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BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER BASED ON SUBORDINATE CONDITIONS INVOLVING HURWITZ-LERCH ZETA FUNCTION

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Abstract. The purpose of the present paper is to introduce and investigate two new subclasses of bi-univalent functions of complex order defined in the open unit disk, which are associated with Hurwitz-Lerch zeta function and satisfying subordinate conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses. Several (known or new) consequences of the results are also pointed out.

1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{2}$$

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A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For some intriguing examples of functions and characterization of the class Σ, one could refer Srivastava *et al.*, [26] and the references stated therein (see also, [11]). Recently there has been triggering , interest to study the bi-univalent functions class Σ (see[11, 18, 21, 27, 29, 30]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$ ($\mathbb{N} := \{1, 2, 3, \dots\}$) is presumably still an open problem.

The study of operators plays an important role in the geometric function theory and its related fields. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better. The convolution or Hadamard product of two functions $f, h \in \mathcal{A}$ is denoted by $f * h$ and is defined as

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{3}$$

where $f(z)$ is given by (1) and $h(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$.

We recall here a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined in [25] by

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{4}$$

$(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C}, \text{when} |z| < 1; \Re(s) > 1 \text{ when } |z| = 1)$ where, as usual, $\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N}$, $(\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}), \mathbb{N} := \{1, 2, 3, ...\})$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [5], and the references stated therein (see also, [7, 10, 16, 22, 28]). Srivastava and Attiya [28] introduced and investigated the linear operator:

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{\mu,b} * f(z) \tag{5}$$

$(z \in \mathbb{U}; b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mu \in \mathbb{C}; f \in \mathcal{A})$, where, for convenience,

$$\mathcal{G}_{\mu,b}(z) := (1 + b)^{\mu} [\Phi(z, \mu, b) - b^{-\mu}] \quad (z \in \mathbb{U}). \tag{6}$$

We recall here the following relationships (given earlier by [22]) which follow easily by using (1), (5) and (6)

$$\mathcal{J}_{\mu,b} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b} \right)^{\mu} a_n z^n. \tag{7}$$

Motivated essentially by the Srivastava-Attiya operator, Murugusundaramoorthy [20] introduced the generalized integral operator

$$\mathcal{J}_{\mu,b}^{m,k} f(z) = z + \sum_{n=2}^{\infty} C_n^m(b,\mu) a_n z^n. \tag{8}$$

where

$$\Psi_n = C_n^m(b,\mu) = \left| \left(\frac{1+b}{n+b}\right)^{\mu} \frac{m!(n+k-2)!}{(k-2)!(n+m-1)!} \right| \tag{9}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mu \in \mathbb{C}, k \geq 2$ and $m > -1$. It is of interest to note that $J_{\mu,b}^{1,2}$ is the Srivastava-Attiya operator and $J_{0,b}^{m,k}$ is the well-known Choi-Saigo-Srivastava operator (see [17]). Suitably specializing the parameters m, k, μ and b in $\mathcal{J}_{\mu,b}^{m,k} f(z)$ we can get various integral operators introduced by Alexander [2]and Bernardi[4]. Further we get the Jung-Kim-Srivastava integral operator [12] closely related to some multiplier transformation studied by Flett [8].

An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there is an analytic function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. Ma and Minda [19] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disk $\mathbb{U}, \phi(0) = 1, \phi'(0) > 0$, and ϕ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f \in \mathcal{A}$ satisfying the subordination $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$.

A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}_{\Sigma}^*(\phi)$ and $\mathcal{K}_{\Sigma}(\phi)$.In the sequel, it is assumed that ϕ is an analytic function with positive real part in the unit disk $\mathbb{U}$, satisfying $\phi(0) = 1, \phi'(0) > 0$, and $\phi(\mathbb{U})$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0). \tag{10}$$

Motivated by the earlier work of Deniz[6], in the present paper we introduce two new subclasses of the function class Σ of complex order $\gamma \in \mathbb{C}\backslash\{0\}$, involving generalized Srivastava operator $\mathcal{J}_{\mu,b}^{m,k}$, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the new subclasses of the function class Σ. Several related classes are also considered, and connection to earlier known results are made.

