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CONVERGENCE OF NEWTON'S METHOD FOR SOLVING A NONLINEAR MATRIX EQUATION

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Abstract. We consider the nonlinear matrix equation $X^p + AX^qB + CXD + E = 0$, where p and q are positive integers, A, B and E are $n \times n$ nonnegative matrices, C and D are arbitrary $n \times n$ real matrices. A sufficient condition for the existence of the elementwise minimal nonnegative solution is derived. The monotone convergence of Newton's method for solving the equation is considered. Several numerical examples to show the efficiency of the proposed Newton's method are presented.

1. Introduction

We consider the nonlinear matrix equation

$$X^p + AX^qB + CXD + E = 0, \tag{1.1}$$

where p and q are positive integers, A, B and E are $n \times n$ nonnegative matrices, C and D are arbitrary $n \times n$ real matrices.

Nonlinear matrix equations play an important role in control theory, ladder networks, dynamic programming, stochastic filtering, and many other areas [1, 3, 9, 11]. The positive definite solutions of nonlinear matrix equation (1.1) has been widely studied, see [6, 15, 16, 17, 22, 23] and the references therein. Many methods have been proposed for the numerical solutions, such as invariant subspace methods [21], fixed-point iteration [15], inversion-free variant iteration [24, 27], and Newton's method [8, 9, 12]. In this paper, we consider the positive (nonnegative) solution of nonlinear matrix equation (1.1) instead of the positive definite solution. Newton's method is widely used for obtaining the positive (nonnegative) solution of a matrix equation or matrix polynomials. Davis [4, 5] considered Newton's method for solving a quadratic matrix equation. Higham

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and Kim [13, 14] incorporated the exact line searches into Newton's method and improved the global convergence. In [26], Seo and Kim considered the convergence of pure and relaxed Newton's methods. More applications of Newton's method to matrix polynomials with degree $n > 2$ is shown in [2, 19, 20]. Guo and Laub [11] and Kim [18] showed that the elementwise minimal positive solutions can be found by Newton's method. As far as we know, the nonnegative solutions of the nonlinear matrix equation (1.1) haven't been studied yet. In this paper, we show the existence of the elementwise minimal nonnegative solution by applying Newton's method.

This paper is organized as follows. In Section 2, we derive a sufficient condition for the existence of the elementwise minimal nonnegative solution and consider the monotone convergence of Newton's method for solving equation (1.1). In Section 3, we apply Newton's method to a special case of nonlinear matrix equation (1.1). Finally, we present several numerical examples to show the efficiency of the proposed Newton's method.

Some relevant definitions and notations throughout this paper are as follows. A real square matrix A is called a Z-matrix if all its off-diagonal elements are nonpositive. It is clear that any Z-matrix A can be written as $sI-B$ with $B \geq 0$. A Z-matrix A is called an M-matrix if $s \geq \rho(B)$, where $\rho(\cdot)$ is the spectral radius; it is a singular M-matrix if $s = \rho(B)$ and a nonsingular M-matrix if $s > \rho(B)$. $\mathbb{R}^{n\times n}$ stands for the set of $n \times n$ matrices with elements on field $\mathbb{R}$. I_n means $n \times n$ identity matrix. For a matrix $A = (a_1, a_2, \cdots a_n) = (a_{ij}) \in \mathbb{R}^{n\times n}$ and a matrix B, $\mathrm{vec}(A)$ is a vector defined by $\mathrm{vec}(A) = (a_1^T, \cdots, a_n^T)^T$; $A \otimes B = (a_{ij}B)$ is a Kronecker product. $A \geq B (A > B)$ means $a_{ij} \geq b_{ij}$ $(a_{ij} > b_{ij})$.

2. Convergence of Newton's method

In this section, we show the monotone convergence of Newton's method and derive a sufficient condition for the existence of the elementwise minimal nonnegative solution of equation (1.1). To this end, we need the following well-known results.

