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THE BASKET NUMBERS OF LINKS OF 6 CROSSINGS OR LESS

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ABSTRACT. In present article, we find a complete classification theorem of links of basket numbers 2 or less. As an application, we study the basket numbers of links of 6 crossings or less.

1. Introduction

A *link* L is an embedding of n copies of $\mathbb{S}^1$ in $\mathbb{S}^3$. If the number of components of the link L is 1, a link is called a *knot*. Throughout the article, we will assume all links are *tame* which means all links can be in a form of a finite union of line segments. Two links are *equivalent* if there is an isotopy between them. In the case of prime knots, this equivalence is the same as the existence of an orientation preserving homeomorphism on $\mathbb{S}^3$, which sends a knot to the other knot. Although the equivalent class of a link L is called a *link type*, throughout the article, a link really means the equivalent class of the link.

A compact orientable surface $\mathcal{F}$ is called a *Seifert surface* of a link L if the boundary of $\mathcal{F}$ is isotopic to L. The existence of such a surface was first proven by Seifert by using an algorithm on a diagram of L, this algorithm was named after him as *Seifert's algorithm* [7]. A Seifert surface $\mathcal{F}_L$ of an oriented link L which is produced by applying Seifert's algorithm to a link diagram and is called a *canonical Seifert surface*.

Some Seifert surfaces feature extra structures. Seifert surfaces obtained by annuli plumbings are the main subjects of this article. Even though higher dimensional plumbings can be defined, but we will only concentrate on *annuli plumbings* called a *Murasugi sum*, which is extensively studied for the fibreness of links and surfaces [4, 8].

The definition of basket surfaces [6] is very technical and so it is difficult to handle but by modifying the work in [3]. A basket surface is one of these plumbing surfaces and it can be presented by two sequential presentations, the first sequence is the flat plumbing basket code found by Furihata, Hirasawa and Kobayashi and the second sequence presents the number of the full twists for each of annuli. For knots, Choi, Chung and the first author [2] found a completely new knot tabulation with respect to

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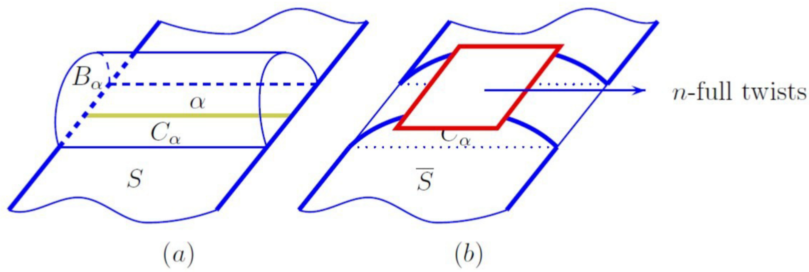


FIGURE 1. (a) A geometric shape of α, B_α and C_α on a Seifert surface S and (b) a new Seifert surface $\bar{S}$ obtained from S by a top A_n plumbing along the path α.

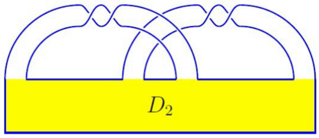
the flat plumbing basket number of knots using Dowker-Thistlethwaite notation and **knotfinder** which is a computer program of knotscape. However, none of these methods can be directly applied for links with more than one components.

In present article, we find a complete classification theorem of links of basket numbers 2 or less. As an application, we study the basket numbers of links of 6 crossings or less. The outline of this paper is as follows. We first provide some preliminary definitions and results in Section 2. We first explain how basket code developed to study the basket number of knot can be applied for links. Then we find a complete classification theorem of links of basket numbers 2 or less. As an application, we study the basket numbers of links of 6 crossings or less in Section 3.

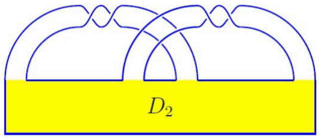
2. Preliminaries

The followings are the exact definitions of the flat plumbing basket surface given by Rudolph [6].