Definition 1. A function $f \in \Sigma$ given by (1) is said to be in the class $\Sigma_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{z(\mathcal{J}_{\mu,b}^{m,k}f(z))'}{(1-\lambda)\mathcal{J}_{\mu,b}^{m,k}f(z)+\lambda z(\mathcal{J}_{\mu,b}^{m,k}f(z))'}-1\right)\prec\phi(z) \tag{11}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(\mathcal{J}_{\mu,b}^{m,k}g(w))'}{(1-\lambda)\mathcal{J}_{\mu,b}^{m,k}g(w)+\lambda w(\mathcal{J}_{\mu,b}^{m,k}g(w))'}-1\right)\prec\phi(w) \quad , \tag{12}$$

where $\gamma\in\mathbb{C}\backslash\{0\};\ 0\leqq\lambda<1;\ z,w\in\mathbb{U}$ and the function g is given by(2).

Definition 2. A function $f \in \Sigma$ given by (1) is said to be in the class $\mathcal{H}_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left((1-\lambda)\frac{\mathcal{J}_{\mu,b}^{m,k}f(z)}{z}+\lambda(\mathcal{J}_{\mu,b}^{m,k}f(z))'-1\right)\prec\phi(z) \tag{13}$$

and

$$1+\frac{1}{\gamma}\left((1-\lambda)\frac{\mathcal{J}_{\mu,b}^{m,k}g(w)}{w}+\lambda(\mathcal{J}_{\mu,b}^{m,k}g(w))'-1\right)\prec\phi(w) \tag{14}$$

where $\gamma\in\mathbb{C}\backslash\{0\};\ 0\leqq\lambda<1;\ z,w\in\mathbb{U}$ and the function g is given by (2).

Specializing the parameters b,k,λ,μ suitably, several (known and) new subclasses can be obtained from the class $\Sigma_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$. We present some of the subclasses of $\Sigma_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$, as given below :

Example 1. For $\lambda=0$ and $\gamma\in\mathbb{C}\backslash\{0\}$, a function $f\in\Sigma$, given by (1) is said to be in the class $\Sigma_{\mu,b}^{m,k}(\gamma,\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{z(\mathcal{J}_{\mu,b}^{m,k}f(z))'}{\mathcal{J}_{\mu,b}^{m,k}f(z)}-1\right)\prec\phi(z) \tag{15}$$

and

$$1+\frac{1}{\gamma}\left(\frac{w(\mathcal{J}_{\mu,b}^{m,k}g(w))'}{\mathcal{J}_{\mu,b}^{m,k}g(w)}-1\right)\prec\phi(w) \tag{16}$$

where $z,w\in\mathbb{U}$ and the function g is given by (2).

Example 2. If $k=2$ and $m=1$ with $\mu=0$, $b=0$, then a function $f\in\Sigma$, given by (1) is said to be in the class $\Sigma_{0,0}^{1,2}(\gamma,\lambda,\phi)\equiv\mathcal{S}_{\Sigma}^{*}(\gamma,\lambda,\phi)$ if the following conditions are satisfied

$$1+\frac{1}{\gamma}\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda f'(z)}-1\right)\prec\phi(z)$$

and

$$1+\frac{1}{\gamma}\left(\frac{wg'(w)}{(1-\lambda)g(w)+\lambda g'(w)}-1\right)\prec\phi(w),$$

where $\gamma \in \mathbb{C}\backslash\{0\},\ 0 \leqq \lambda < 1,\ z, w \in \mathbb{U}$ and the function g is given by (2).

Example 3. If $k = 2$ and $m = 1$ with $\mu = 1, b = 0$, then a function $f \in \Sigma$, given by (1) is said to be in the class $\Sigma_{1,0}^{1,2}(\gamma, \lambda, \phi) \equiv \mathcal{G}_{\Sigma}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{L}f(z))'}{(1-\lambda)\mathcal{L}f(z) + \lambda z(\mathcal{L}f(z))'} - 1\right) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{w(\mathcal{L}g(w))'}{(1-\lambda)\mathcal{L}g(w) + \lambda w(\mathcal{L}g(w))'} - 1\right) \prec \phi(w)$$

where $\gamma \in \mathbb{C}\backslash\{0\},\ 0 \leqq \lambda < 1,\ z, w \in \mathbb{U}$, g is given by (2) and $\mathcal{L}f(z) = \mathcal{J}_{1,0}^{1,2}f(z) = z + \sum\limits_{n=2}^{\infty} \frac{a_n}{n} z^n,\ (z \in \mathbb{U})$ is the Alexander integral operator [2].