Definition 1. ([18]) Let F be a matrix function from $\mathbb{R}^{m\times n}$ to $\mathbb{R}^{m\times n}$. A positive (nonnegative) solution S_1 of the matrix equation $F(X) = 0$ is an elementwise minimal positive (nonnegative) solution and a positive (nonnegative) solution S_2 of $F(X) = 0$ is an elementwise maximal positive (nonnegative) solution if, for any positive (nonnegative) solution S of $F(X) = 0$,

$$S_1 \leq S \leq S_2.$$

Theorem 2.1. ([25], Theorem 2.1.) *For a Z-matrix A, the following are equivalent:*

1) *A is a nonsingular M-matrix.*
2) *A^{-1} is nonnegative.*
3) *$Av > 0$ for some vector $v > 0$.*
4) *All eigenvalues of A have positive real parts.*

Theorem 2.2. *(*[10], Lemma 2.2.*) Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix, then*
1) $Av \geq 0$ implies $v \geq 0$.
2) If B is a Z-matrix and $B \geq A$, then B is also a nonsingular M-matrix.

Lemma 2.3. *(*[26], Lemma 2.3.*) The following statements are true:*

1) *If $Y > X \geq 0$, then*
$$Y^m + (m-1)X^m - \sum_{k=1}^{m} X^{m-k} Y X^{k-1} > 0, \quad m \in \{2, 3, 4, \ldots\}.$$
2) *If $Y \geq X \geq 0$, then*
$$Y^m + (m-1)X^m - \sum_{k=1}^{m} X^{m-k} Y X^{k-1} \geq 0, \quad m \in \mathbb{N}.$$

Let $F(X) = X^p + AX^qB + CXD + E$, we first derive the Fréchet derivative of $F(X)$.

$$\begin{aligned} F(X+H) &= (X+H)^p + A(X+H)^qB + C(X+H)D + E \\ &= F(X) + \sum_{i=1}^{p} X^{p-i}HX^{i-1} + \sum_{j=1}^{q} AX^{q-j}HX^{j-1}B + CHD + O(H^2). \end{aligned}$$

The Fréchet derivative of $F(X)$ at X in the direction H is given by

$$D_X(H) = \sum_{i=1}^{p} X^{p-i}HX^{i-1} + \sum_{j=1}^{q} AX^{q-j}HX^{j-1}B + CHD. \tag{1}$$

Remark 1. We call that the linear operator D_X is regular (see Theorem A3 in [19] and Lemma 3.1 in [14]), if $\|D_X\| > 0$ where
$$\|D_X\| = \inf_{\|H\|=1} \|D_X(H)\|,$$
which means D_X is invertible [19].

Let $X_0 = 0$, Newton's method for solving equation (1.1) is given as following:

$$\begin{cases} D_{X_k}(H_k) = -F(X_k), \\ X_{k+1} = X_k + H_k, \end{cases} \tag{2}$$

where $D_{X_k}(H_k) = \sum_{i=1}^{p} X_k^{p-i}H_kX_k^{i-1} + \sum_{j=1}^{q} AX_k^{q-j}H_kX_k^{j-1}B + CH_kD$ is the Fréchet derivative in (1). Suppose that D_{X_k} is regular, then iteration (2) can be written as
$$X_{k+1} = X_k + D_{X_k}^{-1}(-F(X_k)), \quad k = 0, 1, \ldots$$

By the vec function and the Kronecker product [19, 26], we can get

$$\mathcal{M}_{X_k}\mathrm{vec}(H_k) = \mathrm{vec}(-F(X_k)), \tag{3}$$

where

$$\mathcal{M}_{X_k} = \Big[\sum_{i=1}^{p}(X_k^{i-1})^T \otimes X_k^{p-i} + \sum_{j=1}^{q}(B^T \otimes A)\Big((X_k^{j-1})^T \otimes X_k^{q-j}\Big) + D^T \otimes C\Big].$$

Then from the $n^2 \times n^2$ linear system (3), we can get H_k and consequently X_{k+1}.

Lemma 2.4. *Suppose $A \geq 0, B \geq 0$, and if there are positive matrices X and Y such that $Y > X \geq 0$, $F(X) \geq 0$ and $F(Y) \leq 0$, then $-\mathcal{M}_X$ is a nonsingular M-matrix.*

