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ERROR ESTIMATES FOR A SEMI-DISCRETE MIXED DISCONTINUOUS GALERKIN METHOD WITH AN INTERIOR PENALTY FOR PARABOLIC PROBLEMS

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Abstract. In this paper, we consider a semi-discrete mixed discontinuous Galerkin method with an interior penalty to approximate the solution of parabolic problems. We define an auxiliary projection to analyze the error estimate and obtain optimal error estimates in $L^\infty(L^2)$ for the primary variable u, optimal error estimates in $L^2(L^2)$ for u_t, and suboptimal error estimates in $L^\infty(\boldsymbol{L}^2)$ for the flux variable $\boldsymbol{\sigma}$.

1. Introduction

Discontinuous Galerkin methods with interior penalties which generalized Nitsche method in [11] were introduced to approximate the solutions of elliptic or parabolic problems by several authors [1, 6, 19]. The discontinuous Galerkin methods are widely used for many partial differential equations because of its advantages such as the mesh adaptivity and the local mass conservativeness. There are now a lot of forms and names of the discontinuous Galerkin method. For more details, we refer to [2, 3] and the literatures cited therein.

Riviere and Wheeler [18] introduced semidiscrete and fully discrete locally conservative discontinuous Galerkin methods for nonlinear parabolic equations. They obtained optimal error estimates in $L^2(H^1)$ and suboptimal error estimates in $L^\infty(L^2)$ for semidiscrete approximations and optimal error estimates in $\ell^2(H^1)$ and suboptimal error estimates in $\ell^\infty(L^2)$ for fully discrete approximations. Ohm et. al [12, 13] obtained optimal error estimates in $L^\infty(L^2)$ for semidiscrete approximations and optimal error estimates in $\ell^\infty(L^2)$ for fully discrete approximations which improved the results of Riviere and Wheeler [18]. And using Crank-Nicolson method for time stepping, Ohm et. al [14] introduced fully discrete discontinuous Galerkin method for nonlinear parabolic equations

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and obtained optimal error estimates in $\ell^\infty(L^2)$ for both spatial and temporal directions.

Raviart and Thomas [17] and Nedelec [10] introduced mixed finite element methods to approximate both primary variable and its flux variable, simultaneously. These mixed finite element methods requiring the inf-sup conditions are widely used for elliptic or parabolic problems [5, 7, 9]. And Pani [15] introduced H^1-Galerkin mixed finite element method without inf-sup conditions for parabolic problems. Applications of H^1-Galerkin mixed finite element method can be seen in [8, 16].

Chen [3] introduced a family of mixed discontinuous finite element methods for second-order elliptic equations. Chen and Chen [4] developed a theory for stability and convergence for mixed discontinuous finite element methods in a general form for second-order partial differential problems.

In this paper, we consider a semi-discrete mixed discontinuous Galerkin method with an interior penalty to approximate the solution of parabolic problems and obtain error estimates for both primary variable and its flux variable, simultaneously. In Section 2, we introduce a model problem, semi-discrete mixed discontinuous Galerkin method with an interior penalty for the model problem, and some projections with approximation properties. In Section 3, we define auxiliary projections and give some estimates for the auxiliary projections which will be used in Section 4. And in Section 4, we obtain optimal error estimates in $L^\infty(L^2)$ for the primary variable u, optimal error estimates in $L^2(L^2)$ for u_t, and suboptimal error estimates in $L^\infty(\boldsymbol{L}^2)$ for the flux variable $\boldsymbol{\sigma}$.

2. A model problem and finite element spaces

We consider the following parabolic problem

$$\begin{aligned} &u_t - \nabla\cdot(a(x)\nabla u) = f, && \text{in } \Omega\times(0,T],\\ &u = g_D, && \text{on } \partial\Omega_D\times(0,T],\\ &a(x)\nabla u\cdot\boldsymbol{n} = g_N, && \text{on } \partial\Omega_N\times(0,T],\\ &u(x,0) = u^0(x), && \text{in } \Omega, \end{aligned} \tag{2.1}$$

where $\Omega\subset\mathbb{R}^d$, $1\le d\le 3$, is an open bounded convex domain with the boundary $\partial\Omega = \partial\Omega_D\cup\partial\Omega_N$, $\partial\Omega_D\cap\partial\Omega_N = \phi$ and $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$. Here a is a symmetric, positive definite bounded tensor. And $f\in L^2(\Omega)$, $u^0\in L^2(\Omega)$, $g_D\in H^{1/2}(\partial\Omega_D)$, and $g_N\in H^{-1/2}(\partial\Omega_N)$ are given functions.

Letting $\boldsymbol{\sigma} = a(x)\nabla u$, we obtain the mixed formulation of (2.1)

$$\begin{aligned} & u_t - \nabla\cdot\boldsymbol{\sigma} = f, && \text{in } \Omega\times(0,T], \\ & \boldsymbol{\sigma} = a(x)\nabla u, && \text{in } \Omega\times(0,T], \\ & u = g_D, && \text{on } \partial\Omega_D\times(0,T], \\ & \boldsymbol{\sigma}\cdot\boldsymbol{n} = g_N, && \text{on } \partial\Omega_N\times(0,T], \\ & u(x,0) = u^0(x), && \text{in } \Omega. \end{aligned} \tag{2.2}$$

To introduce the mixed discontinuous Galerkin finite element method for the problem (2.1), let $\{T_h\}_{h>0}$ be a sequence of a regular quasi-uniform partitions of Ω and each subdomain $T\in T_h$ be a triangle or a quadrilateral (a 3-simplex or 3-rectangle) if $d=2$ (if $d=3$, respectively). Let $h_T = diam(T)$ be the diameter of T and $h = max_{T\in T_h} h_T$. From the assumptions of regularity and quasi-uniformity, there exist constants ρ and γ such that each T contains a ball of radius ρh_T and $h\le\gamma h_T$ for all $T\in T_h$. Two adjacent elements in T_h are not required to be matched, i.e., a vertex of one element can lie on the edge or face of another element. For a given T_h, let $\mathcal{E}_h^I$ denote the set of all interior boundaries e of T_h, $\mathcal{E}_h^D$ and $\mathcal{E}_h^N$ be the sets of boundaries e on $\partial\Omega_D$ and $\partial\Omega_N$, respectively, $\mathcal{E}_h^B = \mathcal{E}_h^D\cup\mathcal{E}_h^N$ the set of the boundaries e on $\partial\Omega$, $\mathcal{E}_h^{ID} = \mathcal{E}_h^I\cup\mathcal{E}_h^D$, and $\mathcal{E}_h = \mathcal{E}_h^I\cup\mathcal{E}_h^B$. For $e\in\mathcal{E}_h^B$, $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$. For $e\in\mathcal{E}_h^I$, with $e = T_1\cap T_2$ and T_1, $T_2\in T_h$, the direction of $\boldsymbol{n}$ is associated with the definition of jump across e.