Example 4. If $k = 2$ and $m = 1$ with $b = \nu (\nu > -1)$, and $\mu = 1$, then $f \in \Sigma$ is said to be in $\Sigma_{1,\nu}^{1,2}(\gamma, \lambda, \phi) \equiv \mathcal{B}_{\Sigma}^{\nu}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{F}_\nu f(z))'}{(1-\lambda)\mathcal{F}_\nu f(z) + \lambda z(\mathcal{F}_\nu f(z))'} - 1\right) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{w(\mathcal{F}_\nu g(w))'}{(1-\lambda)\mathcal{F}_\nu g(w) + \lambda w(\mathcal{F}_\nu g(w))'} - 1\right) \prec \phi(w)$$

where $\gamma \in \mathbb{C}\backslash\{0\},\ 0 \leqq \lambda < 1,\ z, w \in \mathbb{U}$, the function g is given by(2)and $\mathcal{F}_\nu f(z) = \mathcal{J}_{1,\nu}^{1,2}f(z) = z + \sum\limits_{n=2}^{\infty} \left(\frac{1+\nu}{n+\nu}\right) a_n z^n\ (z \in \mathbb{U})$ is a Bernardi operator [4]. Note that the operator $\mathcal{F}_1$ was studied earlier by Libera [13] and Livingston [14].

Example 5. If $k = 2$ and $m = 1$ with $b = 1,\ \mu = \sigma\,(\sigma > 0)$, then $f \in \Sigma$ is said to be in $\Sigma_{\sigma,1}^{1,2}(\gamma, \lambda, \phi) \equiv \mathcal{I}_{\Sigma}^{\sigma}(\gamma, \lambda, \phi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{I}^\sigma f(z))'}{(1-\lambda)\mathcal{I}^\sigma f(z) + \lambda z(\mathcal{I}^\sigma f(z))'} - 1\right) \prec \phi(z)$$

and

$$1 + \frac{1}{\gamma}\left(\frac{w(\mathcal{I}^\sigma g(w))'}{(1-\lambda)\mathcal{I}^\sigma g(w) + \lambda w(\mathcal{I}^\sigma g(w))'} - 1\right) \prec \phi(w)$$

where $\gamma \in \mathbb{C}\backslash\{0\},\ 0 \leqq \lambda < 1,\ z, w \in \mathbb{U}$, the function g is given by(2) and $\mathcal{J}_{\sigma,1}^{1,2}f(z) = \mathcal{I}^\sigma f(z) = z + \sum\limits_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n\ (z \in \mathbb{U})$.

Remark 1. For $\lambda = 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$, given by (1), as in Example 1, one can state various analogous subclasses defined in Examples 2 to 5.

Remark 2. From the Definition 2, on specializing the parameters b, k, λ, μ suitably, as listed in Examples 1 to 5 one can define several new subclasses from the class $\mathcal{H}_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$. Further by taking $\lambda = 0 (or \lambda = 1)$ we can state certain analogous subclasses of Σ studied in [9, 26, 24].

Example 6. A function $f \in \Sigma$, given by (1),
(i) $\lambda = 0$, is said to be in the class $\mathcal{H}_{\mu,b}^{m,k}(\gamma,\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left(\frac{\mathcal{J}_{\mu,b}^{m,k}f(z)}{z}-1\right)\prec\phi(z) \quad and \quad 1+\frac{1}{\gamma}\left(\frac{\mathcal{J}_{\mu,b}^{m,k}g(w)}{w}-1\right)\prec\phi(w)$$

and
(ii) $\lambda = 1$, is said to be in the class $\mathcal{G}_{\mu,b}^{m,k}(\gamma,\phi)$ if the following conditions are satisfied:

$$1+\frac{1}{\gamma}\left((\mathcal{J}_{\mu,b}^{m,k}f(z))'-1\right)\prec\phi(z) \quad and \quad 1+\frac{1}{\gamma}\left((\mathcal{J}_{\mu,b}^{m,k}g(w))'-1\right)\prec\phi(w)$$

where $\gamma \in \mathbb{C}\backslash\{0\},\ z, w \in \mathbb{U}$ and the function g is given by (2).

In the following section we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined subclasses $\Sigma_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$ of the function class Σ by employing the techniques used earlier by Deniz[6].

2. Coefficient Bounds for the Function Class $\Sigma_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$

In order to derive our main results, we shall need the following lemma.

