Abstract. We explicitly derive diagrams representing (1,1)-decompositions of rational pretzel knots $K_\beta = M((-2,1),(3,1),(6\beta + 1,\beta))$ from four unknotting tunnels for $\beta = 1, -2$ and 2.

1. Preliminaries

Let $V$ be an unknotted solid torus and let $\alpha$ be a properly embedded arc in $V$. We say that $\alpha$ is trivial in $V$ if and only if there exists an arc $\beta$ in $\partial V$ joining the two end points of $\alpha$, and a 2-disk $D$ in $V$ such that the boundary $\partial D$ of $D$ is a union of $\alpha$ and $\beta$. Such a disk $D$ is called a cancelling disk of $\alpha$. Equivalently $\alpha$ is trivial in $V$ if and only if $H = V - N(\alpha^o$, the complement of the open tubular neighbourhood $N(\alpha)$ of $\alpha$ in $V$ is a handlebody of genus 2. Note that a cancelling disk of the trivial arc $\alpha$ and a meridian disk of $V$ disjoint from $\alpha$ contribute a meridian disk system of the handlebody $H$.

A knot $K$ in $S^3$ is referred to as admitting genus 1, 1-bridge decomposition or (1,1)-decomposition for short if and only if $(S^3,K)$ is split into pairs $(V_i,\alpha_i)$ ($i = 1,2$) of trivial arcs in solid tori determined by a Heegaard torus $T = \partial V_i$ of $S^3$. Namely we have:

$$(S^3,K) = (V_1,\alpha_1) \cup_T (V_2,\alpha_2)$$

We say that two (1,1)-decompositions $(V_1,\alpha_1) \cup_T (V_2,\alpha_2)$ and $(W_1,\beta_1) \cup_U (W_2,\beta_2)$ of $(S^3,K)$ are homeomorphic if and only if there exists an orientation preserving homeomorphism $\Phi : (S^3,K) \rightarrow (S^3,K)$ such that $\Phi(T) = U$.

For each solid torus $V_i$ in a given (1,1)-decomposition $(V_1,\alpha_1) \cup_T (V_2,\alpha_2)$ of $(S^3,K)$ we choose a meridian disk $D_i$ of $V_i$ so that $D_i$ is disjoint from a trivial arc $\alpha_i$. Then an ordered quadruple $(\partial V_1,\partial D_1,\partial D_2,\partial \alpha_1)$ is referred to as a (1,1)-diagram associated with $V_i$-side of a given (1,1)-decomposition of $(S^3,K)$. Likewise an ordered quadruple $(\partial V_2,\partial D_2,\partial D_1,\partial \alpha_2)$ is referred to as a (1,1)-diagram associated with $V_2$-side of a given (1,1)-decomposition of $(S^3,K)$. Similarly an ordered quadruple $(\partial V_2,\partial D_2,\partial D_1,\partial \alpha_2)$ is referred to as a (1,1)-diagram associated with $V_2$-side of a given (1,1)-decomposition of $(S^3,K)$. Likewise an ordered quadruple $(\partial V_2,\partial D_2,\partial D_1,\partial \alpha_2)$ is referred to as a (1,1)-diagram associated with $V_2$-side of a given (1,1)-decomposition of $(S^3,K)$.
as a (1,1)-diagram associated with \( V_2 \)-side of a given (1,1)-decomposition of \((S^3, K)\). The latter (1,1)-diagram is called the dual of the former. In the sequel it is observed that such refined treatment of (1,1)-diagrams of knot \( K \) in \( S^3 \) is useful in associating a (1,1)-diagram to a (1,1)-tunnel. Conversely we may construct a (1,1)-decomposition from an abstractly defined (1,1)-diagram \((T, m_1, m_2, \{P, Q\})\) and its dual as follows. For each \( i = 1, 2 \), take an unknotted solid torus \( V_i \) bounding \( T \), and a meridian disk \( D_i \) of \( V_i \) bounding \( m_i \) so that \( D_i \) may be disjoint from a trivial arc \( \alpha_i \) joining \( P \) and \( Q \) in \( V_i \). Then gluing these two soli tori together along the orientation reversing homeomorphism \( \phi : \partial V_1 \to \partial V_2 \) which sends regions on \( \partial V_1 \) determined by the given (1,1)-diagram to those on \( \partial V_2 \) by its dual. In an unpublished manuscript [11], the author introduced (1,1)-diagrams of knots in \( S^3 \) governed by four tuples of nonnegative integers as shown in Figure 1. Indeed Grasselli-Mulazzani [5] and