Proof. Since $F(X) \geq 0$ and $F(Y) \leq 0$, we have

$$-CXD \leq X^p + AX^qB + E,$$

and

$$CYD + E \leq -Y^p - AY^qB.$$

Then applying Lemma 2.4, we can get

$$\begin{aligned}
&D_X(Y-X)\\
&= \sum_{i=1}^{p} X^{p-i}(Y-X)X^{i-1} + \sum_{j=1}^{q} AX^{q-j}(Y-X)X^{j-1}B + C(Y-X)D\\
&= \sum_{i=1}^{p} X^{p-i}YX^{i-1} - pX^p + \sum_{j=1}^{q} AX^{q-j}YX^{j-1}B - qAX^qB + C(Y-X)D\\
&\leq \sum_{i=1}^{p} X^{p-i}YX^{i-1} - (p-1)X^p + CYD + E\\
&\qquad + \sum_{j=1}^{q} AX^{q-j}YX^{j-1}B - (q-1)AX^qB\\
&\leq \Big(\sum_{i=1}^{p} X^{p-i}YX^{i-1} - (p-1)X^p - Y^p\Big)\\
&\qquad + A\Big(\sum_{j=1}^{q} X^{q-j}YX^{j-1} - (q-1)X^q - Y^q\Big)B\\
&< 0.
\end{aligned}$$

Since $Y > X$, by Theorem 2.2., we can get $-\mathcal{M}_X$ is a nonsingular M-matrix. $\square$

Theorem 2.5. *Suppose $p \geq 2$, $q \geq 2$, $A \geq 0, B \geq 0$, $E \geq 0$, and $-D^T \otimes C$ is a nonsingular M-matrix. If there is a positive matrix Y such that $F(Y) \leq 0$, then equation (1.1) has an elementwise minimal nonnegative solution S and*

$S \le Y$. And the sequence $\{X_m\}$ from the Newton's iteration (2) with $X_0 = 0$ is well-defined and $\lim_{m\to\infty} X_m = S$. Furthermore,

$$-\mathcal{M}_{X_m} = -\Big[\sum_{i=1}^{p}(X_m^{i-1})^T \otimes X_m^{p-i} + \sum_{j=1}^{q}(B^T \otimes A)\Big((X_m^{j-1})^T \otimes X_m^{q-j}\Big) + D^T \otimes C\Big]$$

is a nonsingular M-matrix for each $m = 0, 1, \dots$

Proof. We will proof the following statements

$$X_m \le X_{m+1}, \tag{4}$$

$$X_m < Y, \tag{5}$$

$$F(X_m) \ge 0, \tag{6}$$

and

$$-\mathcal{M}_{X_m} \text{ is a nonsingular } M - \text{matrix} \tag{7}$$

are true for $m = 0, 1, \dots$

Trivially, statements (5)-(7) are true for $m = 0$. Since $E \ge 0$ and $-D^T \otimes C$ is a nonsingular M-matrix, we can get that

$$\mathrm{vec}(X_1) = (-D^T \otimes C)^{-1}\mathrm{vec}(E) \ge 0.$$

Hence, $X_0 \le X_1$. Suppose that statements (4)-(7) are true for $m = k$, it is sufficient to show that they are true for $m = k+1$. Note that $D_{X_k}(H_k) = -F(X_k)$, it yields

$$\begin{aligned}
&\sum_{i=1}^{p} X_k^{p-i} X_{k+1} X_k^{i-1} + \sum_{j=1}^{q} A X_k^{q-j} X_{k+1} X_k^{j-1} B + C X_{k+1} D \\
&= (p-1)X_k^p + (q-1)A X_k^q B - E.
\end{aligned} \tag{8}$$

Then from (8) and Lemma 2.4, we have

$$
\begin{aligned}
&D_{X_k}(Y - X_{k+1})\\
&= \sum_{i=1}^{p} X_k^{p-i}(Y - X_{k+1})X_k^{i-1} + \sum_{j=1}^{q} AX_k^{q-j}(Y - X_{k+1})X_k^{j-1}B\\
&\qquad + C(Y - X_{k+1})D\\
&= \sum_{i=1}^{p} X_k^{p-i}YX_k^{i-1} - \sum_{i=1}^{p} X_k^{p-i}X_{k+1}X_k^{i-1} + \sum_{j=1}^{q} AX_k^{q-j}YX_k^{j-1}B\\
&\qquad - \sum_{j=1}^{q} AX_k^{q-j}X_{k+1}X_k^{j-1}B + CYD - CX_{k+1}D\\
&= \sum_{i=1}^{p} X_k^{p-i}YX_k^{i-1} + CYD + \sum_{j=1}^{q} AX_k^{q-j}YX_k^{j-1}B - (p-1)X_k^p\\
&\qquad - (q-1)AX_k^qB + E\\
&\leq \sum_{i=1}^{p} X_k^{p-i}YX_k^{i-1} - Y^p - AY^qB + \sum_{j=1}^{q} AX_k^{q-j}YX_k^{j-1}B - (p-1)X_k^p\\
&\qquad - (q-1)AX_k^qB\\
&= \Big(\sum_{i=1}^{p} X_k^{p-i}YX_k^{i-1} - (p-1)X_k^p\Big) - Y^p\\
&\qquad + A\Big(\sum_{j=1}^{q} X_k^{q-j}YX_k^{j-1} - (q-1)X_k^q - Y^q\Big)B\\
&< 0.
\end{aligned}
$$