Lemma 2.1. [23]*: If $h \in \mathcal{P}$, then $|c_k| \leqq 2$ for each k, where $\mathcal{P}$ is the family of all functions h, analytic in $\mathbb{U}$, for which*

$$\Re\{h(z)\} > 0 \qquad (z \in \mathbb{U}),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \cdots .$$

It follows that,

$$u(z) := \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots\right]$$

and

$$v(z) := \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \cdots\right].$$

Then $p(z)$ and $q(z)$ are analytic in $\mathbb{U}$ with $p(0) = 1 = q(0)$. Since $u, v : \mathbb{U} \to \mathbb{U}$, the functions $p(z)$ and $q(z)$ have a positive real part in $\mathbb{U}$, and $|p_i| \leq 2$ and $|q_i| \leq 2$ for each i.

Theorem 2.2. *Let the function $f(z)$ given by* (1) *be in the class $\Sigma_{\mu,b}^{m,k}(\gamma, \lambda, \phi)$. Then*

$$|a_2| \leqq \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|[\gamma(\lambda^2-1)B_1^2 + (1-\lambda)^2(B_1 - B_2)]\Psi_2^2 + 2\gamma(1-\lambda)B_1^2\Psi_3|}} \tag{17}$$

and

$$|a_3| \leqq \frac{|\gamma|^2 B_1^2}{(1-\lambda)^2\Psi_2^2} + \frac{|\gamma| B_1}{2(1-\lambda)\Psi_3}, \tag{18}$$

where

$$\Psi_2 = C_2^m(b, \mu) = \left|\left(\frac{1+b}{2+b}\right)^{\mu} \frac{m!(k)!}{(k-2)!(m+1)!}\right|$$

and

$$\Psi_3 = C_3^m(b, \mu) = \left|\left(\frac{1+b}{3+b}\right)^{\mu} \frac{m!(k+1)!}{(k-2)!(m+2)!}\right|.$$

Proof. It follows from (11) and (12) that

$$1 + \frac{1}{\gamma}\left(\frac{z(\mathcal{J}_{\mu,b}^{m,k} f(z))'}{(1-\lambda)\mathcal{J}_{\mu,b}^{m,k} f(z) + \lambda z(\mathcal{J}_{\mu,b}^{m,k} f(z))'} - 1\right) = \phi(u(z)) \tag{19}$$

and

$$1 + \frac{1}{\gamma}\left(\frac{w(\mathcal{J}_{\mu,b}^{m,k} g(w))'}{(1-\lambda)\mathcal{J}_{\mu,b}^{m,k} g(w) + \lambda z(\mathcal{J}_{\mu,b}^{m,k} g(w))'} - 1\right) = \phi(v(w)), \tag{20}$$

where $p(z)$ and $q(w)$ in $\mathcal{P}$ and have the following forms:

$$\phi(u(z)) = \phi\left(\frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots\right]\right) \tag{21}$$

and

$$\phi(v(w)) = \phi\left(\frac{1}{2}\left[q_1 w + \left(q_2 - \frac{q_1^2}{2}\right) w^2 + \cdots\right]\right), \tag{22}$$

respectively. Now, equating the coefficients in (19) and (20), we get

$$\frac{(1-\lambda)}{\gamma}\Psi_2 a_2 = \frac{1}{2}B_1 p_1, \tag{23}$$

$$\frac{(\lambda^2-1)}{\gamma}\Psi_2^2 a_2^2 + \frac{2(1-\lambda)}{\gamma}\Psi_3 a_3 = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2, \tag{24}$$

$$-\frac{(1-\lambda)}{\gamma}\Psi_2 a_2 = \frac{1}{2}B_1 q_1 \tag{25}$$

and

$$\frac{(\lambda^2-1)}{\gamma}\Psi_2^2 a_2^2 + \frac{2(1-\lambda)}{\gamma}\Psi_3(2a_2^2 - a_3) = \frac{1}{2}B_1(q_2 - \frac{q_1^2}{2}) + \frac{1}{4}B_2 q_1^2. \tag{26}$$