For $\ell\ge 0$, we define

$$\begin{aligned} & H^\ell(T_h) = \{v\in L^2(\Omega) : v|_T\in H^\ell(T),\ T\in T_h\}, \\ & \boldsymbol{H}^\ell(T_h) = \{\boldsymbol{w}\in(L^2(\Omega))^d : \boldsymbol{w}|_T\in\boldsymbol{H}^\ell(T) = (H^\ell(T))^d,\ T\in T_h\} \end{aligned}$$

with

$$\begin{aligned} \|v\|_\ell &= \left(\sum_{T\in T_h}\|v\|^2_{H^\ell(T)}\right)^{1/2}, \\ \|\boldsymbol{w}\|_\ell &= \left(\sum_{T\in T_h}\|\boldsymbol{w}\|^2_{\boldsymbol{H}^\ell(T)}\right)^{1/2}. \end{aligned}$$

We simply write $\|\cdot\|$ when $\ell=0$. For $v\in H^\ell(T_h)$ with $\ell>\frac{1}{2}$, the jump of v across $e = \partial T_1\cap\partial T_2\in\mathcal{E}_h^I$ is defined by

$$[v] = v|_{T_2\cap e} - v|_{T_1\cap e}.$$

The average of v on $e = \partial T_1\cap\partial T_2\in\mathcal{E}_h^I$ is defined as

$$\{v\} = \frac{1}{2}\Big(v|_{T_1\cap e} + v|_{T_2\cap e}\Big).$$

As a convention, for $e \in \mathcal{E}_h^B$, the jump and the average are defined as follows:

$$\{v\} = v|_e, \quad [v] = \begin{cases} 0, & e \in \mathcal{E}_h^D, \\ v, & e \in \mathcal{E}_h^N. \end{cases}$$

Let $V = H^1(T_h)$ and $\boldsymbol{W} = \{\boldsymbol{w} \in \boldsymbol{H}^1(T_h) \mid \nabla \cdot \boldsymbol{w} \in L^2(\Omega)\}$. And let $V_h = \{v \in V \mid v|_T \in P_k(T),\ T \in T_h\}$ and $\boldsymbol{W}_h = \{\boldsymbol{w} \in \boldsymbol{W} \mid \boldsymbol{w}|_T \in \boldsymbol{P}_k(T),\ T \in T_h\}$ be the finite element spaces of V and $\boldsymbol{W}$, respectively, where $P_k(T)$ the set of polynomials of total degree $\leq k$ defined on T and $\boldsymbol{P}_k(T) = (P_k(T))^d$. They are defined locally on each element $T \in T_h$, so that $\boldsymbol{W}_h(T) = \boldsymbol{W}_h|_T$ and $V_h(T) = V_h|_T$. Neither continuity constraint nor boundary values are imposed on $\boldsymbol{W}_h \times V_h$.

Now the corresponding semi-discrete mixed discontinuous Galerkin method with an interior penalty of (2.1) is: Find $u_h \in V_h$ and $\boldsymbol{\sigma}_h \in \boldsymbol{W}_h$ such that

$$\begin{aligned} &((u_h)_t, v) + \sum_T (\boldsymbol{\sigma}_h, \nabla v)_T - \sum_{e \in \mathcal{E}_h^{ID}} (\{\boldsymbol{\sigma}_h \cdot \boldsymbol{n}\}, [v])_e + J(u_h, v) \\ &= \sum_{e \in \mathcal{E}_h^N} (g_N, v)_e + \sum_{e \in \mathcal{E}^D} h_e^{-1} (g_D, v)_e + (f, v), \ \ \forall v \in V_h, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &(\alpha(x)\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - \sum_T (\nabla u_h, \boldsymbol{\tau})_T + \sum_{e \in \mathcal{E}_h^{ID}} (\{\boldsymbol{\tau} \cdot \boldsymbol{n}\}, [u_h])_e \\ &= \sum_{e \in D} (g_D, \boldsymbol{\tau} \cdot \boldsymbol{n})_e, \ \ \forall \boldsymbol{\tau} \in \boldsymbol{W}_h, \end{aligned} \tag{2.4}$$

where $J(u, v) = \sum_{e \in \mathcal{E}_h^{IN}} h_e^{-1} \int_e [u][v] ds$, $h_e = |e|$, $\alpha(x) = a(x)^{-1}$, $(\ ,\ )$ denote an L^2 inner product on Ω, $(\ ,\ )_T$ an L^2 inner product on T, and $(\ ,\ )_e$ an L^2 inner product on e. We define the following bilinear forms as follows:

$$\begin{aligned} &A(\boldsymbol{q}, \boldsymbol{r}) = (\alpha(x)\boldsymbol{q}, \boldsymbol{r}), \quad \forall\ \boldsymbol{q}, \boldsymbol{r} \in \boldsymbol{W} \\ &B(\boldsymbol{\tau}, v) = \sum_T (\boldsymbol{\tau}, \nabla v)_T - \sum_{e \in \mathcal{E}_h^{ID}} (\{\boldsymbol{\tau} \cdot \boldsymbol{n}\}, [v])_e, \quad \forall\ \boldsymbol{\tau} \in \boldsymbol{W}, v \in V, \\ &C(u, v) = J(u, v) + \lambda(u, v), \quad \forall\ u, v \in V, \end{aligned} \tag{2.5}$$

where λ is a positive real number. And we define the following broken norms on V and $\boldsymbol{W}$ as follows:

$$\begin{aligned} &\|v\|_C^2 = J(v, v) + \lambda \|v\|^2, \\ &\|v\|_S^2 = \|v\|_1^2 + J(v, v), \\ &\|\boldsymbol{\sigma}\|_{\boldsymbol{W}}^2 = \|\boldsymbol{\sigma}\|^2 + \sum_{T \in T_h} h_T^2 \|\nabla \cdot \boldsymbol{\sigma}\|_T^2, \\ &\|\boldsymbol{\tau}\|_A^2 = A(\boldsymbol{\tau}, \boldsymbol{\tau}), \end{aligned} \tag{2.6}$$

where $\| \|_1$ denotes H^1 norm on V and $\| \|$ denotes L^2 norm on V or $\boldsymbol{W}$. Notice that $\|v\|_C \leq \|v\|_S$ for sufficiently small λ. And also we define the following linear functionals on V as follows:

$$\begin{aligned} F(v) &= (f, v), \\ G_N(v) &= \sum_{e \in \mathcal{E}_h^N} (g_N, v)_e, \\ G_D^1(\boldsymbol{\tau}) &= \sum_{e \in \mathcal{E}_h^D} (g_D, \boldsymbol{\tau} \cdot \boldsymbol{n})_e, \\ G_D^2(v) &= \sum_{e \in \mathcal{E}_h^D} h_e^{-1} (g_D, v)_e. \end{aligned} \tag{2.7}$$

