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TRANSVERSAL HALF LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD OF A QUASI-CONSTANT CURVATURE

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Abstract. We study transversal half lightlike submanifolds of an indefinite Kaehler manifold of a quasi-constant curvature. First, we provide a new result for such a transversal half lightlike submanifold. Next, we investigate a statical half lightlike submanifold M such that (1) the screen distribution $S(TM)$ is totally umbilical, or (2) M is screen homothetic.

1. Introduction

The theory of lightlike submanifolds is an important topic of research in differential geometry due to its application in mathematical physics. The study of such notion was initiated by Duggal-Bejancu [3] and later studied by many authors [5, 6]. Half lightlike submanifold M is a lightlike submanifold of codimension 2 such that $rank\{Rad(TM)\} = 1$, where $Rad(TM) = TM \cap TM^{\perp}$ is the radical distribution of M. It is a special case of general r-lightlike submanifolds [3] such that $r = 1$. Its geometry is more general than that of lightlike hypersurfaces or coisotropic submanifolds which are lightlike submanifolds M of codimension 2 such that $rank\{Rad(TM)\} = 2$. Much of its theory will be immediately generalized in a formal way to general r-lightlike submanifolds.

In the classical theory of Riemannian geometry, Chen-Yano [1] introduced the notion of a *Riemannian manifold of a quasi-constant curvature* as a Riemannian manifold $(\bar{M}, \bar{g})$ endowed with a curvature tensor $\bar{R}$ of the form

$$\begin{aligned} \bar{R}(X,Y)Z = \; & f_1\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ & + f_2\{\theta(Y)\theta(Z)X - \theta(X)\theta(Z)Y \\ & \quad + \bar{g}(Y,Z)\theta(X)\zeta - \bar{g}(X,Z)\theta(Y)\zeta\}, \end{aligned} \tag{1.1}$$

for any vector fields X, Y and Z on $\bar{M}$, where f_1 and f_2 are smooth functions which are called the *curvature functions*, ζ is a unit vector field which is called

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the *characteristic vector field* of $\bar{M}$, and θ is a 1-form associated with ζ by $\theta(X) = \bar{g}(X, \zeta)$. If $f_2 = 0$, then $\bar{M}$ is a space of constant curvature.

In this paper, we study half lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature such that the characteristic vector field ζ of $\bar{M}$ belongs to the transversal vector bundle $tr(TM)$ of M, which M is called a *transversal half lightlike submanifold* of $\bar{M}$. First, we provide a new result for such a transversal half lightlike submanifold. Next, we investigate a statical transversal half lightlike submanifold M such that (1) the screen distribution $S(TM)$ is totally umbilical, or (2) M is screen homothetic.

2. Preliminaries

Let (M, g) be a codimension 2 half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ equipped with the tangent bundle TM, the normal bundle $TM^{\perp}$, the radical distribution $Rad(TM) = TM \cap TM^{\perp}$, a screen distribution $S(TM)$, and a coscreen distribution $S(TM^{\perp})$ such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \qquad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E. Also denote by $(2.6)_1$ the first equation of the two equations in (2.6). We use same notations for any others. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit spacelike vector field, *i.e.*, $\bar{g}(L, L) = 1$, without loss of generality. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$, of rank 3. Certainly the vector fields ξ and L belong to $\Gamma(S(TM)^{\perp})$. Hence we have the following orthogonal decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$, of rank 2. It is known [4] that, for any null section ξ of $Rad(TM)$, there exists a uniquely defined null vector field $N \in \Gamma(S(TM^{\perp})^{\perp})$ satisfying

$$\bar{g}(\xi, N) = 1, \ \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \ \forall\, X \in \Gamma(S(TM)).$$

Denote by $ltr(TM)$ the subbundle of $S(TM^{\perp})^{\perp}$ locally spanned by N. Then we show that $S(TM^{\perp})^{\perp} = Rad(TM) \oplus ltr(TM)$. Let $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$. Then we call N, $ltr(TM)$ and $tr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to the screen distribution $S(TM)$ respectively.