From (23) and (25), we find that

$$a_2 = \frac{\gamma B_1 p_1}{2(1-\lambda)\Psi_2} = \frac{-\gamma B_1 q_1}{2(1-\lambda)\Psi_2}, \tag{27}$$

which implies

$$p_1 = -q_1. \tag{28}$$

and

$$8(1-\lambda)^2\Psi_2^2 a_2^2 = \gamma^2 B_1^2(p_1^2 + q_1^2). \tag{29}$$

Adding (24)and (26), by using(27) and (28) ,we obtain

$$4\left([\gamma(\lambda^2-1)B_1^2 + (1-\lambda)^2(B_1 - B_2)]\Psi_2^2 + 2\gamma(1-\lambda)B_1^2\Psi_3\right)a_2^2 = \gamma^2 B_1^3(p_2+q_2). \tag{30}$$

Thus,

$$a_2^2 = \frac{\gamma^2 B_1^3(p_2+q_2)}{4\left([\gamma(\lambda^2-1)B_1^2 + (1-\lambda)^2(B_1-B_2)]\Psi_2^2 + 2\gamma(1-\lambda)B_1^2\Psi_3\right)}. \tag{31}$$

Applying Lemma 2.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2|^2 \leqq \frac{|\gamma|^2 B_1^3}{|[\gamma(\lambda^2-1)B_1^2 + (1-\lambda)^2(B_1-B_2)]\Psi_2^2 + 2\gamma(1-\lambda)B_1^2\Psi_3|}. \tag{32}$$

Since $B_1 > 0$,the last inequality gives the desired estimate on $|a_2|$ given in (17).

Subtracting (26) from (24), we get

$$\frac{4(1-\lambda)}{\gamma}\Psi_3 a_3 - \frac{4(1-\lambda)}{\gamma}\Psi_3 a_2^2 = \frac{B_1}{2}(p_2 - q_2) + \frac{B_2 - B_1}{4}(p_1^2 - q_1^2). \tag{33}$$

It follows from (27), (28) and (33) that

$$a_3 = \frac{\gamma^2 B_1^2(p_1^2+q_1^2)}{8(1-\lambda)^2\Psi_2^2} + \frac{\gamma B_1(p_2-q_2)}{8(1-\lambda)\Psi_3}.$$

Applying Lemma 2.1 once again for the coefficients p_2 and q_2, we readily get

$$|a_3| \leqq \frac{|\gamma|^2 B_1^2}{(1-\lambda)^2\Psi_2^2} + \frac{|\gamma| B_1}{2(1-\lambda)\Psi_3}, B_1 > 0.$$

This completes the proof of Theorem 2.2. □

Taking $\lambda = 0$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let the function $f(z)$ given by (1) be in the class $\Sigma_{\mu,b}^{m,k}(\gamma,\phi)$. Then*

$$|a_2| \leqq \frac{|\gamma| B_1\sqrt{B_1}}{\sqrt{|[(B_1-B_2) - \gamma B_1^2]\Psi_2^2 + 2\gamma B_1^2\Psi_3|}} \tag{34}$$

and

$$|a_3| \leqq \frac{|\gamma|^2 B_1^2}{\Psi_2^2} + \frac{|\gamma| B_1}{2\Psi_3}. \tag{35}$$

From Example 2, for a function $f \in \Sigma_{0,0}^{1,2}\Sigma(\gamma,\lambda,\phi) \equiv \mathcal{S}_{\Sigma}(\gamma,\lambda,\phi)$, we state the following :

Corollary 2.4. *Let the function $f(z)$ given by (1) be in the class $\mathcal{S}_{\Sigma}(\gamma,\lambda,\phi)$. Then*

$$|a_2| \leqq \frac{|\gamma|B_1\sqrt{B_1}}{(1-\lambda)\sqrt{|(B_1-B_2)+\gamma B_1^2|}} \tag{36}$$

and

$$|a_3| \leqq \frac{|\gamma|^2 B_1^2}{(1-\lambda)^2} + \frac{|\gamma|B_1}{2(1-\lambda)}. \tag{37}$$

Taking $\lambda = 0$, in the Corollary 2.4, we get the following corollary.

