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HEIGHT ESTIMATES FOR DOMINANT ENDOMORPHISMS ON PROJECTIVE VARIETIES

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Abstract. If ϕ is a polarizable endomorphism on a projective variety, then the Weil height machine guarantees that ϕ satisfies Northcott's theorem. In this paper, we show that Northcott's theorem only holds for polarizable endomorphisms and generalize this result to arbitrary dominant endomorphisms: we introduce the height expansion and contraction coefficients which provide weak Northcott's theorem for dominant endomorphisms. We also give some applications of the height expansion and contraction coefficients.

1. Introduction

In this paper, we introduce the height expansion and contraction coefficients which provide weak Northcott's theorem for dominant endomorphisms:

Definition 1. Let X be a projective variety and let $\phi : X \to X$ be a dominant endomorphism defined over $\overline{\mathbb{Q}}$. We define *the height expansion coefficient of ϕ for D* to be

$$\mu_1(\phi, D) := \sup\{\alpha \in \mathbb{R} \mid \phi^*D - \alpha D \text{ is ample}\} = \max\{\alpha \in \mathbb{R} \mid \phi^*D - \alpha D \text{ is nef}\}$$

and *the height contraction coefficient of ϕ for D* to be

$$\mu_2(\phi, D) := \inf\{\alpha \in \mathbb{R} \mid \alpha D - \phi^*D \text{ is ample}\} = \min\{\alpha \in \mathbb{R} \mid \alpha D - \phi^*D \text{ is nef}\}.$$

Main Theorems (Theorem A, Theorem B)**.** *Let X be a projective variety, let $\phi : X \to X$ be a dominant endomorphism defined over $\overline{\mathbb{Q}}$, let D be an ample divisor on X and let $\mu_1 = \mu_1(\phi, D)$, $\mu_2 = \mu_2(\phi, D)$ be the height expansion and contraction coefficients of ϕ for D. Then*

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(A) *ϕ is a polarizable if and only if the following inequality holds:*

$$h_D(P) = \frac{1}{q} h_D\left(\phi(P)\right) + O(1) \tag{1}$$

for all $P \in X(\overline{\mathbb{Q}})$.

(B) *In general, weak Northcott's theorem holds for ϕ: for any $\epsilon > 0$, there are constants $C_{1,\epsilon}, C_{2,\epsilon}$ satisfying*

$$\frac{1}{\mu_1 - \epsilon} h_D\big(\phi(P)\big) + C_{1,\epsilon} \geq h_D(P) \geq \frac{1}{\mu_2 + \epsilon} h_D\big(\phi(P)\big) - C_{2,\epsilon}$$

for all $P \in X(\overline{\mathbb{Q}})$.

(C) *μ_1, μ_2 are the optimal constants which satisfy* (B).

A dynamical system (X, ϕ) consists of a set X and a self map $\phi : X \to X$. We are interested in special points on X like fixed points, so we examine the orbit of points $\mathcal{O}_\phi(P) = \{P, \phi(P), \phi^2(P) = \phi \circ \phi(P), \cdots\}$. We say a point P is *'ϕ-preperiodic'* if $\mathcal{O}_\phi(P)$ is a finite set. The set of preperiodic points has nice properties. For example, consider a dynamical system $(A, [2])$ where A is an abelian variety and $[2]$ is the doubling map. Then preperiodic points are exactly torsion points, which is one of interesting and important topics in algebraic geometry.

If we have a tool which shows the difference of P and $\phi(P)$, then we can find interesting properties of preperiodic points. Northcott's theorem [14] tells that we can use the height functions to study dynamical systems $(\mathbb{P}^n, \phi)$ consisting of endomorphisms on projective spaces. In particular, Northcott's theorem holds for polarizable endomorphisms: we say that an endomorphism ϕ on a projective variety X is *polarizable* if there is an ample divisor $D \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\phi^* D$ is linearly equivalent to qD for some $q > 1$. Then, by the functorial property of the Weil height machine [18, §B.5], an endomorphism ϕ satisfies (1). This result was developed in various directions - the canonical height function [3], dynamical equidistribution [23], etc.

However, not all endomorphisms are polarizable: in general, a polarizable endomorphism is a restriction of an endomorphism ϕ on projective spaces on a ϕ-invariant subvariety. (See [2, 8].) So, any automorphism on a projective variety cannot be polarizable. For example, an automorphism of infinite order on a $K3$ surface introduced on Example 4.4 is not polarizable. We will show that Northcott's theorem only holds for polarizable endomorphisms, so we only expect weaker result for non-polarizable endomorphisms.