From now and in the sequel, let X, Y, Z and W be the vector fields on M, unless otherwise specified. Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of TM on $S(TM)$. Then the local Gauss and Weingarten formulas of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N + D(X,Y)L, \tag{2.1}$$
$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{2.2}$$
$$\bar{\nabla}_X L = -A_L X + \phi(X)N; \tag{2.3}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.4}$$
$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.5}$$

respectively, where ∇ and ∇^* are induced connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local screen second fundamental form* on $S(TM)$, A_N, A^*_ξ and A_L are called the *shape operators*, and τ, ρ and ϕ are 1-forms on TM.

Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free, and B and D are symmetric. The above three local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{2.6}$$
$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{2.7}$$
$$D(X,Y) = g(A_L X, Y) - \phi(X)\eta(Y), \qquad \bar{g}(A_L X, N) = \rho(X), \tag{2.8}$$

here η is a 1-form such that $\eta(X) = \bar{g}(X,N)$. From $(2.6)_1$ and $(2.8)_1$, we get

$$B(X,\xi) = 0, \qquad D(X,\xi) = -\phi(X). \tag{2.9}$$

Both A^*_ξ and A_N are $S(TM)$-valued, and A^*_ξ is self-adjoint such that

$$A^*_\xi \xi = 0. \tag{2.10}$$

The induced connection ∇ of M is not a metric connection and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y). \tag{2.11}$$

We need the following Gauss-Codazzi equations (for a full set of these equations, see [4]). Denote by $\bar{R}$, R and R^* the curvature tensors of $\bar{\nabla}$, ∇ and ∇^* respectively. Using the local Gauss-Weingarten formulas, we have

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ B(X,Z)A_N Y - B(Y,Z)A_N X \\ &+ D(X,Z)A_L Y - D(Y,Z)A_L X \\ + \{(\nabla_X B)&(Y,Z) - (\nabla_Y B)(X,Z) \\ &+ \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &+ \phi(X)D(Y,Z) - \phi(Y)D(X,Z)\}N \\ + \{(\nabla_X D)&(Y,Z) - (\nabla_Y D)(X,Z) + \rho(X)B(Y,Z) \\ &- \rho(Y)B(X,Z)\}L,\end{aligned} \tag{2.12}$$

$$\begin{aligned}\bar{R}(X,Y)N &= -\nabla_X(A_N Y) + \nabla_Y(A_N X) + A_N[X,Y] \qquad (2.13)\\ &+ \tau(X)A_N Y - \tau(Y)A_N X + \rho(X)A_L Y - \rho(Y)A_L X\\ &+ \{B(Y, A_N X) - B(X, A_N Y) + 2d\tau(X,Y)\\ &\qquad + \phi(X)\rho(Y) - \phi(Y)\rho(X)\}N\\ &+ \{D(Y, A_N X) - D(X, A_N Y) + 2d\rho(X,Y)\\ &\qquad + \rho(X)\tau(Y) - \rho(Y)\tau(X)\}L,\end{aligned}$$

$$\begin{aligned}R(X,Y)PZ &= R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A_\xi X \qquad (2.14)\\ &+ \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ)\\ &\qquad - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ)\}\xi.\end{aligned}$$

In the case $R = 0$, we say that M is *flat*.