Corollary 2.5. *Let the function $f(z)$ given by (1) be in the class $\mathcal{S}_{\Sigma}^{*}(\gamma,\phi)$. Then*

$$|a_2| \leqq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|(B_1-B_2)+\gamma B_1^2|}} \tag{38}$$

and

$$|a_3| \leqq |\gamma|^2 B_1^2 + \frac{|\gamma|B_1}{2}. \tag{39}$$

3. Coefficient Bounds for the Function Class $\mathcal{H}_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$

Theorem 3.1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{H}_{\mu,b}^{m,k}(\gamma,\lambda,\phi)$. Then*

$$|a_2| \leqq \frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|\gamma(1+2\lambda)\Psi_3 B_1^2 + (1+\lambda)^2(B_1-B_2)\Psi_2^2|}} \tag{40}$$

and

$$|a_3| \leq \frac{|\gamma|^2 B_1^2}{(1+\lambda)^2\Psi_2^2} + \frac{|\gamma|B_1}{(1+2\lambda)\Psi_3}, \tag{41}$$

where

$$\Psi_2 = C_2^m(b,\mu) = \left|\left(\frac{1+b}{2+b}\right)^{\mu} \frac{m!(k)!}{(k-2)!(m+1)!}\right|$$

and

$$\Psi_3 = C_3^m(b,\mu) = \left|\left(\frac{1+b}{3+b}\right)^{\mu} \frac{m!(k+1)!}{(k-2)!(m+2)!}\right|$$

Proof. Proceeding as in the proof of Theorem 3.1 we can arrive the following relations

$$\frac{(1+\lambda)}{\gamma}\Psi_2 a_2 = \frac{1}{2}B_1 p_1, \tag{42}$$

$$\frac{(1+2\lambda)}{\gamma}\Psi_3 a_3 = \frac{1}{2}B_1(p_2 - \frac{p_1^2}{2}) + \frac{1}{4}B_2 p_1^2, \tag{43}$$

$$\frac{-(1+\lambda)}{\gamma}\Psi_2 a_2 = \frac{1}{2}B_1 q_1 \tag{44}$$

and

$$\frac{(1+2\lambda)}{\gamma}\Psi_3(2a_2^2-a_3)=\frac{1}{2}B_1(q_2-\frac{q_1^2}{2})+\frac{1}{4}B_2q_1^2. \tag{45}$$

From (42) and (44), it follows that

$$p_1=-q_1 \tag{46}$$

and

$$8(1+\lambda)^2\Psi_2^2a_2^2=\gamma^2B_1^2(p_1^2+q_1^2). \tag{47}$$

Adding (43)and (45), it follows that

$$a_2^2=\frac{\gamma^2B_1^3(p_2+q_2)}{4[\gamma(1+2\lambda)\Psi_3B_1^2+(1+\lambda)^2(B_1-B_2)\Psi_2^2]}. \tag{48}$$

Applying Lemma 2.1 for the coefficients p_2 and q_2, we immediately get the desired estimate on $|a_2|$ as asserted in 40.

Subtracting (45) from (43)and using (46)and (47), we get

$$a_3=\frac{\gamma^2B_1^2(p_1^2+q_1^2)}{8(1+\lambda)^2\Psi_2^2}+\frac{\gamma B_1(p_2-q_2)}{4(1+2\lambda)\Psi_3}.$$

Applying Lemma 2.1 for the coefficients p_1, p_2 and q_1, q_2, we immediately get the desired estimate on $|a_2|$ as asserted in 40. □

For the function classes defined in Example6, from Theorem 3.1, we have the following corollary.

Corollary 3.2. *(i) Let the function $f(z)$ given by* (1) *be in the class $\mathcal{H}_{\mu,b}^{m,k}(\gamma,\phi)$. Then*

$$|a_2|\leqq\frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|\gamma\Psi_3B_1^2+(B_1-B_2)\Psi_2^2|}}\quad\text{and}\quad|a_3|\leq\frac{|\gamma|^2B_1^2}{\Psi_2^2}+\frac{|\gamma|B_1}{\Psi_3}.$$

(ii) Let the function $f(z)$ given by (1) *be in the class $\mathcal{G}_{\mu,b}^{m,k}(\gamma,\phi)$. Then*

$$|a_2|\leqq\frac{|\gamma|B_1\sqrt{B_1}}{\sqrt{|3\gamma\Psi_3B_1^2+4(B_1-B_2)\Psi_2^2|}}\quad\text{and}\quad|a_3|\leq\frac{|\gamma|^2B_1^2}{4\Psi_2^2}+\frac{|\gamma|B_1}{3\Psi_3}.$$