Then (2.3)-(2.4) can be rewritten into the system

$$\begin{aligned} ((u_h)_t, v) + B(\boldsymbol{\sigma}_h, v) + C(u_h, v) - \lambda(u_h, v) \\ = G_N(v) + G_D^2(v) + F(v), \quad \forall v \in V_h, \end{aligned} \tag{2.8}$$

and

$$A(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - B(\boldsymbol{\tau}, u_h) = G_D^1(\boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in \boldsymbol{W}_h. \tag{2.9}$$

Obviously, the solution $(u, \boldsymbol{\sigma})$ of the problem (2.2) satisfy the system

$$\begin{aligned} (u_t, v) + B(\boldsymbol{\sigma}, v) + C(u, v) - \lambda(u, v) \\ = G_N(v) + G_D^2(v) + F(v), \quad \forall v \in V, \end{aligned} \tag{2.10}$$

and

$$A(\boldsymbol{\sigma}, \tau) - B(\boldsymbol{\tau}, u) = G_D^1(\boldsymbol{\tau}), \quad \forall \boldsymbol{\tau} \in \boldsymbol{W}. \tag{2.11}$$

Let $P_h : V \to V_h$ and $\boldsymbol{\Pi}_h : \boldsymbol{W} \to \boldsymbol{W}_h$ denote the projections satisfying the following approximation properties:

$$\begin{aligned} &\|v - P_h v\|_i \leq K h^{r-i} \|v\|_r, \ \forall v \in V \cap H^r(T), \ i \leq r \leq k+1, \ i = 0, 1, \\ &\|\boldsymbol{w} - \boldsymbol{\Pi}_h \boldsymbol{w}\| \leq K h^r \|\boldsymbol{w}\|_r, \ \forall \boldsymbol{w} \in \boldsymbol{W} \cap \boldsymbol{H}^r(T), \ \ 1 \leq r \leq k+1, \\ &\|\nabla \cdot (\boldsymbol{w} - \boldsymbol{\Pi}_h \boldsymbol{w})\| \leq K h^r \|\nabla \cdot \boldsymbol{w}\|_r, \ \forall \boldsymbol{w} \in \boldsymbol{W} \cap \boldsymbol{H}^r(T), \ \ 0 \leq r \leq k. \end{aligned} \tag{2.12}$$

Lemma 2.1. *For any $u, v \in V$ and any $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{W}$, the followings hold:*
(1) $\quad A(\boldsymbol{\sigma}, \boldsymbol{\tau}) \leq K \|\boldsymbol{\sigma}\|_A \|\boldsymbol{\tau}\|_A, \quad A(\boldsymbol{\sigma}, \boldsymbol{\tau}) \leq K \|\boldsymbol{\sigma}\|_{\boldsymbol{W}} \|\boldsymbol{\tau}\|_{\boldsymbol{W}}$;
(2) $\quad B(\boldsymbol{\sigma}, v) \leq K \|\boldsymbol{\sigma}\|_{\boldsymbol{W}} \|v\|_S$;
(3) $\quad C(u, v) \leq K \|u\|_C \|v\|_C, \quad C(u, v) \leq K \|u\|_S \|v\|_S$.

Proof. The proofs of (1) and (3) are trivial. So we will prove (2) only.
(2) Let $v \in V$ and $\boldsymbol{\sigma} \in \boldsymbol{W}$. Then

$$\begin{aligned} B(\boldsymbol{\sigma}, v) &= \sum_T (\boldsymbol{\sigma}, \nabla v)_T - \sum_{e \in \epsilon_h^{ID}} (\{\boldsymbol{\sigma} \cdot \boldsymbol{n}\}, [v])_e \\ &\leq \|\boldsymbol{\sigma}\|_{(L^2(\Omega))^d} \left(\sum_T \|\nabla v\|^2_{(L^2(T))^d} \right)^{1/2} \end{aligned}$$

$$
\begin{aligned}
&+\Big(\sum_{e\in\epsilon_h^{ID}} h_e\|\{\boldsymbol{\sigma}\cdot\boldsymbol{n}\}\|^2_{L^2(e)}\Big)^{1/2}\Big(\sum_{e\in\epsilon_h^{ID}} h_e^{-1}\|[v]\|^2_{L^2(e)}\Big)^{1/2}\\
&\le \|\boldsymbol{\sigma}\|_{(L^2(\Omega))^d}\|\nabla v\|_{(L^2(\Omega))^d}\\
&\quad + K\Big[\|\boldsymbol{\sigma}\|^2_{(L^2(\Omega))^d} + \Big(\sum_T h_T^2\|\nabla\cdot\boldsymbol{\sigma}\|^2_{(L^2(T))^d}\Big)\Big]^{1/2} J(v,v)^{1/2}\\
&\le K\Big[\|\boldsymbol{\sigma}\|^2_{(L^2(\Omega))^d} + \Big(\sum_T h_T^2\|\nabla\cdot\boldsymbol{\sigma}\|^2_{(L^2(T))^d}\Big)\Big]^{1/2}\\
&\qquad\cdot\Big[\|\nabla v\|^2_{(L^2(\Omega))^d} + J(v,v)\Big]^{1/2}\\
&\le K\|\boldsymbol{\sigma}\|_{\boldsymbol{W}}\|v\|_S.
\end{aligned}
$$

This completes the proof. □

Lemma 2.2. *For any $v \in V_h$ and any $\boldsymbol{\tau} \in \boldsymbol{W}_h$, the followings hold:*
(1) $A(\boldsymbol{\tau},\boldsymbol{\tau}) \ge K\|\boldsymbol{\tau}\|^2_{\boldsymbol{W}}$;
(2) $C(v,v) \ge K\|v\|^2_S$, *for* $\lambda > 0$.

Proof. The proofs of these results are trivial from the given conditions on a and $\lambda > 0$. □

3. Auxiliary projections and some estimates

For given $(u,\boldsymbol{\sigma}) \in V \times \boldsymbol{W}$, we define $(\tilde{u},\tilde{\boldsymbol{\sigma}}) \in V_h \times \boldsymbol{W}_h$ such that

$$B(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}, v) + C(u-\widetilde{u}, v) = 0, \qquad \forall v \in V_h \tag{3.1}$$

and

$$A(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}}, \boldsymbol{\tau}) - B(\boldsymbol{\tau}, u-\widetilde{u}) = 0, \qquad \forall \boldsymbol{\tau} \in \boldsymbol{W}_h. \tag{3.2}$$

Due to [4], the unique existence of $(\tilde{u},\tilde{\boldsymbol{\sigma}}) \in V_h \times \boldsymbol{W}_h$ follows from Lemmas 2.1 and 2.2.