The *Ricci tensor* of $\bar{M}$, denote it by $\bar{Ric}$, is defined by

$$\bar{Ric}(X,Y) = trace\{Z \to \bar{R}(X,Z)Y\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(T\bar{M}).$$

Denote by $R^{(0,\,2)}$ the induced tensor of type (0, 2) on M such that

$$R^{(0,\,2)}(X,\,Y) = trace\{Z \to R(X,\,Z)Y\}, \quad \forall\, X,\, Y,\, Z \in \Gamma(TM). \qquad (2.15)$$

Due to [7], using (2.6)∼(2.8) and the Gauss equation (2.12), we get

$$\begin{aligned}R^{(0,\,2)}(X,Y) = \;& \bar{Ric}(X,Y) + B(X,Y)tr\, A_N + D(X,Y)tr\, A_L \qquad (2.16)\\ &- g(A_N X,\, A^*_\xi Y) - g(A_L X,\, A_L Y) + \rho(X)\phi(Y)\\ &- \bar{g}(\bar{R}(\xi,Y)X,\, N) - \bar{g}(\bar{R}(L,Y)X,\, L).\end{aligned}$$

Using the lightlike transversal part of (2.13) and the Bianchi's identity, we get

$$R^{(0,\,2)}(X,\,Y) - R^{(0,\,2)}(Y,\,X) = 2d\tau(X,\,Y).$$

This shows that, in general, $R^{(0,\,2)}$ is not symmetric. A tensor field $R^{(0,\,2)}$ of M, given by (2.15), is called its *induced Ricci tensor* and denote it by Ric if it is symmetric. In this case, M is called *Ricci flat* if $Ric = 0$. M is called an *Einstein manifold* if there exists a smooth function κ such that

$$Ric = \kappa g. \qquad (2.17)$$

Let $\nabla^\ell_X N = \pi(\bar{\nabla}_X N)$, where π is the projection morphism of $T\bar{M}$ on $ltr(TM)$. Then ∇^ℓ is a linear connection on $ltr(TM)$. We say that ∇^ℓ is the *lightlike transversal connection* of M. We define a curvature tensor R^ℓ by

$$R^\ell(X,Y)N = \nabla^\ell_X\nabla^\ell_Y N - \nabla^\ell_Y\nabla^\ell_X N - \nabla^\ell_{[X,\,Y]}N.$$

If R^ℓ vanishes identically, then the lightlike transversal connection ∇^ℓ is said to be *flat*. We quote the following result (see [10, 11]).

Theorem 2.1. *Let M be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. The following statements are equivalent:*

(i) *The lightlike transversal connection of M is flat, i.e., $R^\ell = 0$.*

(ii) *The 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*
(iii) *The tensor field $R^{(0,2)}$ of M is an induced Ricci tensor of M.*

Note 1. Suppose τ and $\bar{\tau}$ are 1-forms with respect to the sections ξ and $\bar{\xi}$, respectively, by directed calculation, we get $d\tau = d\bar{\tau}$ [4]. In case $d\tau = 0$, by the cohomology theory, there exists a smooth function f such that $\tau = df$. Thus $\tau(X) = X(f)$. If we take $\bar{\xi} = \lambda\xi$, it follows that $\tau(X) = \bar{\tau}(X) + X(\text{In}\,\lambda)$. Setting $\lambda = \exp(f)$ in this equation, we get $\bar{\tau}(X) = 0$. Thus *if $d\tau = 0$, then we can take a 1-form τ such that $\tau = 0$* [3].

3. Transversal half lightlike submanifolds

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a real even dimensional indefinite Kaehler manifold, where $\bar{g}$ is a semi-Riemannian metric of index $q = 2v$, $0 < v < \frac{1}{2}(\dim \bar{M})$, and J is an almost complex structure on $\bar{M}$ such that, for all $X, Y \in \Gamma(T\bar{M})$,

$$J^2 = -I, \qquad \bar{g}(JX, JY) = \bar{g}(X, Y), \qquad (\bar{\nabla}_X J)Y = 0. \tag{3.1}$$

Let (M, g) be a half lightlike submanifold of an indefinite Kaeler manifold $\bar{M}$, where g is a degenerate metric on M induced by $\bar{g}$. Due to [8, 9], we choose a screen distribution $S(TM)$ such that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are vector subbundles of $S(TM)$. In this case, the screen distribution $S(TM)$ is expressed as follow:

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o,$$

where H_o is a non-degenerate and almost complex distribution with respect to J, *i.e.*, $J(H_o) = H_o$. The tangent bundle TM is decomposed as follow:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})), \tag{3.2}$$

where H is a 2-lightlike almost complex distribution on M such that

$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

Consider two null and one spacelike vector fields $\{U, V\}$ and W such that

$$U = -JN, \quad V = -J\xi, \quad W = -JL, \tag{3.3}$$

respectively. Denote by S the projection morphism of TM on H. By (3.2), for any vector field X on M, the vector field JX is decomposed as

$$JX = FX + u(X)N + w(X)L, \tag{3.4}$$

where u, v and w are 1-forms locally defined on M by

$$u(X) = g(X, V), \qquad v(X) = g(X, U), \qquad w(X) = g(X, W), \tag{3.5}$$

and F is a tensor field of type $(1, 1)$ globally defined on M by $F = J \circ S$.

Applying $\bar{\nabla}_X$ to (3.3) and using the Gauss-Weingarten formulas, we have

$$B(X,U) = C(X,V), \quad C(X,W) = D(X,U), \tag{3.6}$$
$$D(X,V) = B(X,W),$$
$$\nabla_X U = F(A_N X) + \tau(X)U + \rho(X)W, \tag{3.7}$$
$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \phi(X)W, \tag{3.8}$$
$$\nabla_X W = F(A_L X) + \phi(X)U. \tag{3.9}$$

Definition 1. Let M be a half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature. We say that ζ is *transversal* to M if it belongs to the transversal vector bundle $tr(TM) = S(TM^\perp) \oplus ltr(TM)$ of M. In this case, M is called an *transversal half lightlike submanifold.*

For a transversal half lightlike submanifold M, ζ is decomposed as

$$\zeta = lL + \beta N,$$

where $l = \theta(L)$ and $\beta = \theta(\xi)$. As $\bar{g}(\zeta, \zeta) = 1$, we have $l^2 = 1$. We may assume that $l = 1$, without loss of generality. In this case we have

$$\zeta = L + \beta N. \tag{3.10}$$

In this paper, by saying that *transversal half lightlike submanifolds* we shall mean *half lightlike submanifolds satisfying* (3.10) *such that* $\beta \neq 0$.

From (3.10), we see that

$$\theta(X) = \beta\eta(X), \qquad \theta(PX) = 0.$$

Theorem 3.1. *Let M be an transversal half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature. Then the curvature functions f_1 and f_2 are satisfied $f_1 = 0$ and $f_2\theta(X) = 0$ for all $X \in \Gamma(TM)$.*

Proof. Comparing the tangential, lightlike transversal and co-screen components of the two equations (1.1) and (2.12), we get the following equations:

$$\begin{aligned} &R(X,Y)Z \\ &= f_1\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + f_2\{\theta(Y)X - \theta(X)Y\}\theta(Z) \\ &+ B(Y,Z)A_N X - B(X,Z)A_N Y + D(Y,Z)A_L X - D(X,Z)A_L Y, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ &+ \phi(X)D(Y,Z) - \phi(Y)D(X,Z) \\ &= \beta f_2\{g(Y,Z)\theta(X) - g(X,Z)\theta(Y)\}. \end{aligned} \tag{3.12}$$

Taking the scalar product with N to (2.14) and then, substituting (3.11) into the resulting equation and using $(2.7)_2$ and $(2.8)_2$, we get

$$\begin{aligned}&(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ)\\&- \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ)\\&- \rho(X)D(Y, PZ) + \rho(Y)D(X, PZ)\\&= f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}.\end{aligned} \tag{3.13}$$

Applying ∇_X to $(3.6)_1 : B(Y, U) = C(Y, V)$, we have

$$(\nabla_X B)(Y, U) = (\nabla_X C)(Y, V) + g(A_N Y, \nabla_X V) - g(A^*_\xi Y, \nabla_X U).$$