Interestingly, we have a similar result for rational maps: Silverman introduced height expansion coefficient for equidimensional dominant rational maps on [20]. Clearly, a dominant endomorphism is an example of equidimensional dominant rational maps so that we can compare Silverman's height expansion coefficient with μ_1. In section 4, we show that they are exactly same:

Theorem C. *Let $\phi : X \to X$ be a dominant endomorphism defined $\overline{\mathbb{Q}}$, let D be an ample divisor on X, let $\mu_1(\phi, D)$ be the height expansion coefficient of ϕ for*

D and let μ' be Silverman's height expansion coefficient defined on [20]*. Then, the following equality holds:*

$$\mu_1(\phi, D) = \mu'(\phi, D, D) := \liminf_{\substack{h_D(P)\to\infty \\ P\in X}} \frac{h_D\big(\phi(P)\big)}{h_D(P)}.$$

Not only $\mu_1(\phi, D)$ but also $\mu_2(\phi, D)$ has a relation with previous result in arithmetic dynamics. It can be a way of proving Silverman's conjecture [16].

Theorem D. *Let $\phi : X \to X$ be an endomorphism of dynamical degree δ_ϕ. Then, for any ample divisor D on X, the following inequality holds:*

$$\limsup_{n\to\infty} \mu_2(\phi^n, D)^{\frac{1}{n}} \leq \delta_\phi$$

and the equality holds if there is a point whose arithmetic degree is the same with the dynamical degree.

From now on, we will let X be a projective variety, let $\phi : X \to X$ be a dominant endomorphism on X defined over $\overline{\mathbb{Q}}$, let D be an ample divisor on X and let h_D be the Weil height function associated with a divisor D unless state otherwise.

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2. Northcott's theorem and polarizable endomorphisms

In this section, we check that why we cannot expect Northcott's theorem for non-polarizable endomorphisms.

Theorem A. *Let $\phi : X \to X$ be an endomorphism on a projective variety X defined over $\overline{\mathbb{Q}}$. Then, ϕ is a polarizable if and only if*

$$h_D(P) = \frac{1}{q} h_D(\phi(P)) + O(1) \quad \textit{for all } P \in X(\overline{\mathbb{Q}}).$$

Proof. The 'only if' part is easy: suppose that there is an ample divisor D such that $\phi^* D \sim qD$ for some $q > 1$. By the Functorial property of the Weil height machine [18, Theorem B.3.2.(b)], we have

$$h_D\big(\phi(P)\big) = h_{\phi^* D}(P) + O(1) = q h_D(P) + O(1)$$

and get the desired result.

Suppose that Northcott's theorem holds for ϕ: there is an ample divisor D such that

$$h_D(\phi(P)) = q h_D(P) + O(1) \quad \text{for all } P \in X(\overline{\mathbb{Q}}). \tag{2}$$

By [11, 16], the kernel of the Weil height machine is the torsion subgroup of $\mathrm{Pic}(X)$. So, (2) means that the linear equivalence class of $E := qD - \phi^* D$ generates the trivial Weil height function so that the linear equivalence class

$[E]$ is a torsion element in $\mathrm{Pic}(X)$[11, Theorem 3.3]. Let k be the order of $[E]$. Then

$$0 = k[E] = \big[k\big(qD - \phi^* D\big)\big]$$

in $\mathrm{Pic}(X)$. Therefore, $\phi^*(kD)$ is linearly equivalent to $q(kD)$. $\square$

3. Dominant endomorphisms and pull-backs of ample divisors

In previous section, we see that we only expect weaker result for non-polarizable endomorphisms: we expect to find some constants α_i, C_i such that

$$\alpha_1 h_D\big(\phi(P)\big) - C_1 < h_D(P) < \alpha_2 h_D\big(\phi(P)\big) + C_1 \text{ for all } P \in X(\overline{\mathbb{Q}}). \quad (3)$$

In this section, we show that the dominance condition is required for the above inequality. Also, we check some properties of dominant endomorphisms which will be important in the next section.

Lemma 3.1. *Let $\phi : X \to X$ be an endomorphism satisfying the inequality (3). Then ϕ is quasi-finite.*

Proof. Suppose not. Then we have a point P whose inverse image is a subvariety Y. Thus, $h_D(P)$ is a constant while $h_D(Q)$ varies unbounded on Y. Therefore, $h_D(Q)$ cannot be bounded by some multiple of $h_D\big(\phi(Q)\big) = h_D(P)$. $\square$

Usually, a dominant morphism need not be quasi-finite. However, for endomorphisms on projective varieties, the 'quasi-finiteness' condition is equivalent to the 'dominance' condition.

Definition 2. Let $\psi : X \to Y$ be a rational map. We say that ψ *is dominant* if $\overline{\psi(X)} = Y$.

Proposition 3.2. *Let $\phi : X \to X$ be an endomorphism. Then the followings are equivalent;*

(1) *ϕ is dominant.*
(2) *ϕ is quasi-finite.*
(3) *ϕ is finite.*

Proof. (1) $\Rightarrow$ (3) Since X is a projective variety, X is compact and hence ϕ is surjective. Then, [15, §4] says that surjective holomorphic endomorphism on a projective variety is finite.

(3) $\Rightarrow$ (1) It is a property of finite morphisms; if ϕ is not dominant, then ϕ is not quasi-finite and hence not finite.