\[ D(a, b, c, r) \]

the author independently showed that Dunwoody 3-manifolds [2] are cyclic covering spaces of \( S^3 \) branched over (1,1)-knots with (1,1)-diagrams in Figure 1. For computation of Knot Floer Homology groups of (1,1)-knots in \( S^3 \), (1,1)-diagrams appeared bearing with another code \( K(p, q, r, s) \) in [10] (see also [4]). For instance \( K(15, 6, 2, 2) \) corresponds to \( D(6, 2, 1, 4) \) in our notation. In passing the author points out incorrectness of notation for the dual of a (1,1)-diagram announced by Kim and Kim [8]. For instance a torus knot \( t(p, p + 1), p \geq 2 \), should have had a self dual (1,1)-diagram, namely the dual of a (1,1)-diagram is isomorphic to itself. For more details, see [9]. The dual of a (1,1)-diagram should have been taken so that the associated gluing homeomorphism \( \phi : \partial V_1 \to \partial V_2 \) may be orientation reversing ; see Figure 7 in this paper.

In previous study of unknotting tunnels of a (1,1)-knot ([11],[12]), symmetry of a knot induced by its unknotting tunnel or that of its double branched covering space play a central role. As a consequence Heath and Song [6] classified isotopic type of unknotting tunnels of a pretzel knot \( p(-2, 3, 7) \) which turns out to be all four tunnels obtained from its two non-homeomorphic (1,1)-decompositions (Theorem9, [12]). But the author recently realized that importance of Hayshi’s remark on (1,1)-diagrams in p. 378 of [7] was overlooked;
That is, up to homeomorphic type of a (1,1)-decomposition of a knot $S^3$, the associated (1,1)-diagram is uniquely determined! In particular, the transversal intersection number, say it complexity of a (1,1)-diagram denoted by $d = 2a + b + c$ for $D(a, b, c, r)$, of the two meridians in a (1,1) diagram is invariant up to homeomorphic type of a (1,1)-decomposition. In near future the author hopes to investigate relationship between the Cho-McCullough cabling slope invariant [1] and the complexity of a (1,1)-diagram. Based on the Hayashi’s remark we consider a problem of deriving (1,1)-diagrams representing (1,1)-decompositions of rational pretzel knots $K_\beta = M((-2,1),(3,1),(6\beta + 1,\beta))$ from four unknotting tunnels as shown in Figure 2.

**Figure 2.** $K_\beta = M((-2,1),(3,1),(6\beta + 1,\beta))$

To the author’s knowledge, there are two examples of dealing with the problem of this paper in the literature; one is derivation of the (1,1)-diagram of a knot $10_{161}$ from a given (1,1)-tunnel (Figures from 1 to 5 in [4]) and the other is a genus 2 Heegaard diagram of $S^3$ representing (2,0)-decomposition of a Morimoto-Sakuma-Yokota knot $K(5,7,2)$ (Figure 7 in [3]), which can be destabilized to a (1,1) diagram.

**Figure 3**

In the former derivation, we keep the meridian disk $D_1$ (namely the co-core disk of the unknotting tunnel) of solid torus and deform the meridian disk $D_2$
of the complementary solid torus by handle sliding. It is rather inconvenient in
the sense that complexity of a diagram representing a trivial arc and that of a
(1,1)-diagram simultaneously get increased after necessary moves of diagrams
as illustrated in Figure 3.

On the other hand in Figure 4 we deform \( D_1 \) by handle sliding while keeping
\( D_2 \) fixed but both represent the identical (1,1)-diagram described by \( D(4, 4, 1, 7) \).
Note that the dual (1,1)-diagram \( D(3, 4, 3, 6) \) can be easily read from Figure 4.
Thus our method of deriving a (1,1)-diagram from a given tunnel is to get firstly
the dual of the desired one and then to dualize it.