It follows that $-\mathcal{M}_{X_k}\mathrm{vec}(Y - X_{k+1}) > 0$. Since $-\mathcal{M}_{X_k}$ is a nonsingular M-matrix, by Theorem 2.3, we can get $X_{k+1} < Y$.

Now we show that the statements (4), (6) and (7) are true for $m = k + 1$. For convenience, we introduce a function $\Phi : \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ defined in [26]:

$$
\begin{cases}
\Phi(i,0)(X,Y) = X^i, \quad \Phi(0,j)(X,Y) = Y^j, \quad i,j \in \{0\} \cup \mathbb{N}^+,\\
\Phi(i,j)(X,Y) = X\Phi(i-1,j)(X,Y) + Y\Phi(i,j-1)(X,Y), \quad i,j \in \mathbb{N}^+.
\end{cases}
$$

Then

$$\begin{aligned}
F(X_{k+1}) &= (X_k + H_k)^p + A(X_k + H_k)^q B + C(X_k + H_k)D + E \\
&= \sum_{i=0}^{p} \Phi(p-i,i)(X_k, H_k) + \sum_{j=0}^{q} A\Phi(q-j,j)(X_k, H_k)B \\
&\quad\ + CX_kD + E + CH_kD \\
&= F(X_k) + \Phi(p-1,1)(X_k,H_k) + A\Phi(q-1,1)(X_k,H_k)B + CH_kD \\
&\quad + \sum_{i=2}^{p} \Phi(p-i,i)(X_k, H_k) + \sum_{j=2}^{q} A\Phi(q-j,j)(X_k, H_k)B \\
&= F(X_k) + D_{X_k}(H_k) + \sum_{i=2}^{p} \Phi(p-i,i)(X_k, H_k) \\
&\quad\ + \sum_{j=2}^{q} A\Phi(q-j,j)(X_k, H_k)B.
\end{aligned}$$

Since Φ is regarded as the sum of all the possible products of X_k and H_k, and $X_k \geq 0, H_k \geq 0$, then

$$\sum_{i=2}^{p} \Phi(p-i,i)(X_k, H_k) + \sum_{j=2}^{q} A\Phi(q-j,j)(X_k, H_k)B \geq 0,$$

it yields

$$\begin{aligned}
F(X_{k+1}) &\geq F(X_k) + D_{X_k}(H_k) \\
&= F(X_k) + D_{X_k}(D_{X_k}^{-1}(-F(X_k))) \\
&= 0.
\end{aligned}$$

So (6) is true for $m = k+1$. Then by Lemma 2.6, we can get statement (7) is also true. From $D_{X_{k+1}}(X_{k+2} - X_{k+1}) = -F(X_k)$ it follows

$$\mathrm{vec}(X_{k+2} - X_{k+1}) = -\mathcal{M}_{X_{k+1}}^{-1}\mathrm{vec}(F(X_{k+1})) \geq 0,$$

which implies $X_{k+2} \geq X_{k+1}$.

Since the sequence $\{X_m\}$ is monotone increasing and bounded above, it is convergent. Let $\lim_{m\to\infty} X_m = S$, then S satisfies equation (1.1). Since for any positive solution Y, $S \leq Y$. Therefore, S is the elementwise minimal nonnegative solution. $\square$

Corollary 2.6. *If $p = 1$ and $q \geq 2$, $A, B \geq 0$ $(A \neq 0, B \neq 0)$, $E \geq 0$, and $-(I_{2n} + D^T \otimes C)$ is a nonsingular M-matrix. Suppose that there is a positive matrix Y such that $F(Y) \leq 0$, then equation (1.1) has an elementwise minimal nonnegative solution S and $S \leq Y$. And the sequence $\{X_m\}$ from the Newton's* *iteration (2) with* $X_0 = 0$ *is well-defined and* $\lim_{m\to\infty} X_m = S$. *Furthermore,*