Using (3.1), (3.4) and (3.6)∼(3.8), the last equation is reduced to

$$\begin{aligned}&(\nabla_X B)(Y, U)\\&= (\nabla_X C)(Y, V) - 2\tau(X)C(Y, V) - \phi(X)D(Y, U) - \rho(X)D(Y, V)\\&- g(A^*_\xi X, F(A_N Y)) - g(A^*_\xi Y, F(A_N X)).\end{aligned}$$

Substituting this equation into (3.12) such that $Z = U$, we get

$$\begin{aligned}&(\nabla_X C)(Y, V) - (\nabla_Y C)(X, V) - \tau(X)C(Y, V) + \tau(Y)C(X, V)\\&- \rho(X)D(Y, V) + \rho(Y)D(X, V)\\&= \beta f_2\{\theta(X)v(Y) - \theta(Y)v(X)\}.\end{aligned}$$

Comparing this equation with (3.13) such that $PZ = V$, we get

$$\beta f_2\{\theta(X)v(Y) - \theta(Y)v(X)\} = f_1\{\eta(X)u(Y) - \eta(Y)u(X)\}. \tag{3.14}$$

Taking $X = \xi$; $Y = U$ and $X = \xi$; $Y = V$ to (3.14) by turns, we get $f_1 = 0$ and $\beta f_2 = 0$. Using the second result, $f_2\theta(X) = \beta f_2\eta(X) = 0$ for all $X \in \Gamma(TM)$.

4. Totally umbilical screen distribution

Let M be an transversal half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature. As $\bar{R} = 0$ by Theorem 3.1, the general form (2.16) of the Ricci type tensor $R^{(0,2)}$ is reduced to

$$\begin{aligned}R^{(0,2)}(X, Y) =\ & B(X, Y)tr\, A_N + D(X, Y)tr\, A_L + \rho(X)\phi(Y)\\&- g(A_N X,\, A^*_\xi Y) - g(A_L X, A_L Y).\end{aligned} \tag{4.1}$$

Definition 2. A half lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called *statical* [12, 13] if $\bar{\nabla}_X L \in \Gamma(S(TM))$ for any $X \in \Gamma(TM)$.

From (2.3) and $(2.8)_2$, we show that the above definition is equivalent to the following two conditions: $\phi = 0$ and $\rho = 0$. Note that the first condition $\phi = 0$ is equivalent to the conception that M is *irrotational*, *i.e.*, $\bar{\nabla}_X \xi \in \Gamma(TM)$ [14].

Definition 3. A screen distribution $S(TM)$ is called *totally umbilical* [3, 9] in M if there exists a smooth function γ such that $A_N X = \gamma PX$, or equivalently,

$$C(X, PY) = \gamma g(X, Y). \tag{4.2}$$

In case $\gamma = 0$, we say that $S(TM)$ is *totally geodesic* in M.

Theorem 4.1. *Let M be an irrotational transversal half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature. If $S(TM)$ is totally umbilical, then the following properties are satisfied*

(1) *$S(TM)$ is totally geodesic and parallel distribution,*
(2) *M is locally a product manifold $\mathcal{C}_\xi \times M^*$, where $\mathcal{C}_\xi$ is a null geodesic tangent to $Rad(TM)$ and M^* is a leaf of $S(TM)$,*
(3) *the curvature tensor R of M is of the form*

$$R(X,Y)Z = D(Y,Z)A_{L}X - D(X,Z)A_{L}Y,$$

(4) *$d\tau = 0$, $R^{(0,2)}$ is symmetric and the transversal connection is flat.*
(5) *Moreover, if M is an Einstein manifold, then M is Ricci flat.*

Proof. Applying ∇_X to $C(Y, PZ) = \gamma g(Y, PZ)$ and using (2.11), we have

$$(\nabla_X C)(Y, PZ) = (X\gamma)g(Y, PZ) + \gamma B(X, PZ)\eta(Y).$$

Substituting this and (4.2) into (3.13) such that $f_1 = 0$, we obtain

$$\begin{aligned}&\{X\gamma - \gamma\tau(X)\}g(Y, PZ) - \{Y\gamma - \gamma\tau(Y)\}g(X, PZ)\\&+ \gamma\{B(X, PZ)\eta(Y) - B(Y, PZ)\eta(X)\}\\&- \rho(X)D(Y, PZ) + \rho(Y)D(X, PZ) = 0.\end{aligned}$$

