \neq 16$. Thus, by Lemma 3 we conclude that $(M, g)$ is of constant curvature 0 or 1, when $n \neq 16$. Conversely, by Propositions 1 and 2, we see that for the unit tangent sphere bundle over a space of constant curvature $c = 0$ or $c = 1$, the characteristic Jacobi operator $\ell$ is pseudo-parallel with $L = 0$ or $L = 1$, respectively. This completes the proof of Main Theorem.

**Corollary 4.** If $\ell$ of $T_1M$ is semi-parallel, that is, $L = 0$, then $M$ is locally flat.

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