Lemma 3.1. *For any $u \in V \cap H^{k+1}(T_h)$ and any $\boldsymbol{\sigma} \in \boldsymbol{W} \cap \boldsymbol{H}^{k+1}(T_h)$, we have*

$$\|u-\widetilde{u}\|_C + \|\boldsymbol{\sigma}-\widetilde{\boldsymbol{\sigma}}\|_A \le Kh^k(\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_{k+1}).$$

Proof. From (3.1)-(3.2), together with $v = v_h$ and $\boldsymbol{\tau} = \boldsymbol{\tau}_h$, we obtain the following system

$$
\begin{aligned}
&B(\boldsymbol{\Pi}_h\boldsymbol{\sigma}-\widetilde{\boldsymbol{\sigma}}, v_h) + C(P_h u-\widetilde{u}, v_h),\\
&\qquad = B(\boldsymbol{\Pi}_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, v_h) + C(P_h u-u, v_h),
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
&A(\boldsymbol{\Pi}_h\boldsymbol{\sigma}-\widetilde{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) - B(\boldsymbol{\tau}_h, P_h u-\widetilde{u})\\
&\qquad = A(\boldsymbol{\Pi}_h\boldsymbol{\sigma}-\boldsymbol{\sigma}, \boldsymbol{\tau}_h) - B(\boldsymbol{\tau}_h, P_h u-u).
\end{aligned} \tag{3.4}
$$

Let $v_h = P_h u - \widetilde{u}$ and $\boldsymbol{\tau}_h = \boldsymbol{\Pi}_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}$ in (3.3)-(3.4). Then adding both sides of (3.3)-(3.4), we get

$$\begin{aligned} &|||P_h u - \widetilde{u}|||_C^2 + \|\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A^2 \\ &= C(P_h u - \widetilde{u}, P_h u - \widetilde{u}) + A(\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}) \\ &= B(\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma}, P_h u - \widetilde{u}) + C(P_h u - u, P_h u - \widetilde{u}) \\ &\quad + A(\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma}, \Pi_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}) - B(\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, P_h u - u) \\ &= \sum_{i=1}^{4} I_i. \end{aligned} \tag{3.5}$$

By (2.12), we have for $\epsilon > 0$

$$\begin{aligned} I_1 &= B(\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma}, P_h u - \widetilde{u}) \\ &= \sum_T \Big(\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma}, \nabla(P_h u - \widetilde{u})\Big)_T - \sum_{e \in \mathcal{E}_h^{ID}} \Big(\{(\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma}) \cdot n\}, [P_h u - \widetilde{u}]\Big)_e \\ &\le K h^{-1} \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\| \Big(\sum_T h_T^2 \|\nabla(P_h u - \widetilde{u})\|_T^2 \Big)^{\frac{1}{2}} \\ &\quad + K \Big(\sum_{e \in \mathcal{E}^{ID}} h_e \|\{\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma}) \cdot n\}\|_e^2 \Big)^{\frac{1}{2}} \Big(\sum_{e \in \mathcal{E}^{ID}} h_e^{-1} \|[P_h u - \widetilde{u}]\|_e^2 \Big)^{\frac{1}{2}} \\ &\le K h^{-1} \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\| \|P_h u - \widetilde{u}\| \\ &\quad + K(\|\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma}\|^2 + h^2 \|\nabla \cdot (\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma})\|^2)^{\frac{1}{2}} J(P_h u - \widetilde{u}, P_h u - \widetilde{u})^{\frac{1}{2}} \\ &\le K \Big[h^{-2} \|\boldsymbol{\sigma} - \boldsymbol{\Pi}_h \boldsymbol{\sigma}\|^2 + h^2 \|\nabla \cdot (\boldsymbol{\Pi}_h \boldsymbol{\sigma} - \boldsymbol{\sigma})\|^2 \Big] + \epsilon |||P_h u - \widetilde{u}|||_C^2 \\ &\le K h^{2k} \|\boldsymbol{\sigma}\|_{k+1}^2 + \epsilon |||P_h u - \widetilde{u}|||_C^2. \end{aligned}$$

Since

$$\begin{aligned} &C(P_h u - u, P_h u - u) \\ &= \sum_{e \in \mathcal{E}_h^{ID}} h_e^{-1} \|P_h u - u\|_e^2 + \lambda \|P_h u - u\|^2 \\ &\le K \sum_T \Big[h_T^{-1} \|P_h u - u\|_T^2 + \|\nabla(P_h u - u)\|_T^2 \Big] + \lambda \|P_h u - u\|^2 \\ &\le K h^{2k} \|u\|_{k+1}^2, \end{aligned}$$

by (2.12), we get for $\epsilon > 0$

$$\begin{aligned} I_2 &= C(P_h u - u, P_h u - \widetilde{u}) \\ &\le C(P_h u - u, P_h u - u) + \epsilon |||P_h u - \widetilde{u}|||_C^2 \\ &\le K h^{2k} \|u\|_{k+1}^2 + \epsilon |||P_h u - \widetilde{u}|||_C^2. \end{aligned}$$

And by (2.12), we have the following estimates: for $\epsilon > 0$

$$\begin{aligned} I_3 = A(\mathbf{\Pi}_h\boldsymbol{\sigma}, \mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}) &\le K\|\mathbf{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}\|_A^2 + \epsilon\|\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A^2 \\ &\le Kh^{2(k+1)}\|\boldsymbol{\sigma}\|_{k+1}^2 + \epsilon\|\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A^2 \end{aligned}$$

and

$$\begin{aligned} I_4 &= -B(\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, P_h u - u) \\ &= -\sum_T \Big(\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \nabla(P_h u - u)\Big)_T + \sum_{e\in\mathcal{E}_h^{ID}} \Big(\{(\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}})\cdot n\}, [P_h u - u]\Big)_e \\ &\le \|\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|\|\nabla(u - P_h u)\| \\ &\quad + \Big(\sum_{e\in\mathcal{E}_h^{ID}} h_e\|\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_e^2\Big)^{\frac{1}{2}} J(u - P_h u, u - P_h u)^{\frac{1}{2}} \\ &\le K\Big[\|\nabla(u - P_h u)\|^2 + J(u - P_h u, u - P_h u)\Big] + \epsilon\|\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A^2 \\ &\le Kh^{2k}\|u\|_{k+1}^2 + \epsilon\|\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A^2. \end{aligned}$$

Therefore, substituting the bounds for $I_1 - I_4$ into (3.5) and taking $\epsilon > 0$ sufficiently small, we obtain

$$|\!|\!| P_h u - \widetilde{u} |\!|\!|_C^2 + \|\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A^2 \le Kh^{2k}(\|\boldsymbol{\sigma}\|_{k+1}^2 + \|u\|_{k+1}^2). \tag{3.6}$$