(2) $\Leftrightarrow$ (3) If ϕ is a finite endomorphism, then it is clearly quasi-finite. For the other direction, [5, §8.11.1] says that ϕ is finite if ϕ is proper, locally of finite presentation and quasi-finite. Since X is a projective variety, ϕ is automatically projective and hence proper and locally of finite presentation. Therefore, if ϕ is quasi-finite, then ϕ is finite. $\square$

Now assume that ϕ is a dominant endomorphism defined over $\overline{\mathbb{Q}}$. Is it enough to get the inequality (3)? To compare values of $h_D(P)$ and $h_D\big(\phi(P)\big)$, it is essential to observe the relation between D and ϕ^*D because of the functorial property of the Weil height machine [18, Theorem B.3.2.(b)]:

$$h_D\big(\phi(P)\big) = h_{\phi^*D}(P) + O(1).$$

If $\phi : X \to X$ is polarizable, then, there is an ample divisor E such that $qE \sim \phi^*E$, which implies that ϕ^*E is ample. It is also true for general dominant endomorphisms because ϕ is quasi-finite.

Proposition 3.3. *Let $\phi : X \to X$ be an endomorphism. Then the followings are equivalent:*

(1) *ϕ is dominant.*
(2) *ϕ^*E is ample for some ample divisor E.*
(3) *ϕ^*E is ample for all ample divisors E.*

Proof. (1) $\Rightarrow$ (3) Suppose that ϕ^*E is not ample for an ample divisor E. Then, by Kleiman's criterion, there is a nonzero pseudo-effective 1-cycle C (a limit of effective cycle) such that $C \cdot \phi^*E \leq 0$. More precisely, since E is ample and hence nef, ϕ^*E is also nef. So $C \cdot \phi^*E = 0$. Because of the projection formula, we get

$$\phi_*C \cdot E = C \cdot \phi^*E = 0.$$

Since ϕ_* is a graded ring homomorphism on the Chow ring so that ϕ_*C is a pseudo-effective 1-cycle. Moreover, $\phi_*C \cdot E = 0$ so that $\phi(C)$ should be trivial 1-cycle, a union of finite points. However, ϕ is dominant and hence is quasi-finite. Therefore, the preimage of a finite set of points is a finite set of points again, so we again have a contradiction.

(3) $\Rightarrow$ (2) It is trivial.

(2) $\Rightarrow$ (1) If ϕ is not dominant, then $\dim \phi(X) < \dim X$. So, there is a subvariety $Y \subset X$ such that $\phi(Y) = Q \in X$. Therefore, for an ample divisor E,

$$h_{\phi^*E}(P) = h_E\big(\phi(P)\big) + O(1) = h_E(Q) + O(1) \quad \text{for all } P \in Y.$$

Thus, the height corresponding ϕ^*E is bounded on a variety Y and hence ϕ^*E is not ample. $\square$

4. weak Northcott's theorem

In this section, we check that the height expansion and contraction coefficients in Definition 1 are well-defined and prove Theorem B. We start with lemmas, which show the motivation of the height expansion and contraction coefficients.

Lemma 4.1. *Let E_1, E_2 be ample divisors on X. Suppose that there is a constant such that $h_{E_1}(P) < h_{E_2}(P) + C$ for all $P \in X(\overline{\mathbb{Q}})$. Then $E_2 - E_1$ is nef.*

Proof. From the inequality $h_{E_1} < h_{E_2} + C$, we can calculate the fractional limit, defined on [11]:

$$\mathrm{Flim}_{E_1}(E_2 - E_1, X) = \liminf_{\substack{h_{E_1}(P)\to\infty \\ P\in X}} \frac{h_{E_2-E_1}(P)}{h_{E_1}(P)} \geq 0.$$

We know that E is nef if and only if $\mathrm{Flim}_F(E, X) \geq 0$ for any ample divisor F [11, Thaorem A.(2)]. Therefore, $E_2 - E_1$ is nef. $\square$

Since the nef cone is the closure of the ample cone, the height expansion and contraction coefficients are the optimal number for $\phi^*D - \alpha D$, $\beta\phi^*D - D$ to be nef.

Lemma 4.2. *Let $\phi : X \to X$ be a dominant endomorphism on a projective variety, let D an an ample divisor on X. Then,*

$$\{\alpha \in \mathbb{R} \mid \phi^*D - \alpha D \textit{ is ample}\} = (-\infty, \mu_1(\phi, D)] \textit{ or } (-\infty, \mu_1(\phi, D))$$

and

$$\{\alpha \in \mathbb{R} \mid \alpha\phi^*D - D \textit{ is ample}\} = [\mu_2(\phi, D), \infty) \textit{ or } (\mu_2(\phi, D), \infty).$$

Moreover, $\mu_1(\phi, D), \mu_2(\phi, D)$ are positive numbers.

Proof. Let $\mu_i = \mu_i(\phi, D)$ be the height expansion and contraction coefficients. It is clear that $\phi^*D - \alpha D$ is ample if α is nonpositive. Also, $\{\alpha \in \mathbb{R} \mid \phi^*D - \alpha D \text{ is ample}\}$ is connected since $\phi^*D - \beta_0 D$ is ample then $\phi^*D - \beta D$ is ample for all $\beta < \beta_0$.