$$-\mathcal{M}_{X_m} = -\Big[\sum_{i=1}^{q}(B^T \otimes A)\Big((X_m^{i-1})^T \otimes X_m^{q-i}\Big) + I_{2n} + D^T \otimes C\Big]$$

is a nonsingular M-matrix for each $m = 0, 1, \dots$

Corollary 2.7. *If* $p \geq 2$ *and* $q = 1$, $E \geq 0$, *and* $-(B^T \otimes A + D^T \otimes C)$ *is a nonsingular M-matrix. Suppose that there is a positive matrix* Y *such that* $F(Y) \leq 0$, *then equation (1.1) has an elementwise minimal nonnegative solution* S *and* $S \leq Y$. *And the sequence* $\{X_m\}$ *from the Newton's iteration (2) with* $X_0 = 0$ *is well-defined and* $\lim_{m\to\infty} X_m = S$. *Furthermore,*

$$-\mathcal{M}_{X_m} = -\Big[\sum_{i=1}^{p}(X_m^{i-1})^T \otimes X_m^{p-i} + B^T \otimes A + D^T \otimes C\Big]$$

is a nonsingular M-matrix for each $m = 0, 1, \dots$

Corollary 2.8. *If* $p = q = 1$, $E \geq 0$, *and* $-(I_{2n} + B^T \otimes A + D^T \otimes C)$ *is a nonsingular M-matrix. Suppose that there is a positive matrix* Y *such that* $F(Y) \leq 0$, *then equation (1.1) has an elementwise minimal nonnegative solution* S *and* $S \leq Y$. *And the sequence* $\{X_m\}$ *from the Newton's iteration (2) with* $X_0 = 0$ *is well-defined and* $\lim_{m\to\infty} X_m = S$.

3. A special Case

In this section, we consider two special nonlinear matrix equations

$$X^p \pm CXD + E = 0, \tag{9}$$

where p is a positive integer, $C, D \in \mathbb{R}^{n\times n}$, $E \geq 0$.

Without loss of generality, we consider the matrix equation

$$G(X) = X^p - CXD + E = 0. \tag{10}$$

Newton's method for solving equation (10) is

$$\begin{cases} D'_{X_k}(H_k) = -G(X_k), \\ X_{k+1} = X_k + H_k, \end{cases} \tag{11}$$

where $D'_{X_k}(H_k) = \sum_{i=1}^{p} X_k^{p-i} H_k X_k^{i-1} - CH_kD$ is the Fréchet derivative of $G(X)$ at X_k in the direction H_k .

Theorem 3.1. *Suppose* $p \geq 2$, $E \geq 0$, *and* $D^T \otimes C$ *is a nonsingular M-matrix. If there is a positive matrix* Y *such that* $G(Y) \leq 0$, *then equation (10) has an elementwise minimal nonnegative solution* S *and* $S \leq Y$. *And the sequence* $\{X_m\}$ *from the Newton's iteration (11) with* $X_0 = 0$ *is well-defined and* $\lim_{m\to\infty} X_m = S$. *Furthermore,*

$$-\mathcal{M'}_{X_m} = -\Big[\sum_{i=1}^{p}(X_m^{i-1})^T \otimes X_m^{p-i} - D^T \otimes C\Big]$$

is a nonsingular M-matrix for each $m = 0, 1, \dots$

Proof. The proof is similar with that of Theorem 2.7. $\square$

Corollary 3.2. *If* $p = 1$, $E \geq 0$, $-(I_{2n} + D^T \otimes C)$ *is a nonsingular M-matrix, suppose that there is a positive matrix* Y *such that* $G(Y) \leq 0$, *then equation (10) has an elementwise minimal nonnegative solution* S *and* $S \leq Y$. *And the sequence* $\{X_m\}$ *from the Newton's iteration (11) with* $X_0 = 0$ *is well-defined and* $\lim_{m\to\infty} X_m = S$.