Spaces, maps, etc., are piecewise smooth unless stated differently. Let M be an oriented manifold. $-M$ denotes M with its orientation reversed and when notation requires it, $+M$ denotes M. For a suitable subset $S \subset M$, $N_M(S)$ denotes a closed regular neighborhood of S in $(M, \partial M)$ where an ordered pair (S,T) stands a condition $T \subset S$ and a map between ordered pairs $f : (S,T) \to (U,V)$ is a map $f : S \to U$ which requires to preserve subsets so that $f(T) \subset V$. For a suitable codimension -1 submanifold $S \subset M$ (resp., submanifold pair $(S, \partial S) \subset (M, \partial M)$), an emphcollaring is an orientation-preserving embedding $S \times [0,1] \to M$ (resp., $(S, \partial S) \times [0,1] \to (M, \partial M)$) extending $id_S = id_{S\times\{0\}}$; a *collar* of S in M (resp., of $(S, \partial S)$ in $(M, \partial M)$) is the image $col_M(S)$ (resp., $col_{(M,\partial M)}(S, \partial S)$) of a collaring. The push-off of S determined by a collaring of S or $(S, \partial S)$, denoted by S^+, is the image by the collaring of $S \times \{1\}$ with the orientation of S; let $S^- := -S^+$ such that S and S^- are oriented submanifolds of the boundary of $col_M(S)$.



(a)



(b)

FIGURE 2. Basket surfaces of (a) the trefoil knot and (b) the figure eight knot whose basket numbers are 2 [5].

Definition 2.1. ([6]) Let α be a proper arc on a Seifert surface S. Let C_α be $\mathrm{col}_{(S,\partial S)}(\alpha, \partial\alpha)$ which is called the gluing region. Let B_α be $\mathrm{col}_{(S,\partial S)}(\alpha, \partial\alpha)$ (so B_α is a 3-*cell* in top(S), that is the positive normal to S along $C_\alpha = S \cap B_\alpha \subset \partial B_\alpha$ points into B_α) as depicted in Figure 1. Let $A_n \subset B_\alpha$ be an n-full twisted annulus such that $A_n \cap \partial B_\alpha = C_\alpha$. Then top plumbing on S along a path α is the new surface $\overline{S} = S \cup A_n$ where A_n, C_α, B_α satisfy the previous conditions.

Definition 2.2. ([6]) A Seifert surface $\mathcal{F}$ is a basket surface if $\mathcal{F} = D_2$ or if $\mathcal{F} = \mathcal{F}_{0*\alpha}A_n$ which can be constructed by plumbing A_n to a basket $\mathcal{F}_0$ along a proper arc $\alpha \subset D_2 \subset \mathcal{F}_0$ where A_n is an annulus with n full twists. We say that a surface $\mathcal{F}$ is a basket surface of a link L if it is a basket surface and $\partial\mathcal{F}$ is equivalent to L. The minimum number of plumbings among all basket surfaces of a link L is called the basket number of the link L, denoted by $bk(L)$.

A previous study by the first author found an upper bound for the basket number of a link using a braid presentation of the link as follow.

Theorem 2.1. *([5]) Let L be a link which is the closure of a braid $\beta \in B_n$ where the length of the braid β is m. Then the basket number of L is less than or equal to $m - n + 1$, i.e.,*

$$bk(L) \leq m - n + 1.$$

Example 2.1. ([5]) The basket number of the trefoil knot and figure eight knot is 2 as illustrated in Figure 2.

3. Classification and applications

The basket code of a basket surface is two sequences $(a_1a_2 \ldots a_{2m} : b_1, b_2, \ldots, b_m)$. A detailed description can be found in [1]. The *basket code of a basket surface* is two sequences $(a_1a_2\ldots a_{2m} : b_1, b_2, \ldots, b_m)$ where the first part $a_1a_2\ldots a_{2m}$ is the flat plumbing basket code defined in [3] and modified in [2] and the second part $b_1, b_2, \ldots, b_m$ presents the number of full twists for annulus connecting two i which appear exactly twice in the flat plumbing basket code.