We need to show to show that $E_1 = \phi^*D - (\mu_1 - \epsilon)D$ and $E_2 = (\mu_2 + \epsilon)D - \phi^*D$ are ample for any $\epsilon > 0$. By definition of μ_1, for any $\epsilon > 0$, there is an real number $\alpha \in [\mu_1 - \epsilon, \mu_1]$ such that $\phi^*D - \alpha D$ is ample. Therefore, E_1 is a sum of two ample divisors $\phi^*D - \alpha D$ and $\big(\alpha - (\mu_1 - \epsilon)\big)D$ and hence it is ample, too. Similarly, E_2 is ample.

For positivity, we can use the fractional limit (see Remark 1) or a basic properties of ample divisors (see [18, Theorem A.3.2.3].) $\square$

Lemma 4.2 guarantees the well-definedness of $\mu_i(\phi, D)$. Once well-defined, the height expansion and contraction coefficients will provides weak Northcott's theorem.

Theorem B. *Let X be a projective variety, let $\phi : X \to X$ be a dominant endomorphism defined over $\overline{\mathbb{Q}}$, let D be an ample divisor on X and let $\mu_1 = \mu_1(\phi, D)$, $\mu_2 = \mu_2(\phi, D)$ be the height expansion and contraction coefficients of ϕ for D. Then, for any $\epsilon > 0$, there are constants $C_{1,\epsilon}, C_{2,\epsilon}$ satisfying*

$$\frac{1}{\mu_1 - \epsilon} h_D\big(\phi(P)\big) + C_{1,\epsilon} \geq h_D(P) \geq \frac{1}{\mu_2 + \epsilon} h_D\big(\phi(P)\big) - C_{2,\epsilon}$$

for all $P \in X(\overline{\mathbb{Q}})$. Moreover, μ_1, μ_2 are the optimal constants to satisfy the above inequality.

Proof. By Lemma 4.2, both $E_1 = \phi^*D - (\mu_1 - \epsilon)D$ and $E_2 = (\mu_2 + \epsilon)D - \phi^*D$ are ample for any $\epsilon > 0$. Thus, h_{E_1} and h_{E_2} are bounded below because of properties of the Weil height machine [18, Theorem B.5.3]. Therefore, we get

$$h_D\big(\phi(P)\big) - (\mu_1 - \epsilon)h_D(P) \geq h_{E_1}(P) + C_1 > C_1$$

and

$$(\mu_2 + \epsilon)h_D(P) - h_D\big(\phi(P)\big) \geq h_{E_2}(P) + C_2 > C_2.$$

Therefore, we can find $C_{1,\epsilon}, C_{2,\epsilon}$ such that

$$\frac{1}{\mu_1 - \epsilon}h_D\big(\phi(P)\big) + C_{1,\epsilon} \geq h_D(P) \geq \frac{1}{\mu_2 + \epsilon}h_D\big(\phi(P)\big) - C_{2,\epsilon}.$$

Moreover, by Lemma 4.2, μ_1 is the largest number such that $\phi^*D - \alpha D$ is nef. So, $\phi^*D - (\mu_1 + \epsilon)D$ is not nef and hence $(\mu_1 + \epsilon)h_D(P) \leq h_D\big(\phi(P)\big) + C'$ does not hold for any constant C' by Lemma 4.1. Symmetrically, μ_2 is also the optimal choice. $\square$

Example 4.3. *Let E be an elliptic curve defined over $\overline{\mathbb{Q}}$ and let Q be a non-torsion point on E. Define an endomorphism*

$$\phi(R) = [2]R - Q.$$

It is not polarizable by any ample divisor: take any non-torsion point P. We have $\phi^*(P) \sim 4(P + Q)$. We can easily check that the divisor $\phi^*(P) - q(P)$ is ample if and only if $q < 4$ and hence $\mu_1(\phi, (P)) = 4$. However, we get

$$\lim_{M\to\infty} \frac{1}{4}h_{\phi^*(P)}([-M]Q) - h_{(P)}([-M]Q) = \lim_{M\to\infty} -\frac{M}{2}\widehat{h}(Q) = -\infty$$

where $\widehat{h}$ is the Néron-Tate height function on E. Therefore, we only expect that

$$\frac{1}{4-\epsilon}h_{(P)}\big(\phi(R)\big) + C_{1,\epsilon} \geq h_{(P)}(R) \geq \frac{1}{4+\epsilon}h_{(P)}(R) - C_{2,\epsilon}.$$

Here are some examples on $K3$ surfaces. For details of $K3$ surfaces like the definition of involutions, we refer [1, 9, 12, 17] and [19, § 7.4] to the reader.

