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A ROLE OF SYMBOLS OF MINIMUM TYPE IN EXPONENTIAL CALCULUS

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Abstract. We introduce formal symbols of product type and of minimum type and show that the formal power series representation for e^p is a formal symbol of product type, where p is a formal symbol of minimum type.

1. Introduction

Pseudodifferential operators are essential for studying the theory of partial differential equations and itself have been studied in various respects. A pseudodifferential operator can be represented as an equivalent class of some function defined on the cotangent bundle. We call this function a symbol of pseudodifferential operator. We can represent the operations of addition, scalar multiplication, composition, formal adjoint and change of variables of pseudodifferential operators by using these symbols and so can treat pseudodifferential operators concretely. Symbolic calculus means calculating pseudodifferential operators by using symbols. Exponential calculus means symbolic calculus on pseudodifferential operators expressed as symbols of exponential functions. A symbol of exponential function means that the symbol can be expressed as a form of exponential function. We can analyze (pseudo) differential operators of infinite order or (pseudo) differential equations of infinite order by studying the pseudodifferential operators expressed as symbols of exponential functions. In particular, Sato, M., T. Kawai and M. Kashiwara (ref. [16]) won epoch-making results in the theory of transformation of the system of linear partial (pseudo) differential equations by using pseudodifferential operators of infinite order. T. Aoki accomplished exponential calculus of analytic pseudodifferential operators(ref. [1] – [8]). The author considers the case of a kind of the direct product structure (ref. [12] – [15]). That is, we introduce and study calculus of analytic pseudodifferential operators of product type and generalize the theory of T. Aoki in some sense. This study is deeply connected with exponential calculus of positive definite operators of infinite order which have deep relation to the energy method in the hyperfunction theory(ref. [9] – [11]). In this article, we introduce

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formal symbols of product type and of minimum type and show that the formal power series representation for e^p is a formal symbol of product type, where p is a formal symbol of minimum type. Such formal symbols play a decisive role in exponential calculus.

2. Product Type and Minimum Type

Let $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ be domains. Then, the cotangent bundles T^*X and T^*Y are identified with $X \times \mathbb{C}^n$ and $Y \times \mathbb{C}^m$, respectively. We set

$$S^*X := (T^*X - X)/\mathbb{R}^+, \ S^*Y := (T^*Y - Y)/\mathbb{R}^+$$

and define the mapping γ as

$$\gamma\colon \overset{\circ\circ}{T^*}(X \times Y) \ni (z, w; \xi, \eta) \longmapsto \left(z; \frac{\xi}{|\xi|}\right) \times \left(w; \frac{\eta}{|\eta|}\right) \in S^*X \times S^*Y,$$

where

$$\overset{\circ\circ}{T^*}(X \times Y) := T^*(X \times Y) \backslash \{(T^*X \times Y) \cup (X \times T^*Y)\}.$$

For $d_1, d_2 > 0$ and an open subset U of $S^*X \times S^*Y$, we use the notation

$$\gamma^{-1}(U; d_1, d_2) := \gamma^{-1}(U) \cap \{|\xi| > d_1, |\eta| > d_2\}.$$

Hereafter we write (z, ξ, w, η) for coordinates $(z, w; \xi, \eta)$.
Let K be a compact subset of $S^*X \times S^*Y$.

Definition 2.1. A function $\Lambda : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ is said to be infra-linear if the following conditions hold:

(1) Λ is continuous.
(2) For each $\alpha > 1$, $\Lambda(\alpha t) \le \alpha\Lambda(t)$ on $(0, \infty)$.
(3) Λ is increasing.
(4) $\displaystyle\lim_{t\to\infty} \frac{\Lambda(t)}{t} = 0$.

Definition 2.2. A formal series $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$ is called a formal symbol of product type on K if the following hold:

(1) There are some constants $d > 0, 0 < A < 1$, and an open set $U \supset K$ in $S^*X \times S^*Y$ such that $P_{j,k}$ is holomorphic in $\gamma^{-1}(U; (j + 1)d, (k + 1)d)$ for each $j, k \ge 0$.
(2) There exists an infra-linear function Λ such that

$$|P_{j,k}(z, \xi, w, \eta)| \le A^{j+k} e^{\Lambda(|\xi|+|\eta|)} \quad \text{on } \gamma^{-1}(U; (j + 1)d, (k + 1)d) \tag{1}$$

for each $j, k \ge 0$.