Algoritm([1])

- Step 1. For a give link L, we find its braid representation β, the closed braid $\overline{\beta} = L$.
- Step 2. Apply the method in [3] to obtain a basket surface $\mathcal{F}$ while allowing either A_2 or A_{-2} plumbings.
- Step 3. Find a basket code $(a_1a_2\ldots a_{2m} : b_1, b_2, \ldots, b_m)$ of the basket surface.

This algorithm and basket code can be demonstrated in the following Example 3.1.

Example 3.1. A basket code of the link 6^3_3 is $(24123413 : 0, 1, 1, 0)$.

Proof. For the link 6^3_3, we first present it as a closed braid $\overline{\sigma_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}}$ on three strings as illustrated in Figure 3(a). Based on the theorem in [3], we choose the first $\sigma_2\sigma_1$ to have a disc D_2 which is the union of three discs, bounded by three Seifert circles, joined by two half twisted bands presented by $\sigma_2\sigma_1$ as indicated by the dashed purple line in Figure 3(b). We read the other generators $\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}$ from the red point in the direction indicated in Figure 3(b). Since σ_2^{-1} and σ_1^{-1} has the different sign to σ_2 and σ_1 which were used for the disc D_2, we need two flat plumbings. However $\sigma_1\sigma_2$ has the same signs, so we need two A_2-plumbings. Since we were using $\sigma_2\sigma_1$ for the disc D_2, the order of plumbings are plumbing connecting $2-6, 5-8, 1-4$ and $3-7$. Therefore, we find $(24123413 : 0, 1, 1, 0)$ as the basket code of the basket surface of the link 6^3_3 as depicted in Figure 3(c). □

Now, we are set to prove the main theorem which completely classifies the links of the flat plumbing basket numbers 2 or less.

Theorem 3.1. *(1) A link L has the basket number* 0 *if and only if L is the trivial knot.*
(2) A link L has the basket number 1 *if and only if L is a closed two braid of the form* $(\sigma_1)^{2n}$ *for some integer n.*
(3) A link L has the basket number 2 *if and only if L is either a rational knot with continued fraction of the form* $\frac{4mn-1}{2n}$ *or a composite link of the form* $(\sigma_1)^{2m}\#(\sigma_1)^{2n}$ *for some integers* m, n*.*

Proof. (1) The basket surface with 0 plumbing must be a 2-disc. Thus one can easily see that a link L has the basket number 0 if and only if L is the trivial knot.
(2) The basket surface with 1 plumbing must be a 2-disc with a single annulus of an n-full twists. A link L has the basket number 1 if and only if L is a closed two braid of the form $(\sigma_1)^{2n}$ for some integer n.
(3) Consider a link of the basket number 2. There are only three possible flat plumbing basket code (1122), (1212) and (2121). The first one produces a composite link of

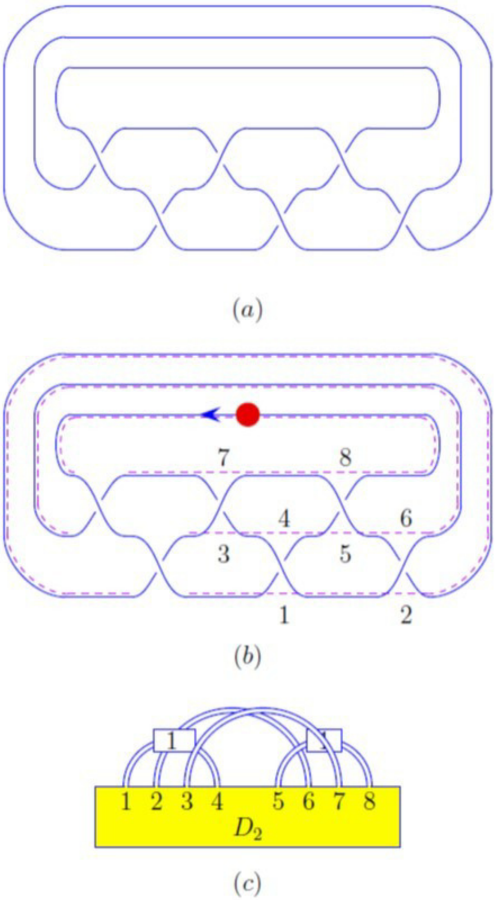


FIGURE 3. (a) The link 6_3^3 as a closed braid, (b) a basket surface of 6_3^3 for which the basket code is $(24123413 : 0, 1, 1, 0)$.