Example 4.4. *Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a $K3$-surface defined by intersection of hypersurfaces of bidegree $(1, 1)$ and $(2, 2)$, let $\imath_1$, $\imath_2$ be involutions on V, let D_1, D_2 be pullbacks of $H \times \mathbb{P}^2$ and $\mathbb{P}^2 \times H$ and let $E_+ = -D_1 + \beta D_2$, $E_- = D_2 + \beta^{-1}D_1$ where $\beta = 2 + \sqrt{3}$. Then divisor $D = aE_+ + bE_-$ is ample if and only if $a, b > 0$.*

We get $\imath_1^*(aE_+ + bE_-) = \beta(aE_-) + \beta^{-1}(bE_+)$. Thus,

$$\begin{aligned} \mu_1(\imath_1, E_+ + E_-) &= \sup\{\alpha \mid \beta^{-1} - \alpha > 0, \beta - \alpha > 0\} \\ &= \min\left(\beta^{-1}, \beta\right) \\ &= \beta^{-1}. \end{aligned}$$

Let $\phi = \imath_2 \circ \imath_1$. Then it is dominant and satisfies

$$\phi^*(aE_+ + bE_-) = \imath_1^*\big(\imath_2^*(aE_+ + bE_-)\big) = \imath_1^*(\beta aE_- + \beta^{-1}bE_+) = \beta^{-2}aE_+ + \beta^2 bE_-.$$

Thus, we get

$$\phi^*(aE_+ + bE_-) - \alpha(aE_+ + bE_-) = a(\beta^{-2} - \alpha)E_+ + b(\beta^2 - \alpha)E_-.$$

Therefore, $\mu_1(\phi, aE_+ + bE_-) = \beta^{-2}$ *and hence* $\mu(\phi) = \beta^{-2}$*. Similarly,* $\mu_2(\phi, aE_+ + bE_-) = \beta^2$*.*

Example 4.5. *Let* $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *be a generic hypersurface of tridegree* $(2,2,2)$ *of the Picard number* 3 *and let* $\imath_1$*,* $\imath_2$ *and* $\imath_3$ *be involutions on* X*. Then, the ample cone is the light cone*

$$\mathcal{L}^+ = \{E \in \mathrm{Pic}(V) \mid E^2 > 0, E \cdot D_0 > 0\}$$

where D_0 *is an arbitrary ample divisor. Let* E_i *be pullbacks of hyperplane* H_i *of* i*-th component. Since the Picard number of* X *is three,* $\{E_1, E_2, E_3\}$ *is a generator of* $\mathrm{Pic}(X)$*. Moreover,* $E_a = E_1 + E_2 + E_3$ *is the very ample divisor corresponding to the Segre embedding and the table of intersection numbers of* $\{E_1, E_2, E_3\}$ *is*

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$

Then the ample cone is described as follows:

$$\left\{ \sum a_i E_i \;\middle|\; \sum_{i \neq j} a_i a_j > 0, \sum a_i > 0 \right\}.$$

Since, $\imath_1^* \left(\sum a_i E_i\right) = -a_1 E_1 + (2a_1 + a_2)E_2 + (2a_1 + a_3)E_3$*, we get*

$$\begin{aligned} \mu_1(\imath_1, E_a) &= \sup\{\alpha \mid (-1-\alpha)E_1 + (3-\alpha)E_2 + (3-\alpha)E_3 : \textit{ample}\} \\ &= \sup\{\alpha \mid (5-3\alpha) > 0, (\alpha-3)(3\alpha-1) > 0\} \\ &= \frac{1}{3} \end{aligned}$$

and

$$\begin{aligned} \mu_2(\imath_1, E_a) &= \inf\{\alpha \mid (\alpha+1)E_1 + (\alpha-3)E_2 + (\alpha-3)E_3 : \textit{ample}\} \\ &= \inf\{\alpha \mid (3\alpha-5) > 0, (\alpha-3)(3\alpha-1) > 0\} \\ &= 3. \end{aligned}$$

Let $\phi_{1,2} = \imath_2 \circ \imath_1$*. It is dominant because it is an automorphism. By symmetry, we have* $\imath_2^* \left(\sum a_i E_i\right) = (2a_2 + a_1)E_1 - a_2 E_2 + (2a_2 + a_3)E_3$ *and hence*

$$\phi_{1,2}^* E_a = \imath_1^*\big(\imath_2^* E_a\big) = \imath_1^*(3E_1 - E_2 + 3E_3) = -3E_1 + 5E_2 + 9E_3.$$

Thus,

$$\begin{aligned} \mu_1(\phi_{1,2}, E_a) &= \sup\{\alpha \mid (-3-\alpha)E_1 + (5-\alpha)E_2 + (9-\alpha)E_3 : \textit{ample}\} \\ &= \sup\{\alpha \mid (11-3\alpha) > 0, 3\alpha^2 - 22\alpha + 3 > 0\} \\ &= \frac{11 - \sqrt{112}}{3}. \end{aligned}$$