We denote by $\widehat{S}(K)$ the set of such formal symbols on K.
We often write a formal power series $\sum_{j,k=0}^{\infty} t_1^j t_2^k P_{j,k}(z, \xi, w, \eta)$ with indeterminates t_1 and t_2 instead of $\sum_{j,k=0}^{\infty} P_{j,k}(z, \xi, w, \eta)$.

Let $\Lambda_1(t)$, $\Lambda_2(t^*)$ be infra-linear functions of t , t^*, respectively and put

$$\tilde{\Lambda}(\xi, \eta) = \min\{\Lambda_1(|\xi|), \Lambda_2(|\eta|)\}.$$

Definition 2.3. A formal series $\sum_{j,j'=0}^{\infty} {t_1}^j {t_2}^{j'} p_{j,j'}(z,\xi,w,\eta)$ is called a formal symbol of minimum type defined on K if there exist some positive constants C, d, $0<A<1$, $\tilde{\Lambda}$, and some open set $U \supset K$ such that the following hold:

(1) $p_{j,j'}$ is holomorphic in $\gamma^{-1}(U;(j+1)d,(j'+1)d)$ for each $j,j' \geq 0$.
(2) The inequality

$$|p_{j,j'}(z,\xi,w,\eta)| \leq CA^{j+j'}\tilde{\Lambda}(\xi,\eta) \tag{2}$$

holds on $\gamma^{-1}(U;(j+1)d,(j'+1)d)$ for each $j,j' \geq 0$.

Theorem 2.1. *If $p = \sum_{j,j'=0}^{\infty} {t_1}^j {t_2}^{j'} p_{j,j'}(z,\xi,w,\eta)$ is a formal symbol of minimum defined on K, the formal power series representation for e^p is a formal symbol of product type on K.*

Proof. To prove that e^p belongs to $\widehat{S}(K)$ we express e^p as a formal power series with indeterminates t_1 and t_2.

$$\sum_{j,j'=0}^{\infty} {t_1}^j {t_2}^{j'} e_{j,j'}(z,\xi,w,\eta) := e^p$$

Then, we can obtain the following coefficients.

$$e_{0,0} = e^{p_{0,0}}. \tag{3}$$

$$e_{j,0} = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j_1+\cdot+j_k=j} \prod_{\nu=1}^{k} p_{j_\nu,0}, \quad j \geq 1. \tag{4}$$

$$e_{0,j'} = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j'_1+\cdot+j'_k=j'} \prod_{\nu=1}^{k} p_{0,j'_\nu}, \quad j' \geq 1. \tag{5}$$

$$e_{j,j'} = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\substack{j_1+\cdots+j_k=j \\ j'_1+\cdots+j'_k=j'}} \prod_{\nu=1}^{k} p_{j_\nu,j'_\nu}, \quad j \geq 1, j' \geq 1. \tag{6}$$

The following inequality holds on $\gamma^{-1}(U;d,d)$ in the case of (3).

$$|e_{0,0}(z,\xi,w,\eta)| \leq e^{\tilde{\Lambda}(\xi,\eta)} \tag{7}$$

We need Lemma 2.2 to deal with the cases of (4) and (5).