Here is our main result:

**Theorem 1.1.** The rational pretzel knots \( K_\beta \) (\( \beta = 1, -2, 2 \)) have the following
(1,1)-Heegaard diagrams with respect to the four tunnels \( \tau_1, \tau_2, \tau_3 \) and \( \tau_4 \) in
Figure 2;

\[
\begin{array}{|c|c|c|c|c|}
\hline
   & \tau_1 & \tau_2 & \tau_3 & \tau_4 & \beta \\
\hline
  9 & D(1, 4, 3, 2) & D(3, 1, 2, 4) & D(2, 4, 1, 5) & D(2, 1, 4, 6) & 1 \\
15 & D(2, 8, 3, 4) & D(6, 1, 2, 11) & D(4, 1, 6, 7) & D(3, 8, 1, 6) & -2 \\
17 & D(2, 8, 5, 4) & D(7, 1, 2, 4) & D(4, 8, 1, 9) & D(4, 1, 8, 10) & 2 \\
\hline
\end{array}
\]

In a forthcoming paper the author would apply the method of using a train-
track for derivation of (1,1)-diagrams of the ordinary pretzel (1,1)-knots in [4]
to \( K_\beta \) (hopefully for all \( \beta \)).

2. Derivation of (1,1)- diagrams of \( K_\beta \) from their (1,1)-tunnels for
\( K_1 = p(-2, 3, 7) \), \( K_2^- \) and \( K_2 \)

Here we briefly sketch the process of obtaining a (1,1)-diagram from a knot \( K \)
with a (1,1)- tunnel \( \tau \). For more rigorous argument supporting our sketch, see
Proposition 2.3 [4]. Although a complete characterization of a (1,1)-tunnel in
terms of a constituent knot of a \( \theta \)- curve \( K \cup \tau \) remains open, for simplicity and
practical application we assume that a (1,1)-tunnel \( \tau \) of a knot \( K \) is positioned
so that we may identify an unknotted constituent knot of a \( \theta \)- curve containing
Then a regular neighbourhood of the $\theta$-cub contains an unknotted solid torus $V_1$ such that $\alpha_1 = K \cap V_1$ and $\alpha_2 = K \cap V_2$ form trivial arcs in solid tori $V_1$ and $V_2 = S^3 - V^\circ$ respectively. Now take a pair of meridian disks $D_i$ of $V_i$ so that $D_i \cap \alpha_i = \emptyset$ for $i = 1, 2$. Then we have a (1,1)-desired (1,1) diagram by deforming $D_1$ (or equivalently $D_2$) while bringing the trivial arc $\alpha_2$ into standardly embedded position via its handle sliding along $\partial V_1$.

**Figure 5.** $(K_1, \tau_1): D(1, 4, 3, 2)$

**Figure 6.** $(K_1, \tau_2): D(3, 1, 2, 4)$

In Figure 6 we can easily obtain the code $D(1, 4, 3, 2)$ of the dual of the desired (1,1)-diagram. Thus dualizing it as illustrated in Figure 7, we have the code $D(3, 1, 2, 4)$ for the (1,1)-diagram induced by $(K_1, \tau_2)$. In figure 7 considering signature changes in intersection pattern of oriented meridians we can figure out number of bigons for the dual (1,1)-diagram. On the other hand observing that the orientation reversing gluing map should carry the bigonal region containing each distinguished point to itself we can figure out the region of the dual (1,1)-diagram corresponding to each region of a given (1,1)-diagram.
Figure 7. $D(3, 1, 2, 4)$: the dual of $D(1, 4, 3, 2)$

Figure 8. $(K_1, \tau_3):D(2, 4, 1, 5)$

Figure 9. $(K_1, \tau_4)$

Figure 10. $(K_1, \tau_4):D(2, 1, 4, 6)$
Figure 11. \((K_{-2}, \tau_2): D(6, 1, 2, 11)\)

Figure 12. \((K_{-2}, \tau_3): D(4, 1, 6, 7)\)

Figure 13. \((K_2, \tau_2): D(7, 1, 2, 4)\)
Figure 14. \((K_2, \tau_3) : D(4, 8, 1, 9)\)

References


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