4. Concluding remarks

For the class of strongly starlike functions, the function ϕ is given by

$$\phi(z)=\left(\frac{1+z}{1-z}\right)^\alpha=1+2\alpha z+2\alpha^2z^2+\cdots\quad(0<\alpha\leq1), \tag{49}$$

which gives

$$B_1=2\alpha\quad\text{and}\quad B_2=2\alpha^2.$$

On the other hand if we take

$$\phi(z)=\frac{1+(1-2\beta)z}{1-z}=1+2(1-\beta)z+2(1-\beta)^2z^2+\cdots\quad(0\leq\beta<1), \tag{50}$$

then

$$B_1=B_2=2(1-\beta).$$

Remark 3. When $\phi(z)$ given by (49) and from Theorem 2.2, we state the following results

$$|a_2| \leqq \frac{2|\gamma|\alpha}{\sqrt{|[2\gamma\alpha(\lambda^2-1)+(1-\lambda)^2(1-\alpha)]\Psi_2^2+4\gamma\alpha(1-\lambda)\Psi_3}}$$

and

$$|a_3| \leqq \frac{4|\gamma|^2\alpha^2}{(1-\lambda)^2\Psi_2^2}+\frac{|\gamma|\alpha}{(1-\lambda)\Psi_3}.$$

When $\phi(z)$ given by (49) and from Theorem 3.1, we state the following results

$$|a_2| \leqq \frac{2|\gamma|\alpha}{\sqrt{|2\gamma\alpha(1+2\lambda)\Psi_3+(1+\lambda)^2(1-\alpha)\Psi_2^2}} \quad and \quad |a_3| \leqq \frac{4|\gamma|^2\alpha^2}{(1+\lambda)^2\Psi_2^2}+\frac{2|\gamma|\alpha}{(1+2\lambda)\Psi_3}.$$

Similarly we can state the corresponding results for the case when $B_1 = B_2 = 2(1-\beta)$.

From Corollary 2.4, when $\gamma = 1$, and $\phi(z)$ given by (49)(or by (50)) we state the following results given in [21].

Remark 4. Let $f(z)$ given by (1) be in the class $\mathcal{S}_\Sigma^*(\alpha,\lambda)$, and $B_1 = 2\alpha,\ B_2 = 2\alpha^2$ $(0<\alpha\leq 1; 0\leq\lambda<1)$. Then

$$|a_2| \leq \frac{2\alpha}{(1-\lambda)\sqrt{1+\alpha}} \quad and \quad |a_3| \leq \frac{4\alpha^2}{(1-\lambda)^2}+\frac{\alpha}{1-\lambda}.$$

Let $f(z)$ given by (1) be in the class $\mathcal{S}_\Sigma^*(\beta,\lambda)$, $B_1 = B_2 = 2(1-\beta)(0\leq\beta< 1; 0\leq\lambda< 1)$. Then

$$|a_2| \leq \frac{\sqrt{2(1-\beta)}}{1-\lambda} \quad and \quad |a_3| \leq \frac{4(1-\beta)^2}{(1-\lambda)^2}+\frac{(1-\beta)}{1-\lambda}.$$

Remark 5. From Corollary 2.5, when $\gamma = 1,\ \lambda = 0$ and $B_1 = 2\alpha,\ B_2 = 2\alpha^2$ and for $f \in \mathcal{S}_\Sigma^*(\alpha)$[3], we get

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}} \qquad \text{and} \qquad |a_3| \leq 4\alpha^2+\alpha.$$

and $B_1 = B_2 = 2(1-\beta)$, for $f \in \mathcal{S}_\Sigma^*(\beta)$, we get

$$|a_2| \leq \sqrt{2(1-\beta)} \qquad \text{and} \qquad |a_3| \leq 4(1-\beta)^2+(1-\beta).$$

Remark 6. If we assume $k = 2$ and $m = 1$ with $\mu = 0$, $b = 0$, and putting $\gamma = 1$ in Corollary 2.5, we obtain the corresponding results obtained in [1]. Also by taking $\gamma = 1,\ \lambda = 0(or\lambda = 1)$ and $B_1 = 2\alpha,\ B_2 = 2\alpha^2$ (or $B_1 = B_2 = 2(1-\beta)$) from Corollary 3.2, we obtain the corresponding results obtained in [9, 26]. Further, suitably specializing the parameters k, m, μ, b, various results can be derived easily, (which are asserted by Theorem 2.2, 3.1 above)for the new classes defined in Examples 1 to 5 and other interesting corollaries as consequences of our main results, the details involved may be left as an exercise for the interested reader.