Lemma 2.2. *For each $j \geq 1$ and each real number s*

$$\sum_{k=1}^{\infty} \binom{j+k-1}{j} \frac{s^k}{k!} = e^s \sum_{l=0}^{j-1} \binom{j-1}{l} \frac{s^{l+1}}{(l+1)!}.$$

Proof. We consider the identity

$$\sum_{k=0}^{\infty} \frac{s^k}{k!} = e^s. \tag{8}$$

If we differentiate both sides of the identity (8) , we obtain the following identity

$$\sum_{k=1}^{\infty} \frac{ks^{k-1}}{k!} = e^s. \tag{9}$$

If we multiply both sides of the identity (9) by s^j,

$$\sum_{k=1}^{\infty} \frac{ks^{j+k-1}}{k!} = s^j e^s. \tag{10}$$

If we calculuate the $(j-1)-th$ derivatives of both sides in (10),

$$\sum_{k=1}^{\infty} \frac{(j+k-1)\cdots(k+1)ks^k}{k!} = \sum_{m=0}^{j-1} \binom{j-1}{m} j(j-1)\cdots(j-m+1)s^{j-m}e^s. \tag{11}$$

If we divide both sides of (11) by $j!$,

$$\sum_{k=1}^{\infty} \frac{(j+k-1)\cdots(k+1)ks^k}{j!k!} = \sum_{m=0}^{j-1} \binom{j-1}{j-m-1} \frac{j(j-1)\cdots(j-m+1)}{j!} s^{j-m}e^s \tag{12}$$

$$= \sum_{m=0}^{j-1} \binom{j-1}{j-m-1} \frac{s^{j-m}}{(j-m)!} e^s. \tag{13}$$

If we put $l = j-m-1$ in (13), we can complete the proof of Lemma 2.2 □

(continued) If we put $c = 1 - A$ and $B = A(1+c)$, then $0 < c < 1$ and $0 < A < B < 1$. the following inequalities hold on $\gamma^{-1}(U; (j+1)d, d)$ for each $j \geq 1$ with the aid of Lemma 2.2.

$$\begin{aligned}
|e_{j,0}(z,\xi,w,\eta)| &\leq \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j_1+\cdot+j_k=j} A^j (\tilde{\Lambda}(\xi,\eta))^k \\
&= A^j \sum_{k=1}^{\infty} \binom{j+k-1}{j} \frac{(\tilde{\Lambda}(\xi,\eta))^k}{k!} \\
&= A^j e^{\tilde{\Lambda}(\xi,\eta)} \sum_{l=0}^{j-1} \binom{j-1}{l} \frac{(\tilde{\Lambda}(\xi,\eta))^{l+1}}{(l+1)!} \\
&\leq A^j e^{\tilde{\Lambda}(\xi,\eta)} e^{\tilde{\Lambda}(\xi,\eta)/c} \sum_{l=0}^{j-1} \binom{j-1}{l} c^{l+1} \\
&\leq A^j (1+c)^j e^{\tilde{\Lambda}(\xi,\eta)} e^{\tilde{\Lambda}(\xi,\eta)/c} = B^j e^{\tilde{\Lambda}(\xi,\eta)(1+c)/c}
\end{aligned}$$

In like manners, we can show that the following inequality holds on $\gamma^{-1}(U; d, (j'+1)d)$ for each $j' \geq 1$, with the aid of Lemma 2.2.

$$|e_{0,j'}(z,\xi,w,\eta)| \leq B^{j'} e^{\tilde{\Lambda}(\xi,\eta)(1+c)/c}$$

We also need Lemma 2.3 to estimate the case of (6).

Lemma 2.3. *For each $j \geq 1, j' \geq 1$, and each real number s*

$$\sum_{k=1}^{\infty} \binom{j+k-1}{j} \binom{j'+k-1}{j'} \frac{s^k}{k!} = e^s \sum_{l=0}^{j-1} \binom{j-1}{l} \frac{s^{l+1}}{(l+1)!} \sum_{r=0}^{j'} \binom{l+j'}{l+r} \frac{s^r}{r!}.$$

Proof. If we multiply both sides of the identity in Lemma 2.2 by $s^{j'-1}$,

$$\sum_{k=1}^{\infty} \binom{j+k-1}{j} \frac{s^{j'+k-1}}{k!} = \sum_{l=0}^{j-1} \binom{j-1}{l} \frac{1}{(l+1)!} s^{l+j'} e^s. \tag{14}$$