Taking $Y = \xi$ and $PZ = U$ and using (2.9), (3.5), (3.6) and (4.2), we have

$$\gamma^2 u(X) + \gamma\rho(\xi)w(X) = \{\xi\gamma - \gamma\tau(\xi)\}v(X).$$

(1) Replacing X by U to this, we get $\gamma = 0$. Thus $S(TM)$ is totally geodesic. As $C = 0$, from (2.3) we see that $S(TM)$ is a parallel distribution.

(2) As $S(TM)$ is a parallel distribution, $Rad(TM)$ is also an auto-parallel distribution by (2.5) and (2.10), and $TM = Rad(TM) \oplus S(TM)$, by the decomposition theorem of de Rham [2], M is locally a product manifold $\mathcal{C}_\xi \times M^*$, where $\mathcal{C}_\xi$ is a null geodesic tangent to $Rad(TM)$ and M^* is a leaf of $S(TM)$.

(3) As $f_1 = f_2\theta = A_{N} = 0$, from (3.11), the curvature tensor R is given by

$$R(X,Y)Z = D(Y,Z)A_{L}X - D(X,Z)A_{L}Y.$$

(4) As $A_{N} = \phi = 0$, (4.1) is reduced to

$$R^{(0,2)}(X,Y) = D(X,Y)tr\, A_{L} - g(A_{L}X, A_{L}Y). \tag{4.3}$$

Thus $R^{(0,2)}$ is symmetric induced Ricci tensor of M. By Theorem 2.1, $d\tau = 0$ and the transversal connection is flat.

(5) As $C = 0$, using (2.8) and $(3.6)_2$, we have

$$D(X,U) = 0, \qquad A_{L}X = \rho(X)\xi.$$

Substituting (2.17) into (4.3) with $X = V$ and $Y = U$ and using the last equations, we obtain $\kappa = 0$. Therefore, M is Ricci flat.

Denote by $\mathcal{G} = J(Rad(TM)) \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o$. Then $\mathcal{G}$ is a complementary vector subbundle to $J(ltr(TM))$ in $S(TM)$ and we have

$$S(TM) = J(ltr(TM)) \oplus \mathcal{G}.$$

Theorem 4.2. *Let M be a statical transversal half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature. If $S(TM)$ is totally umbilical, then M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_\xi$ and $\mathcal{C}_U$ are null geodesics tangent to $Rad(TM)$ and $J(ltr(TM))$ respectively and $M^\sharp$ is a leaf of the distribution $\mathcal{G}$ of M.*

Proof. By (4) of Theorem 4.1, we get $d\tau = 0$. Thus we can take $\tau = 0$ by Note 1, without loss generality. Also as $A_N = \rho = 0$, from (3.7), we have

$$\nabla_X U = 0. \tag{4.4}$$

Thus $J(ltr(TM))$ is parallel. From (2.5) and (2.10), $Rad(TM)$ is also parallel. For any $X \in \Gamma(\mathcal{G})$ and $Y \in \Gamma(H_o)$, using (4.4), we derive

$$g(\nabla_X Y, U) = 0, \qquad g(\nabla_X V, U) = 0, \qquad g(\nabla_X W, U) = 0.$$

Thus $\mathcal{G}$ is also parallel. By the decomposition theorem of de Rham [2], M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_U \times M^\sharp$, where $\mathcal{C}_\xi$ and $\mathcal{C}_U$ are null geodesics tangent to $Rad(TM)$ and $J(ltr(TM))$ respectively and $M^\sharp$ is a leaf of $\mathcal{G}$.