Thus, using (2.12) and the triangular inequality, we get

$$|\!|\!| u - \widetilde{u} |\!|\!|_C + \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A \le Kh^k(\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_{k+1}), \tag{3.7}$$

which completes the proof. □

Lemma 3.2. *For any $u \in V \cap H^{k+1}(T_h)$ and any $\boldsymbol{\sigma} \in \boldsymbol{W} \cap \boldsymbol{H}^{k+1}(T_h)$,*

$$|\!|\!| \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}} |\!|\!|_{\boldsymbol{W}} \le Kh^k(\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_{k+1}).$$

Proof. From (2.12), the local inverse property, and Lemma 3.1, we have

$$\begin{aligned} \|\nabla\cdot(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}})\| &\le \|\nabla\cdot(\boldsymbol{\sigma} - \mathbf{\Pi}_h\boldsymbol{\sigma})\| + \|\nabla\cdot(\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}})\| \\ &\le Kh^k\|\boldsymbol{\sigma}\|_{k+1} + Kh^{-1}\|\mathbf{\Pi}_h\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\| \\ &\le Kh^k\|\boldsymbol{\sigma}\|_{k+1} + Kh^{-1}(\|\mathbf{\Pi}_h\boldsymbol{\sigma} - \boldsymbol{\sigma}\| + \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|) \\ &\le Kh^k\|\boldsymbol{\sigma}\|_{k+1} + Kh^{-1}\|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\| \\ &\le Kh^{k-1}\Big(\|\boldsymbol{\sigma}\|_{k+1} + \|u\|_{k+1}\Big). \end{aligned} \tag{3.8}$$

Therefore, using the definition of $|\!|\!| \cdot |\!|\!|_{\boldsymbol{W}}$, Lemma 3.1 and (3.8), we get

$$|\!|\!| \boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}} |\!|\!|_{\boldsymbol{W}}^2 = \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|^2 + \sum_T h_T^2\|\nabla\cdot(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}})\|_T^2$$

$$\leq K\Big(\|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A^2 + h^2 \|\nabla \cdot (\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}})\|^2 \Big)$$
$$\leq K h^{2k} (\|\boldsymbol{\sigma}\|_{k+1}^2 + \|u\|_{k+1}^2),$$

which completes the proof. □

Lemma 3.3. *For any $u_t \in V \cap H^{k+1}(T_h)$ and any $\boldsymbol{\sigma}_t \in \boldsymbol{W} \cap \boldsymbol{H}^{k+1}(T_h)$,*

$$\|u_t - \widetilde{u}_t\|_C + \|\boldsymbol{\sigma}_t - \widetilde{\boldsymbol{\sigma}}_t\|_A \leq K h^k \Big(\|u_t\|_{k+1} + \|\boldsymbol{\sigma}_t\|_{k+1} \Big),$$
$$\|\boldsymbol{\sigma}_t - \widetilde{\boldsymbol{\sigma}}_t\|_{\boldsymbol{W}} \leq K h^k \Big(\|u_t\|_{k+1} + \|\boldsymbol{\sigma}_t\|_{k+1} \Big).$$

Proof. The proofs of these results are similar to those of Lemma 3.1 and Lemma 3.2. □

Lemma 3.4. *For any $u \in V \cap H^{k+1}(T_h)$ and any $\boldsymbol{\sigma} \in \boldsymbol{W} \cap \boldsymbol{H}^{k+1}(T_h)$,*

$$\|u - \widetilde{u}\| \leq K h^{k+1} \Big(\|u\|_{k+1} + \|\boldsymbol{\sigma}\|_{k+1} \Big),$$
$$\|u_t - \widetilde{u}_t\| \leq K h^{k+1} \Big(\|u_t\|_{k+1} + \|\boldsymbol{\sigma}_t\|_{k+1} \Big).$$

Proof. Define $\phi \in H^2(\Omega)$ and $\boldsymbol{\psi} \in (H^1(\Omega))^d$ satisfying

$$\begin{aligned} \nabla \phi - \alpha(x) \boldsymbol{\psi} &= 0, && \text{in } \Omega, \\ -\nabla \cdot \boldsymbol{\psi} + \lambda \phi &= u - \tilde{u}, && \text{in } \Omega, \\ \phi &= 0, && \text{on } \partial\Omega_D, \\ \boldsymbol{\psi} \cdot \boldsymbol{n} &= 0, && \text{on } \partial\Omega_N. \end{aligned} \tag{3.9}$$

Then, by the property of elliptic regularity, we have

$$\|\phi\|_2 + \|\boldsymbol{\psi}\|_1 \leq K \|u - \widetilde{u}\|.$$

By (3.9), the integration by parts, and the definition of $B(\cdot, \cdot)$, we obviously have

$$\begin{aligned} (-\nabla \cdot \boldsymbol{\psi}, u - \widetilde{u}) = (\boldsymbol{\psi}, \nabla(u - \widetilde{u})) \\ - \sum_{e \in \mathcal{E}_h^I} \Big(([\boldsymbol{\psi} \cdot \boldsymbol{n}], \{u - \widetilde{u}\})_e + (\{\boldsymbol{\psi} \cdot \boldsymbol{n}\}, [u - \widetilde{u}])_e \Big) \\ - \sum_{e \in \mathcal{E}_h^D} (\boldsymbol{\psi} \cdot \boldsymbol{n}, u - \widetilde{u})_e - \sum_{e \in \mathcal{E}_h^N} (\boldsymbol{\psi} \cdot \boldsymbol{n}, u - \widetilde{u})_e \\ = B(\boldsymbol{\psi}, u - \widetilde{u}). \end{aligned} \tag{3.10}$$

Since $\phi \in H^2(\Omega) \subset C(\Omega)$, $\boldsymbol{\psi} \in (H^1(\Omega))^d$, $\phi = 0$ on $\partial\Omega_D$, and $\boldsymbol{\psi}\cdot\boldsymbol{n} = 0$ on $\partial\Omega_N$, we get the followings:

$$\begin{aligned}(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \nabla\phi) &= (\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \nabla\phi) - \sum_{e\in\mathcal{E}_h^I} \Big(\{(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}})\cdot\boldsymbol{n}\}, [\phi]\Big)_e \\ &\quad - \sum_{e\in\mathcal{E}_h^D} \Big((\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}})\cdot\boldsymbol{n}, \phi\Big)_e \\ &= B(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \phi)\end{aligned} \tag{3.11}$$

and

$$J(u - \widetilde{u}, \phi) = \sum_{e\in\mathcal{E}_h^{ID}} h_e^{-1}([u - \widetilde{u}], [\phi])_e = 0. \tag{3.12}$$