If we calculate the $j'-th$ derivatives of both sides in (14),

$$\begin{aligned}
&\sum_{k=1}^{\infty} \binom{j+k-1}{j} \frac{(j'+k-1)\cdots(k+1)ks^{k-1}}{k!} \\
&= \sum_{l=0}^{j-1} \binom{j-1}{l} \frac{1}{(l+1)!} \frac{d^{j'}}{ds^{j'}} (s^{l+j'} e^s) \\
&= \sum_{l=0}^{j-1} \binom{j-1}{l} \frac{1}{(l+1)!} \sum_{m=0}^{j'} \binom{j'}{m} (l+j')\cdots(l+j'-m+1) s^{l+j'-m} e^s \\
&= \sum_{l=0}^{j-1} \binom{j-1}{l} \frac{1}{(l+1)!} \sum_{m=0}^{j'} \frac{j'!}{(j'-m)!m!} (l+j')\cdots(l+j'-m+1) s^{l+j'-m} e^s
\end{aligned}$$

If we divide both sides of the identity by $j'!$ and put $r = j'-m$, we can complete the proof of Lemma 2.3. □

(continued) The following inequalities hold on $\gamma^{-1}(U;(j+1)d,(j'+1)d)$ for each $j\geq 1, j'\geq 1$, with the aid of Lemma 2.3.

$$
\begin{aligned}
|e_{j,j'}(z,\xi,w,\eta)| &\leq A^{j+j'}\sum_{k=1}^{\infty}\frac{1}{k!}\sum_{\substack{j_1+\cdots+j_k=j\\ j'_1+\cdots+j'_k=j'}}(\tilde{\Lambda}(\xi,\eta))^k \\
&= A^{j+j'}\sum_{k=1}^{\infty}\binom{j+k-1}{j}\binom{j'+k-1}{j'}\frac{(\tilde{\Lambda}(\xi,\eta))^k}{k!} \\
&= A^{j+j'}e^{\tilde{\Lambda}(\xi,\eta)}\sum_{l=0}^{j-1}\binom{j-1}{l}\frac{(\tilde{\Lambda}(\xi,\eta))^{l+1}}{(l+1)!}\sum_{r=0}^{j'}\binom{l+j'}{l+r}\frac{(\tilde{\Lambda}(\xi,\eta))^r}{r!} \\
&\leq A^{j+j'}e^{\tilde{\Lambda}(\xi,\eta)}e^{\tilde{\Lambda}(\xi,\eta)/c}\sum_{l=0}^{j-1}\binom{j-1}{l}\frac{(\tilde{\Lambda}(\xi,\eta))^{l+1}}{(l+1)!c^l}\sum_{r=0}^{j'}\binom{l+j'}{l+r}c^{l+r} \\
&\leq A^{j+j'}e^{\tilde{\Lambda}(\xi,\eta)}e^{\tilde{\Lambda}(\xi,\eta)/c}\sum_{l=0}^{j-1}\binom{j-1}{l}\frac{(\tilde{\Lambda}(\xi,\eta))^{l+1}}{(l+1)!c^l}(1+c)^{l+j'} \\
&\leq A^{j+j'}(1+c)^{j'}e^{\tilde{\Lambda}(\xi,\eta)}e^{\tilde{\Lambda}(\xi,\eta)/c}\sum_{l=0}^{j-1}\binom{j-1}{l}\frac{1}{(l+1)!}(\tilde{\Lambda}(\xi,\eta)(1+c)/c)^{l+1} \\
&\leq A^{j+j'}(1+c)^{j'}e^{\tilde{\Lambda}(\xi,\eta)}e^{\tilde{\Lambda}(\xi,\eta)/c}e^{\tilde{\Lambda}(\xi,\eta)(1+c)/c^2}\sum_{l=0}^{j-1}\binom{j-1}{l}c^{l+1} \\
&\leq A^{j+j'}(1+c)^{j'}e^{\tilde{\Lambda}(\xi,\eta)}e^{\tilde{\Lambda}(\xi,\eta)/c}e^{\tilde{\Lambda}(\xi,\eta)(1+c)/c^2}(1+c)^j \\
&= B^{j+j'}e^{\tilde{\Lambda}(\xi,\eta)(1+c)^2/c^2}
\end{aligned}
$$

This completes the proof of the Theorem 2.1. □

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