4. Numerical examples

In this section, we present three numerical examples to show the efficiency of the proposed Newton's method. Our experiments were done in MATLAB 7.10.0 with machine precision around 10^{-16} and the iterations terminate if the relative residual $\rho_1(X_k)$ and $\rho_2(X_k)$ satisfy

$$\rho_1(X_k) = \frac{\|fl(X_k^p + AX_k^q B + CX_k D + E)\|_F}{\|X_k\|_F^p + \|A\|_F \|X_k\|_F^q \|B\| + \|C\|_F \|X_k\|_F \|D\|_F + \|E\|_F} \leq n \times 10^{-16},$$

and

$$\rho_2(X_k) = \frac{\|fl(X_k^p - CX_k D + E)\|_F}{\|X_k\|_F^p + \|C\|_F \|X_k\|_F \|D\|_F + \|E\|_F} \leq n \times 10^{-16}.$$

Example 4.1. *Let* $p = 1, 2, 3, 4$, $q = 3$, $A = rand(3)/5$, $B = rand(3)/4$, $E = rand(3)/3$, *and*

$$C = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}, \; D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.9 \end{pmatrix}.$$

We apply Newton's method (2) on equation (1.1). The results are shown in Figure 1.

From Figure 1, for $p < q, p = q, p > q$, we can see that the Newton's method converges to a positive solution of equation (1.1). And the bigger the p is, the smaller the number of iteration is.

Example 4.2. *Let*

$$C = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix},$$

$E = eye(4)$ *and* $p = 2, 3, 4, 6$. *We apply Newton's method (11) on equation (10), and the results are shown in Figure 2. It shows that for different p, the Newton's method (11) works well and when p becomes bigger, the number of iteration becomes smaller.*

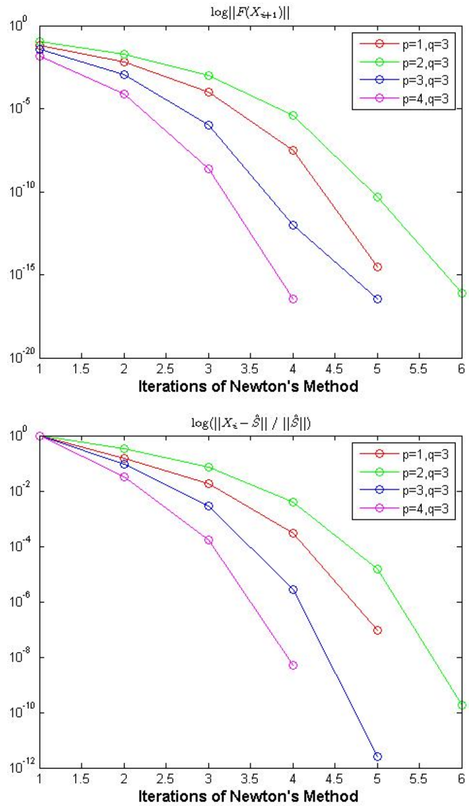

FIGURE 1. Convergence of Newton's method (2)

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Example 4.3. *Let* $p = q = 1,\ E = rand(2)/4,$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},\ B = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, C = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix},\ D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$-(I_{2n} + B^T \otimes A + D^T \otimes C) = \begin{pmatrix} -4 & 3 & 0 & 0 \\ 3 & -3 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 \end{pmatrix}$$

is a nonsingular M-matrix. Applying Newton's method (2) with relative residual $\rho_1(X_k) = 1.388 \times 10^{-16}$, we can get the positive solution

$$S = \begin{pmatrix} 0.1637 & 0.2896 \\ 0.2179 & 0.2330 \end{pmatrix}.$$

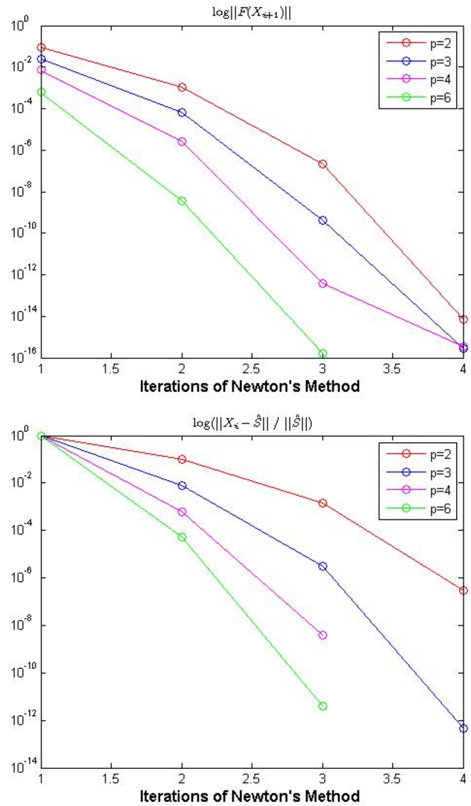

FIGURE 2. Convergence of Newton's method (11)

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