By (3.9)-(3.12), and the definition of $A(\cdot,\cdot)$, we get

$$\begin{aligned}\|u - \widetilde{u}\|^2 &= (-\nabla\cdot\boldsymbol{\psi}, u - \widetilde{u}) + \lambda(\phi, u - \widetilde{u}) \\ &\quad + (\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \nabla\phi) - (\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \alpha(x)\boldsymbol{\psi}) \\ &= B(\boldsymbol{\psi}, u - \widetilde{u}) + \lambda(\phi, u - \widetilde{u}) + B(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \phi) - A(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \boldsymbol{\psi}) \\ &= B(\boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi}, u - \widetilde{u}) + B(\boldsymbol{\Pi}_h\boldsymbol{\psi}, u - \widetilde{u}) \\ &\quad + \lambda(\phi - P_h\phi, u - \widetilde{u}) + \lambda(P_h\phi, u - \widetilde{u}) + B(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \phi - P_h\phi) \\ &\quad + B(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, P_h\phi) - A(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi}) - A(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \boldsymbol{\Pi}_h\boldsymbol{\psi}).\end{aligned} \tag{3.13}$$

Notice that by (2.12), we get

$$\begin{aligned}|\!|\!| u - P_h u |\!|\!|_S^2 &= \|u - P_h u\|_1^2 + \sum_{e\in\mathcal{E}_h^{ID}} h_e^{-1} \int_e |[u - P_h u]|^2 ds \\ &\le K h^{2k} \|u\|_{k+1}^2\end{aligned} \tag{3.14}$$

and for $v \in V_h$

$$\begin{aligned}&B(\boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi}, v) \\ &\quad = \sum_T (\boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi}, \nabla v)_T - \sum_{e\in\mathcal{E}_h^{ID}} (\{(\boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi})\cdot\boldsymbol{n}\}, [v])_e \\ &\quad = - \sum_{e\in\mathcal{E}_h^{ID}} (\{(\boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi})\cdot\boldsymbol{n}\}, [v])_e \\ &\quad \le K h \|\boldsymbol{\psi}\|_1 |\!|\!| v |\!|\!|_C.\end{aligned} \tag{3.15}$$

By applying (3.1), (3.2), (3.12), (3.14), and (3.15) to (3.13), we get

$$\begin{aligned}\|u - \tilde{u}\|^2 &= B(\boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi}, u - \widetilde{u}) + \lambda(\phi - P_h\phi, u - \widetilde{u}) + B(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \phi - P_h\phi) \\ &\quad - A(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi}) - J(u - \widetilde{u}, P_h\phi) \\ &= B(\boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi}, u - P_h u) + B(\boldsymbol{\psi} - \boldsymbol{\Pi}_h\boldsymbol{\psi}, P_h u - \widetilde{u}) \\ &\quad + \lambda(\phi - P_h\phi, u - \widetilde{u}) + B(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \phi - P_h\phi)\end{aligned}$$

$$
\begin{aligned}
&- A(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}, \boldsymbol{\psi} - \boldsymbol{\Pi}_h \boldsymbol{\psi}) + J(u - \widetilde{u}, \phi - P_h \phi) \\
&\leq K \Bigg[\|\!|\boldsymbol{\psi} - \boldsymbol{\Pi}_h \boldsymbol{\psi}\|\!|_{\boldsymbol{W}} \|\!|u - P_h u\|\!|_S \\
&\quad + \|\!|P_h u - \widetilde{u}\|\!|_C \sup_{v \in V_h, v \neq 0} \frac{B(\boldsymbol{\psi} - \boldsymbol{\Pi}_h \boldsymbol{\psi}, v)}{\|\!|v\|\!|_C} \\
&\quad + \|\phi - P_h \phi\| \|u - \widetilde{u}\| + \|\!|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|\!|_{\boldsymbol{W}} \|\!|\phi - P_h \phi\|\!|_S \\
&\quad + \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A \|\boldsymbol{\psi} - \boldsymbol{\Pi}_h \boldsymbol{\psi}\|_A + \|\!|u - \widetilde{u}\|\!|_C \|\!|\phi - P_h \phi\|\!|_C \Bigg] \\
&\leq K \Bigg[h^{k+1} \|u\|_{k+1} \|\boldsymbol{\psi}\|_1 + h \|\!|P_h u - \widetilde{u}\|\!|_C \|\boldsymbol{\psi}\|_1 + h \|\phi\|_2 \|\!|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|\!|_{\boldsymbol{W}} \\
&\quad + h \|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_A \|\boldsymbol{\psi}\|_1 + h \|\phi\|_2 \|\!|u - \widetilde{u}\|\!|_C \Bigg] + K h^2 \|\phi\|_2 \|u - \widetilde{u}\| \\
&\leq K h \left(h^k \|u\|_{k+1} + \|\!|P_h u - \widetilde{u}\|\!|_C + \|\!|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|\!|_{\boldsymbol{W}} + \|\!|u - \widetilde{u}\|\!|_C \right) \|u - \widetilde{u}\| \\
&\quad + K h^2 \|u - \widetilde{u}\|^2
\end{aligned}
$$

and hence for sufficiently small $h > 0$ we have

$$\|u - \widetilde{u}\| \leq \; K h \left(h^k \|u\|_{k+1} + \|\!|P_h u - \widetilde{u}\|\!|_C + \|\!|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|\!|_{\boldsymbol{W}} + \|\!|u - \widetilde{u}\|\!|_C \right).$$

Therefore, by Lemma 3.2, (3.6), and (3.7), we obtain

$$\|u - \widetilde{u}\| \leq \; K h^{k+1} (\|u\|_{k+1} + \|\boldsymbol{\sigma}\|_{k+1}),$$

which completes the proof of the first result. The proof of the second result is similar to one of the first result. □

4. Error estimates

Theorem 4.1. *If* $(u, \boldsymbol{\sigma}) \in (V \cap H^{k+1}(T_h)) \times (\boldsymbol{W} \cap \boldsymbol{H}^{k+1}(T_h))$ *is the solution of (2.2) and* $(u_h, \boldsymbol{\sigma}_h) \in V_h \times \boldsymbol{W}_h$ *is the solution of (2.3)-(2.4), then*

$$
\begin{aligned}
&\|u - u_h\|_{L^\infty(L^2)} + h \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^\infty(\boldsymbol{L}^2)} \\
&\leq K h^{k+1} \Bigg(\|u\|_{L^2(H^{k+1})} + \|u_t\|_{L^2(H^{k+1})} + \|\boldsymbol{\sigma}\|_{L^2(\boldsymbol{H}^{k+1})} + \|\boldsymbol{\sigma}_t\|_{L^2(\boldsymbol{H}^{k+1})} \Bigg).
\end{aligned}
$$