References

[1] R. M. Ali, S.K. Lee, V. Ravichandran, S. Supramaniam, Coefficient estimates for bi-univalent Ma-Minda star-like and convex functions,*Appl. Math. Lett.* **25** (2012) 344–351.

[2] J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Ann. Math.* **17**(1915), 12–22.

[3] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, *Studia Univ. Babeş-Bolyai Math.* **31**(2) (1986), 70–77.

[4] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.* 135 (1969), 429–446.

[5] J. Choi and H. M. Srivastava, Certain families of series associated with the Hurwitz-Lerch Zeta function, *Appl. Math. Comput.* **170** (2005), 399–409.

[6] E. Deniz, Certain subclasses of bi-univalent functions satisfying subordinate conditions, *Jour. Class. Anal.* **2**(1) (2013), 49–60.

[7] C. Ferreira and J. L. Lopez, Asymptotic expansions of the Hurwitz-Lerch Zeta function, *J. Math. Anal. Appl.*, **298** (2004), 210–224.

[8] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.* **38** (1972), 746–765

[9] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.* **24** (2011), 1569–1573.

[10] M. Garg, K. Jain and H. M. Srivastava, Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions, *Integral Transform. Spec. Funct.* **17** (2006), 803–815.

[11] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, *Pan Amer. Math. J.* **22**(4) (2012), 15–26.

[12] I. B. Jung, Y. C. Kim AND H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, *J. Math. Anal. Appl.* **176** (1993), 138–147.

[13] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.* 16 (1965), 755–758.

[14] A. E. Livingston, On the radius of univalence of certain analytic functions, *Proc. Amer. Math. Soc.* 17 (1966), 352–357.

[15] S.-D. Lin and H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, *Appl. Math. Comput.* **154** (2004), 725–733.

[16] S.-D. Lin, H. M. Srivastava and P.-Y. Wang, Some espansion formulas for a class of generalized Hurwitz-Lerch Zeta functions, *Integral Transform. Spec. Funct.* **17** (2006), 817–827.

[17] Y. Ling and F.-S. Liu, The Choi-Saigo-Srivastava integral operator and a class of analytic functions, *Appl. Math. Comput.* **165** (2005), 613–621.

[18] X.-F. Li and A.-P. Wang, Two new subclasses of bi-univalent functions, *Internat. Math. Forum* **7** (2012), 1495–1504.

[19] W.C. Ma, D. Minda, A unified treatment of some special classes of functions, *in: Proceedings of the Conference on Complex Analysis, Tianjin,* 1992, 157–169, Conf. Proc. Lecture Notes Anal. 1. Int. Press, Cambridge, MA, 1994.

[20] G.Murugusundaramoorthy, Subordination results for spirallike functions associated with Hurwitz-Lerch zeta function, *Integral Transform. Spec. Funct.* **23**(2) (2012) 97–103

[21] G. Murugusundaramoorthy , N. Magesh and V.Praemala, Coefficient bounds for certain subclasses of bi-univalent function, *Abst. Appl. Anal.* **2013**, Article ID 573017, 3 pages.

[22] J. K. Prajapat and S. P. Goyal, Applications of Srivastava-Attiya operator to the classes of strongly starlike and strongly convex functions, *J. Math. Inequal.* **3** (2009), 129-?37.

[23] C. Pommerenke, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

[24] T. Panigarhi and G. Murugusundaramoorthy, Coefficient bounds for Bi- univalent functions analytic functions associated with Hohlov operator, *Proc. Jangjeon Math. Soc.* **16** (1) (2013), 91–100.

[25] H. M. Srivastava and J. Choi, *Series associated with the Zeta and related functions*, Dordrecht, Boston, London: Kluwer Academic Publishers, 2001.

[26] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* **23** (2010), 1188–1192.

[27] H. M. Srivastava, G. Murugusundaramoorthy and N. Magesh,, Certain subclasses of bi-univalent functions associated with the Hohlov operator, *Global Jour. Math. Anal.* **1**(2) (2013) 67–73.

[28] H. M. Srivastava and A. A. Attiya, *An integral operator associated with the Hurwitz-Lerch Zeta function and differential subordination*, Integral Transform. Spec. Funct. **18** (2007), 207–216.

[29] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.* **25** (2012), 990–994.

[30] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.* **218** (2012), 11461–11465.

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