Similarly,

$$\mu_2(\phi_{1,2}, E_a) \quad = \quad \frac{11+\sqrt{112}}{3}.$$

Example 4.6. *Let* $\mathbb{X} := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ *where* $n_i < n_{i+1}$ *and let* ϕ *be a dominant endomorphism on* $\mathbb{X}$ *defined* $\overline{\mathbb{Q}}$. *Then* $\phi = (\phi_1, \cdots, \phi_k)$ *where* $\phi_i : \mathbb{P}^{n_i} \to \mathbb{P}^{n_i}$ *is an endomorphism. Let* $\pi_i : \mathbb{X} \to \mathbb{P}^{n_i}$ *be a projection map, let* $\iota_i : \mathbb{P}^{n_i} \to \mathbb{X}$ *be a closed embedding map and let* $E_i = \pi_i^* H_i$ *where* H_i *is a hyperplane of* $\mathbb{P}^{n_i}$. *Then a divisor* $D = \sum_{i=1}^{k} a_i E_i$ *is ample if and only if* $a_i > 0$ *for all* i. *Furthermore,* $\phi^* E_i = \deg \phi_i \cdot E_i$ *and hence*

$$\mu_1(\phi, D) = \min \deg \phi_i \quad \mu_2(\phi, D) = \max \deg \phi_i.$$

5. Silverman's height expansion coefficient

In this section, we compare μ_1 with Silverman's height expansion coefficient. Silverman [20] introduces the height expansion coefficient for equidimensional dominant rational maps:

Definition 3. Let $\psi : X \dashrightarrow Y$ be a dominant rational map between quasi-projective varieties of the same dimension, all defined over $\overline{\mathbb{Q}}$. Fix height functions h_{D_Y} and h_{D_X} on Y and X respectively, corresponding to ample divisors D_Y and D_X. *The height expansion coefficient of* ψ (relative to chosen ample divisors D_Y and D_X) is the quantity

$$\mu'(\psi, D_X, D_Y) = \sup_{\emptyset \neq U \subset X} \liminf_{P \in U(\overline{\mathbb{Q}})} \frac{h_{D_Y}\big(\psi(P)\big)}{h_{D_X}(P)},$$

where the sup runs over all nonempty Zariski dense open subsets of X.

We can see that the relation between Definition 1 and Definition 3.

Theorem C. *Let* $\phi : X \to X$ *be a dominant endomorphism defined over* $\overline{\mathbb{Q}}$, *let* D *be an ample divisor on* X, *let* $\mu_1(\phi, D)$ *be the height expansion coefficient of* ϕ *for* D *and let* μ_S *be the Silverman's height expansion coefficient defined on* [20]. *Then, the following equality holds:*

$$\mu_1(\phi, D) = \mu'(\phi, D, D) := \liminf_{\substack{h_D(P) \to \infty \\ P \in X}} \frac{h_D\big(\phi(P)\big)}{h_D(P)}.$$

Proof. For a dominant endomorphism $\phi : X \to X$, ϕ is defined on entire X. Thus, the supremum comes from the biggest open set of X, which is X itself:

$$\mu'(\phi, D, D) = \sup_{\emptyset \neq U \subset X} \liminf_{\substack{h_D(P) \to \infty \\ P \in U}} \frac{h_D\big(\phi(P)\big)}{h_D(P)} = \liminf_{\substack{h_D(P) \to \infty \\ P \in X}} \frac{h_D\big(\phi(P)\big)}{h_D(P)}.$$

Let $\mu_1 = \mu_1(\phi, D)$, $\mu' = \mu'(\phi, D, D)$ and let $\epsilon > 0$ be any positive number. Then, by Lemma 4.2, $\phi^* D - (\mu_1 - \epsilon) D$ is ample. Thus,

$$h_{\phi^* D}(P) - (\mu_1 - \epsilon) h_D(P) \geq O(1).$$

Therefore,

$$\frac{h_{\phi^* D}(P) - O(1)}{h_D(P)} \geq \mu_1 - \epsilon \quad \text{and hence} \quad \liminf_{\substack{h_D(P) \to \infty \\ P \in X}} \frac{h_{\phi^* D}(P)}{h_D(P)} \geq \mu_1 - \epsilon. \tag{4}$$

On the other hand, let $E = \phi^* D - (\mu_1 + \epsilon) D$. Then, E is not nef divisor because of Lemma 4.2. So, by Kleiman's criterion, there is an irreducible curve C such that $C \cdot E < 0$. By [18, Theorem B.3.2(7)], we get

$$\lim_{\substack{h_D(P) \to \infty \\ P \in C}} \frac{h_E(P)}{h_D(P)} = \frac{E \cdot C}{D \cdot C} < 0.$$

Thus, we get

$$\liminf_{\substack{h_D(P) \to \infty \\ P \in X}} \frac{h_E(P)}{h_D(P)} \leq \lim_{\substack{h_D(P) \to \infty \\ P \in C}} \frac{h_E(P)}{h_D(P)} < 0.$$