Proof. From (2.8)-(2.11), we obtain the system of error equations

$$
\begin{aligned}
(u_t - (u_h)_t, v_h) + B(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v_h) + C(u - u_h, v_h) &= \lambda(u - u_h, v_h), \quad \forall v_h \in V_h, \\
A(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) - B(\boldsymbol{\tau}_h, u - u_h) &= 0, \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{W}_h.
\end{aligned}
$$

And using (3.1)-(3.2) in the system of error equations, we get

$$
\begin{aligned}
&(\widetilde{u}_t - (u_h)_t, v_h) + B(\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h, v_h) + C(\widetilde{u} - u_h, v_h) \\
&\qquad = (\widetilde{u}_t - u_t, v_h) + \lambda(u - u_h, v_h), \quad \forall v_h \in V_h
\end{aligned}
\tag{4.1}
$$

and

$$A(\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)-B(\boldsymbol{\tau}_h,\widetilde{u}-u_h)=0,\ \ \forall\boldsymbol{\tau}_h\in\boldsymbol{W}_h. \tag{4.2}$$

Letting $v_h=\widetilde{u}-u_h,\ \boldsymbol{\tau}_h=\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h$ in (4.1)-(4.2), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\widetilde{u}-u_h\|^2+|\!|\!|\widetilde{u}-u_h|\!|\!|_C^2+\|\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h\|_A^2\\
&\le\|\widetilde{u}_t-u_t\|\|\widetilde{u}-u_h\|+\lambda(u-u_h,\widetilde{u}-u_h)\\
&\le\|\widetilde{u}_t-u_t\|\|\widetilde{u}-u_h\|+\lambda(\|u-\widetilde{u}\|+\|\widetilde{u}-u_h\|)\|\widetilde{u}-u_h\|\\
&\le K\Big[\|u_t-\widetilde{u}_t\|^2+\|u-\widetilde{u}\|^2+\|\widetilde{u}-u_h\|^2\Big].
\end{aligned}\tag{4.3}$$

Now we integrate both sides of (4.3) with respect to t from 0 to $t\le T$ to get

$$\begin{aligned}
&\frac{1}{2}\|(\widetilde{u}-u_h)(t)\|^2+\int_0^t|\!|\!|\widetilde{u}-u_h|\!|\!|_C^2+\|\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h\|_A^2ds\\
&\le\frac{1}{2}\|(\widetilde{u}-u_h)(0)\|^2\\
&\qquad+K\int_0^t\|(u_t-\widetilde{u}_t)(s)\|^2+\|(u-\widetilde{u})(s)\|^2+\|(\widetilde{u}-u_h)(s)\|^2ds
\end{aligned}$$

and hence by Gronwall's inequality we get

$$\begin{aligned}
&\|\widetilde{u}-u_h\|_{L^\infty(L^2)}+|\!|\!|\widetilde{u}-u_h|\!|\!|_{L^2(C)}+\|\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h\|_{L^2(A)}\\
&\le K\Big(\|u-\widetilde{u}\|_{L^2(L^2)}+\|u_t-\widetilde{u}_t\|_{L^2(L^2)}+\|(\widetilde{u}-u_h)(0)\|\Big)\\
&\le Kh^{k+1}\Big(\|u\|_{L^2(H^{k+1})}+\|u_t\|_{L^2(H^{k+1})}+\|\boldsymbol{\sigma}\|_{L^2(\boldsymbol{H}^{k+1})}+\|\boldsymbol{\sigma}_t\|_{L^2(\boldsymbol{H}^{k+1})}\Big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\|u-u_h\|_{L^\infty(L^2)}+h\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^\infty(L^2)}\\
&\le Kh^{k+1}\Big(\|u\|_{L^2(H^{k+1})}+\|u_t\|_{L^2(H^{k+1})}+\|\boldsymbol{\sigma}\|_{L^2(\boldsymbol{H}^{k+1})}+\|\boldsymbol{\sigma}_t\|_{L^2(\boldsymbol{H}^{k+1})}\Big).
\end{aligned}$$

This completes the proof. □

Theorem 4.2. *If $(u,\boldsymbol{\sigma})\in(V\cap H^{k+1}(T_h))\times(\boldsymbol{W}\cap\boldsymbol{H}^{k+1}(T_h))$ is the solution of (2.2) and $(u_h,\boldsymbol{\sigma}_h)\in V_h\times\boldsymbol{W}_h$ is the solution of (2.3)-(2.4), then*

$$\begin{aligned}
&\|u_t-(u_h)_t\|_{L^2(L^2)}\\
&\le Kh^{k+1}\Big(\|u\|_{L^2(H^{k+1})}+\|u_t\|_{L^2(H^{k+1})}+\|\boldsymbol{\sigma}\|_{L^2(\boldsymbol{H}^{k+1})}+\|\boldsymbol{\sigma}_t\|_{L^2(\boldsymbol{H}^{k+1})}\Big).
\end{aligned}$$

Proof. Differentiating (4.2) with respect to t, we obtain

$$A(\widetilde{\boldsymbol{\sigma}}_t-(\boldsymbol{\sigma}_h)_t,\boldsymbol{\tau}_h)-B(\boldsymbol{\tau}_h,\widetilde{u}_t-(u_h)_t)=0,\ \ \forall\boldsymbol{\tau}_h\in\boldsymbol{W}_h. \tag{4.4}$$

Letting $v_h = \widetilde{u}_t - (u_h)_t,\ \boldsymbol{\tau}_h = \widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h$ in (4.1) and (4.4) and adding the resulting equations, we get

$$\begin{aligned}&\|\widetilde{u}_t - (u_h)_t\|^2 + C(\widetilde{u} - u_h, \widetilde{u}_t - (u_h)_t) + A(\widetilde{\boldsymbol{\sigma}}_t - (\boldsymbol{\sigma}_h)_t, \widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h)\\ &= (\widetilde{u}_t - u_t, \widetilde{u}_t - (u_h)_t) + \lambda(u - u_h, \widetilde{u}_t - (u_h)_t).\end{aligned}$$