three components and one may find that it is a connected sum of two closed two braids as in the form of $(\sigma_1)^{2m}\#(\sigma_1)^{2n}$ for some integers m, n. Next two cases are indeed the same because the disc D_2 does not exists if we only consider the boundary which is a knot. One may easily find that it is a rational knot. Because we only use full twists for the plumbings, its continued fraction must be a form $\frac{4mn-1}{2n}$. Let us remark that this case was already proved in [1]. Therefore, it completes the proof. $\square$

Although, we have found the classification theorem of links of the basket number 2 or less, it can be used to determine the basket number of a link which is either 3 or 4 as follows. Let us remind that the boundary of a basket surface of n annuli has at most $n+1$ components, and the number of components is always congruent to $n+1$ modulo

Name of Link	Basket number	Basket Code
O_1^1 = Unknot O	0	∅
$O_1^2 = O \uplus O$	1	(11:0)
$O_1^3 = O \uplus O \uplus O$	2	(1122:0,0)
$O_1^4 = O \uplus O \uplus O \uplus O$	3	(112233:0,0,0)
$O_1^5 = O \uplus O \uplus O \uplus O \uplus O$	4	(11223344:0,0,0,0)
$2_1^2(L2a1)$	1	(11:1)
$4_1^2(L4a1)$	1	(11:2)
$5_1^2(L5a1)$	3	(123213:-1,1,1)
$6_1^2(L6a3)$	1	(11:3)
$6_2^2(L6a2)$	3-5	(1254312543:0,0,1,1,1)
$6_3^2(L6a1)$	3	(123123:1,1,-1)
$6_1^3(L6a5)$	4	(12432143:-1,-1,0,0)
$6_2^3(L6a4)$	4	(14231243:-1,-1,1,1)
$6_3^3(L6n1)$	4	(13412342:1,1,0,0)

TABLE 1. The basket numbers of links of 6 crossings or less.

2. In fact, if one find a basket surface of 3 annuli whose boundary is a link L of two components which is not listed in the classification theorem, then the basket number of the link L must be 3. Further if one find a basket surface of 4 annuli whose boundary is a link L of three components which is not listed in the classification theorem, then the basket number of the link L must be 4. Using these ideas, we find the following corollary.

Corollary 3.2. *The basket numbers of links of more than one components and of 6 crossings or less are listed in Table 1, where* $\uplus$ *presents the unlinked disjoint union.*

Proof. First, the boundary of a basket surface of n annuli has at most $n+1$ components and $n+1$ components can be obtained only for the basket code of the form $(1122\cdots nn : **\cdots*)$. Thus, the basket number of the trivial links O_1^n of n components must be $n-1$. One may check the boundary of the basket surface of the given basket code in Table 1 is the given link in the Table 1. The basket number of the other nontrivial link can be found as follows. The links $2_2^1, 4_1^2$ and 6_1^2 are nontrivial and closed 2 braids, by Theorem 3.2 (2), their basket numbers are 1. The links 5_1^2 and 6_3^2 are nontrivial and are not closed 2 braids, by Theorem 3.2(2), their basket numbers are 3. We have found a basket surface of 6_2^2 with 5 plumbings and it is nontrivial and are not closed 2 braids. Thus, its basket number is either 3 or 5. The links $6_4^4, 6_5^4$ and 6_6^4 are nontrivial prime links of three components, by Theorem 3.2 (3), their basket numbers are 4. □

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