We decompose $E = \phi^* D - (\mu_1 + \epsilon) D$ to get an upper bound of $\mu'(\phi, D, D)$:

$$0 > \liminf_{\substack{h_D(P) \to \infty \\ P \in X}} \frac{h_E(P)}{h_D(P)} = \liminf_{\substack{h_D(P) \to \infty \\ P \in X}} \frac{h_{\phi^* D}(P)}{h_D(P)} - \lim_{\substack{h_D(P) \to \infty \\ P \in X}} \frac{h_{(\mu_1 + \epsilon) D}(P)}{h_D(P)}$$

and hence we obtain

$$\mu' = \liminf_{\substack{h_D(P) \to \infty \\ P \in X}} \frac{h_{\phi^* D}(P)}{h_D(P)} < \mu_1 + \epsilon. \tag{5}$$

We combine (4) and (5) and get

$$\mu_1 - \epsilon \leq \mu' \leq \mu_1 + \epsilon$$

for any $\epsilon > 0$. Therefore, we get the desired result. $\square$

Remark 1. Note that Silverman's height expansion coefficient for dominant endomorphisms is the fractional limit defined in [11]:

$$\mu'(\phi, D, D) := \liminf_{\substack{h_D(P) \to \infty \\ P \in X}} \frac{h_D\big(\phi(P)\big)}{h_D(P)} = \mathrm{Flim}_D(\phi^* D, X).$$

Since $\phi^* D$ is ample because of Proposition 3.3, we get $\mathrm{Flim}_D(\phi^* D, X)$ is positive beause of arithmetic Kleiman's Criterion [11, Theorem A(1)]. It is another proof that μ_1 is a positive number. Similarly, we can show that

$$\mu_2(\phi, D) = \mathrm{Flim}_{\phi^* D}(D, X)$$

so that μ_2 is also positive.

6. Applications

6.1. Arithmetic dynamics

The height expansion coefficient has an application in arithmetic dynamics. We know that $\mathrm{Preper}(\phi)$ is of bounded height when ϕ is polarizable with $q > 1$. Recall that $q = \mu_1(\phi, D)$. Thus, it is not surprising that dominant endomorphisms have the similar result.

Definition 4. Let $\phi : X \to X$ be a dominant endomorphism defined over $\overline{\mathbb{Q}}$. We define *the global height expansion coefficient of* ϕ to be

$$\mu(\phi) := \sup_{D:\ ample} \mu_1(\phi, D).$$

Theorem 6.1. *Let $\phi : X \to X$ be a dominant endomorphism and E be an ample divisor. Suppose that the global height expansion coefficient $\mu(\phi) > 1$. Then, the set of preperiodic points is of bounded height by h_E.*

Proof. Let $\mu(\phi) > 1$. Then, there is an ample divisor D such that $\mu_1(\phi, D) > 1$. Suppose that $\epsilon = \frac{\mu_1(\phi,D)-1}{2}$. Then,

$$\frac{1}{\mu_1(\phi, D) - \epsilon} h_D\big(\phi(P)\big) = \frac{1}{1+\epsilon} h_D\big(\phi(P)\big) \geq h_D(P) - C.$$

By telescoping sum, we have

$$\lim_{n\to\infty} \left(\frac{1}{1+\epsilon}\right)^n h_D\big(\phi^n(P)\big) \geq h_D(P) - \frac{1}{1 - \frac{1}{1+\epsilon}} C.$$

Therefore, if $P \in \mathrm{Preper}(\phi)$, then the left hand side goes to zero so that $h_D(P)$ is bounded.

Moreover, if E is another ample divisor then $\alpha D - E$ is ample for sufficiently large $\alpha > 0$. Since the Weil height corresponding the ample divisor is bounded below and hence

$$\alpha \cdot h_D(P) + O(1) > h_E(P)$$

for all $P \in X$. Therefore, $h_E\big(\mathrm{Preper}(\phi)\big)$ is also bounded. □

Example 6.2. *Let $\phi_i : \mathbb{P}^n \to \mathbb{P}^n$ be an endomorphism of degree $d_i > 1$. Then, a morphism*

$$\phi = \prod \phi_i : (\mathbb{P}^n)^m \to (\mathbb{P}^n)^m$$

is a dominant endomorphism of $\mu(\phi) = \min d_i > 1$. Thus, $\mathrm{Preper}(\phi)$ *is a set of bounded height.*

Theorem 6.3 (Fakhruddin)**.** *Let D be an ample divisor on X and let $\phi : X \to X$ be a dominant endomorphism such that $\phi^* D - D$ is ample. Then the subset of $X(\overline{\mathbb{Q}})$ containing periodic points of ϕ is Zariski dense.*

Proof. See [8, Theorem 5.1]. □

Corollary 6.4. *Let $\phi : X \to X$ be a dominant morphism with $\mu(\phi) > 1$. Then, $\overline{\mathrm{Preper}(\phi)} = X$.*

Proof. By the definition of μ, for any $\epsilon > 0$, there is an ample divisor D such that $\mu_1(\phi, D) > \mu(\phi) - \epsilon$. Take $\epsilon = \dfrac{1}{2}\big(\mu_1(\phi) - 1\big)$. Then, $\mu_1(\phi, D) > 1$. Now, by definition of μ_1, $\phi^*D - (\mu_1(\phi, D) - \delta) \cdot D$ is ample for any $\delta > 0$. Because $\mu_1(\phi) > 1$, take $\delta = \mu_1(\phi, D) - 1$ and get an ample divisor $\phi^*D - D$. $\square$

6.2. Dynamical degree and Silverman's conjecture

We need some definition from arithmetic dynamics for this subsection. We refer [10, 21] for details.