Since

$$\begin{aligned}C(\widetilde{u} - u_h, \widetilde{u}_t - (u_h)_t) &= J(\widetilde{u} - u_h, \widetilde{u}_t - (u_h)_t) + \lambda(\widetilde{u} - u_h, \widetilde{u}_t - (u_h)_t)\\ &= \frac{1}{2}\frac{d}{dt}\sum_{e\in\mathcal{E}_h^{ID}} h_e^{-1}\|[\widetilde{u} - u_h]\|_e^2 + \frac{\lambda}{2}\frac{d}{dt}\|\widetilde{u} - u_h\|^2\\ &= \frac{1}{2}\frac{d}{dt}\|\!|\widetilde{u} - u_h|\!\|_C^2\end{aligned}$$

and

$$A(\widetilde{\boldsymbol{\sigma}}_t - (\boldsymbol{\sigma}_h)_t, \widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h) = \frac{1}{2}\frac{d}{dt}\|\alpha^{\frac{1}{2}}(x)(\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h)\|^2,$$

we obtain

$$\begin{aligned}&\|\widetilde{u}_t - (u_h)_t\|^2 + \frac{1}{2}\frac{d}{dt}\left(\|\!|\widetilde{u} - u_h|\!\|_C^2 + \|\alpha^{\frac{1}{2}}(x)(\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h)\|^2\right)\\ &\le \|\widetilde{u}_t - u_t\|\|\widetilde{u}_t - (u_h)_t\| + \lambda\|u - u_h\|\|\widetilde{u}_t - (u_h)_t\|\end{aligned}$$

and so

$$\begin{aligned}&\|\widetilde{u}_t - (u_h)_t\|^2 + \frac{d}{dt}\left(\|\!|\widetilde{u} - u_h|\!\|_C^2 + \|\alpha^{\frac{1}{2}}(x)(\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h)\|^2\right)\\ &\le K(\|\widetilde{u}_t - u_t\|^2 + \|u - u_h\|^2).\end{aligned}$$

Now we integrate both sides of the above inequality with respect to t from 0 to $t \le T$ to get

$$\begin{aligned}&\|\widetilde{u}_t - (u_h)_t\|_{L^2(L^2)}^2 + \sup_{[0,T]}\|\!|\widetilde{u} - u_h|\!\|_C^2 + \|\alpha^{\frac{1}{2}}(x)(\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h)\|_{L^\infty(\boldsymbol{L}^2)}^2\\ &\le K(\|\widetilde{u}_t - u_t\|_{L^2(L^2)}^2 + \|u - u_h\|_{L^2(L^2)}^2)\end{aligned}$$

and so, by Theorem 4.1 and Lemma 3.4, we get

$$\begin{aligned}&\|\widetilde{u}_t - (u_h)_t\|_{L^2(L^2)}^2 + \|\widetilde{u} - u_h\|_{L^\infty(L^2)}^2 + \|(\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h)\|_{L^\infty(\boldsymbol{L}^2)}^2\\ &\le K(\|\widetilde{u}_t - u_t\|_{L^2(L^2)}^2 + \|u - u_h\|_{L^2(L^2)}^2)\\ &\le Kh^{2(k+1)}\left(\|\boldsymbol{\sigma}\|_{L^2(\boldsymbol{H}^{k+1})}^2 + \|\boldsymbol{\sigma}_t\|_{L^2(\boldsymbol{H}^{k+1})}^2 + \|u\|_{L^2(H^{k+1})}^2 + \|u_t\|_{L^2(H^{k+1})}^2\right).\end{aligned}$$

Therefore, using the triangular inequality and Lemma 3.4, we have

$$\begin{aligned}&\|u_t - (u_h)_t\|_{L^2(L^2)}\\ &\le Kh^{k+1}\left(\|\boldsymbol{\sigma}\|_{L^2(\boldsymbol{H}^{k+1})} + \|\boldsymbol{\sigma}_t\|_{L^2(\boldsymbol{H}^{k+1})} + \|u\|_{L^2(H^{k+1})} + \|u_t\|_{L^2(H^{k+1})}\right).\end{aligned}$$

This completes the proof. □

References

[1] D. N. Arnold, An interior penalty finite element method with discontinuous elements, SIAM J. Numer. Anal. 19 (1982), 724 - 760.

[2] D. N. Arnold, F. Breezi, B. Cockburn, L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems, SIAM J. Numer. Anal. 39 (2001), 1749 - 1779.

[3] Z. Chen, On the relationship of various discontinuous finite element methods for second-order elliptic equations, East-West J. Numer. Math. 9 (2001), 99 - 122.

[4] H. Chen, Z. Chen, Stability and convergence of mixed discontinuous finite element methods for second-order diffferential problems, J. Numer. Math. 11 (2003), 253 - 287.

[5] Z. Chen, J. Douglas, Approximation of coefficients in hybrid and mixed methods for nonlinear parabolic problems, Math. Applic. Comp. 10 (1991), 137 - 160.

[6] J. Douglas, T. Dupont, Interior penalty procedures for elliptic and parabolic Galerkin methods, Lect. Notes. Phys. 58 (1976), 207 - 216.

[7] J. Douglas, J. E. Roberts, Global estimates for mixed methods for second order elliptic problems, Math. Comp. 44 (1985), 39 - 52.

[8] I. Guo, H. Z. Chen, H^1-Galerkin mixed finite element methods for the Sobolev equation, Systems Sci. Math. Sci. 26 (2006), 301 - 314.

[9] C. Johnson, V. Thom*é*e, Error estimates for some mixed finite element methods for parabolic type problems, RAIRO Anal. Numer. 14 (1981), 41 - 78.

[10] J. -C. Nedelec, Mixed finite elements in $\mathbb{R}^3$, Numer. Math. 35 (1980), 315 - 341.

[11] J. Nitsche, Über ein Variationspringzip zvr Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. Abh. Math. Sem. Univ. Hamburg, 36 (1971), 9 - 15.

[12] M. R. Ohm, H. Y. Lee, J. Y. Shin, Error estimates for discontinuous Galerkin method for nonlinear parabolic equations, J. Math. Anal. Appl. 315 (2006), 132 - 143.

[13] M. R. Ohm, H. Y. Lee, J. Y. Shin, Error estimates for fully discrete discontinuous Galerkin method for nonlinear parabolic equations, J. Appl. Math. & Informatics 28 (2010), 953 - 966.

[14] M. R. Ohm, H. Y. Lee, J. Y. Shin, Higher order discontinuous Galerkin finite element methods for nonlinear parabolic equations, J. Korean Soc. Ind. Appl. Math. 18 (2014), 337 - 350.

[15] A. K. Pani, An H^1-Galerkin mixed finite element methods for parabolic partial differential equations, SIAM J. Numer. Anal. 25 (1998), 712 - 727.

[16] A. K. Pani, G. Fairweather, H^1-Galerkin mixed finite element methods for parabolic integro-differential equations, IMA J. Numer. Anal. 22 (2002), 231-252.

[17] R. Raviart, J. Thomas, A mixed finite element method for second order elliptic problems, Lecture Notes in Mathematics 606 (1977), 292 - 315.

[18] B. Rivire, M. F. Wheeler, A discontinuous Galerkin method applied to nonlinear parabolic equations, Discontinuous Galerkin methods:theory, computation and applications [Eds by B. Cockburn, G. E. Karniadakis and C. -W. Shu], Lect. Notes Comput. Sci. Eng. 11 (2000), 231 - 244.

[19] M. F. Wheeler, An elliptic collocation-finite element method with interior penalties, SIAM J. Numer. Anal. 15 (1978), 152 - 161.

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