5. Screen homothetic submanifolds

Definition 4. A half lightlike submanifold M is called *screen homothetic* [5] if there exists a non-zero constant φ such that $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi B(X, Y). \tag{5.1}$$

Note 2. As $\bar{R} = 0$, the form (2.16) of the tensor field $R^{(0,2)}$ is reduced to

$$\begin{aligned} R^{(0,2)}(X, Y) = B(X, Y) tr\, A_N + D(X, Y) tr\, A_L + \rho(X)\phi(Y) \\ - \varphi g(A^*_\xi X, A^*_\xi Y) - g(A_L X, A_L Y). \end{aligned} \tag{5.2}$$

It follows that if M is statical, then $R^{(0,2)}$ is symmetric. Thus $d\tau = 0$ and the transversal connection is flat. As $d\tau = 0$, we can take $\tau = 0$ by Note 1.

As $\{U, V\}$ is a null basis of $J(Rad(TM)) \oplus J(ltr(TM))$, let

$$\mu = U - \varphi V, \qquad \nu = U + \varphi V,$$

$\{\mu, \nu\}$ form an orthogonal basis of $J(Rad(TM)) \oplus J(ltr(TM))$. From (2.6), (2.8), (3.6), (5.1) and the fact that $\rho = 0$, we see that

$$B(X, \mu) = 0, \qquad D(X, \mu) = 0, \qquad A^*_\xi \mu = 0, \qquad A_L \mu = 0. \tag{5.3}$$

Let $\mathcal{H}' = Span\{\mu\}$. Then $\mathcal{H} = H_o \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} Span\{\nu\}$ is a complementary vector subbundle to $\mathcal{H}'$ in $S(TM)$ and we have

$$S(TM) = \mathcal{H}' \oplus_{orth} \mathcal{H}. \tag{5.4}$$

Theorem 5.1. *Let M be a statical transversal screen homothetic half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of quasi-constant curvature. Then M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\natural$, where $\mathcal{C}_\xi$ and $\mathcal{C}_\mu$ are null and non-null geodesics tangent to $Rad(TM)$ and $\mathcal{H}'$, respectively and $M^\natural$ is a leaf of the distribution $\mathcal{H}$ of M.*

Proof. Using (3.7), (3.8) and the fact that F is linear operator, we have

$$\nabla_X \mu = 0. \tag{5.5}$$

This implies that $\mathcal{H}'$ is parallel. From (2.5) and (2.10), $Rad(TM)$ is also parallel. For any $X \in \Gamma(\mathcal{H})$ and $Y \in \Gamma(D_o)$, using (5.5), we derive

$$g(\nabla_X Y, \mu) = 0, \qquad g(\nabla_X V, \mu) = 0, \qquad g(\nabla_X W, \mu) = 0.$$

Thus $\mathcal{H}$ is also a parallel distribution. By the decomposition theorem [2], M is locally a product manifold $\mathcal{C}_\xi \times \mathcal{C}_\mu \times M^\natural$, where $\mathcal{C}_\xi$ and $\mathcal{C}_\mu$ are null and non-null geodesics tangent to $Rad(TM)$ and $\mathcal{H}'$ respectively and $M^\natural$ is a leaf of $\mathcal{H}$.

Theorem 5.2. *Let M be a statical transversal screen homothetic Einstein half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ of a quasi-constant curvature. Then M is Ricci flat, i.e., $\kappa = 0$.*

Proof. Since M is Einstein manifold, (5.2) is reduced to

$$\begin{aligned} &g(A_{_L}X, A_{_L}Y) + \varphi g(A_\xi^* X, A_\xi^* Y) \\ &- g(A_\xi^* X, Y) tr\, A_{_N} - g(A_{_L}X, Y) tr\, A_{_L} + \kappa g(X, Y) = 0. \end{aligned}$$

Put $X = Y = \mu$ and using $(5.3)_{3,\,4}$, we have $\kappa = 0$. Thus M is Ricci flat.

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