Definition 5. Let $\phi : X \to X$ be a dominant endomorphism. We define *the dynamical degree* to be

$$\delta_\phi = \lim_{n\to\infty} \rho((\phi^n)^*, \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R})^{\frac{1}{n}}$$

where $\mathrm{NS}(X)$ is the Néron-Severi group of X and ρ is the spectral radius of a linear operator.

Definition 6. Let $\phi : X \to X$ be a dominant endomorphism and let D be an ample divisor. We define *the arithmetic degree of P* to be

$$\alpha_\phi(P) = \lim_{n\to\infty} \max\big(1, h_D\big(\phi^n(P)\big)\big)^{\frac{1}{n}}$$

if the limit exists.

Recently, Silverman conjecture a relation between arithmetic degree and dynamical degree [21]. And Kawaguchi-Silverman prove some height inequality [10]. Actually, we assume that ϕ is a dominant rational map but we introduce endomorphism case for convenience.

Conjecture 1. *$\alpha_\phi(P)$ exists for all P, and $\alpha_\phi(P) = \delta_\phi$ if the orbit of P is Zariski dense in X.*

Theorem 6.5 (Kawaguchi-Silverman)**.** *Let $\phi : X \to X$ be an endomorphism and let D be an ample divisor. There is a constant C such that the following inequality holds for all $P \in X(\overline{\mathbb{Q}})$: for any $\epsilon > 0$,*

$$h_D\big(\phi^n(P)\big) < (\delta_\phi + \epsilon)^n h_D(P) + C$$

holds for sufficiently large n.

We can find an interesting fact: the height contraction coefficient is a lower bound of the dynamical degree.

Theorem 6.6. *Let $\phi : X \to X$ be an endomorphism and let D be an ample divisor. Then we have*

$$\limsup \mu_2(\phi^n, D)^{\frac{1}{n}} \le \delta_\phi.$$

Proof. By Theorem B, we have an height inequality

$$h_D\big(\phi^n(P)\big) < (\mu_2(\phi^n, D) + \epsilon)h_D(P) + C'$$

and $\mu_2(\phi^n, D)$ is the optimal one. Thus we have

$$(\mu_2(\phi^n, D) + \epsilon)h_D(P) < (\delta_\phi + \epsilon)^n h_D(P) + C''$$

Therefore, we have the desired result. □

Corollary 6.7. *Suppose that Silverman's conjecture is true and there is a point P with Zariski dense orbit. Then*

$$\limsup \mu_2(\phi^n, D)^{\frac{1}{n}} = \delta_\phi$$

for all ample divisors D on X.

Proof. By definition, we have

$$\alpha_\phi(P) = \lim_{n\to\infty} h_D\big(\phi^n(P)\big)^{\frac{1}{n}} \le \limsup_{n\to\infty}(\mu_2(\phi^n, D)+\epsilon)^{\frac{1}{n}}(h_D(P)+C)^{\frac{1}{n}} = \limsup_{n\to\infty}(\mu_2(\phi^n, D)+\epsilon)^{\frac{1}{n}}.$$

Since we have the following inequality,

$$(\mu_2(\phi^n, D) + \epsilon)h_D(P) < (\delta_\phi + \epsilon)^n h_D(P) + C''$$

Silverman's conjecture tells

$$\alpha_\phi(P) = \limsup_{n\to\infty}(\mu_2(\phi^n, D) + \epsilon)^{\frac{1}{n}} = \delta_\phi.$$

□

6.3. Seshadri Constant

The height expansion coefficient has a relation with the Seshadri constant. Demailly [4] defined the Seshadri constant.

Definition 7. Let Y be a closed subscheme of X whose underlying subvariety is of codimension $r > 1$, let $\widetilde{X}$ be a blowup of X along Y and let L be a numerically effective divisor of X. Then, we define the *generalized Seshadri constant*

$$\epsilon(L, Y) = \sup\{\alpha \mid \pi^*L - \alpha E : \text{ numerically effective}\}.$$

Similarly, we define the *s-invariant*

$$s_L(Y) = \min\{s \mid s \cdot \pi^*L - E : \text{ numerically effective}\}.$$

Theorem 6.8. *Let $\phi : W \to W$ be a dominant morphism and let D be a ample divisor. Then,*

$$\epsilon(\phi^*D, D) \ge \mu_1(\phi, D).$$

Proof. The closure of the ample cone of V is the nef cone and hence

$$\begin{aligned}\mu_1(\phi, D) &= \sup\{\alpha \mid \widetilde{\phi}^* D - \alpha D \text{ is ample.}\} \\ &= \sup\{\alpha \mid \widetilde{\phi}^* D - \alpha D \text{ is numerically effective.}\} \\ &= \epsilon(\phi^* D, D).\end{aligned}$$

□

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