STABILIZATION FOR THE VISCOELASTIC KIRCHHOFF TYPE EQUATION WITH A NONLINEAR SOURCE

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ABSTRACT. In this paper, we study the viscoelastic Kirchhoff type equation with a nonlinear source

$$u'' - M(x, t, \|\nabla u(t)\|^2)\triangle u + \int_0^t h(t - \tau)div[a(x)\nabla u(\tau)]d\tau + |u|^\gamma u = 0.$$

Under the smallness condition with respect to Kirchhoff coefficient and the relaxation function and other assumptions, we prove the uniform decay rate of the Kirchhoff type energy.

1. Introduction

In the present work, we are concerned with the following problem:

$$u''(x, t) - M(x, t, \|\nabla u(t)\|^2)\Delta u(x, t) \tag{1}$$
$$+ \int_0^t h(t - \tau)div[a(x)\nabla u(\tau)]d\tau + |u|^\gamma u = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+,$$
$$u(x, t) = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+, \tag{2}$$
$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in} \quad \Omega, \tag{3}$$

where Ω be a bounded open set of $\mathbb{R}^N (N \geq 1)$ with a smooth boundary $\Gamma, \gamma > 0$, and other conditions such as M, h, a be in next section. In fact, u_0, u_1 are initially given functions and $u(x, t)$ is the transversal displacement of the strip at spatial coordinate x and time t in the real world application.

This problem has its origin in the mathematical description of system in real world from the mathematical modeling for axially moving viscoelastic materials. It is well known that viscoelastic materials exhibit natural damping, which is due to the special property of these materials to retain a memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators. For these reasons, there are not exist weak or strong damping term in our problem (1)-(3). Recently, problems with Timoshenko or basic hyperbolic type for viscoelastic materials have been

considered by many authors (See [1, 2]). Besides, many engineering devices involve the transverse vibration of axially moving strings. Axially moving string is a typical model that is widely used, especially when the subject is long and narrow enough and has a negligible flexural rigidity, to represent threads, wires, magnetic tapes, belts, band saws, and cables. Various mathematical models and simulations have been established for a better understanding with linear or nonlinear dynamic behavior of these moving continua [3, 4, 5, 6, 7, 8, 9]. The mathematical model for axially moving strings was first introduced by Kirchhoff [10] (and see Carrier [3]), and the original equation is given in the form of

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left(p_0 \; + \; \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right) \frac{\partial^2 u}{\partial x^2}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x,t)$ is the lateral displacement at the space coordinate x and time t; E, the young's modulus; ρ, the mass density; h, the cross section area; L, the length; and p_0, the initial axial tension. Recently, problems with the extended Kirchhoff type equation which is concerning axially moving heterogeneous or non heterogeneous materials (nonlinear vibrations of beams, strings, plates, and membranes) have been considered by many authors (See [11, 12, 13]).

In this paper, we will mainly concern on an aspect of decay rate of the Kirchhoff type energy of the system. We get its proof by using the smallness condition functions with respect to Kirchhoff coefficient and the relaxation function. In fact, the difference of the energy consist in Kirchhoff type potential energy.

This paper organized as follows. In Section 2, we will present some notations, material needed (assumptions, lemmas and so on) for our work and state a global existence and energy decay rate theorem (main result). Section 3 contains the proof of our main result.

2. Preliminaries and main results

We first introduce the elementary bracket pairing in $\Omega \subset \mathbb{R}^N$

$$\langle \varphi, \psi \rangle \equiv \int_\Omega (\varphi, \psi) dx,$$

provided that $(\varphi, \psi) \in L^1(\Omega)$. And we set the norms as follows.

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

To simplify the notations, we denote $\|u\|_{L^2(\Omega)}, \|u\|_{L^1(0,+\infty)}, \; \|v\|_{L^\infty(0,+\infty)}$ by $\|u\|, \|v\|_{L^1}, \|v\|_{L^\infty}$ respectively.

In the sequel we state the general hypotheses.

(A$_1$) $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function satisfying $h(0) > 0$, and there exists positive constant $t_0, \zeta_1, \zeta_2, \zeta_3$ such that

$$-\zeta_1 \leq h'(t) \leq -\zeta_2 h(t), \quad \forall t > t_0,$$

$$0 \leq h''(t) \leq \zeta_3 h(t), \quad \forall t > t_0.$$

(A_2) $a : \Omega \to \mathbb{R}^+$ is a nonnegative bounded function and $a(x) \geq a_0 > 0$ on Ω with

$$\frac{m_0}{a_0} \geq 1 - \|a\|_\infty \int_0^\infty h(s)ds = l > 0,$$

where m_0 is in (B_2). And also, the following smallness condition satisfy

$$\epsilon_7 < a_0^2 \int_0^t h(s)ds.$$

(A_3) γ satisfies

$$0 \leq \gamma \leq \frac{2}{n-2}, \quad n \geq 3,$$

$$\gamma \geq 0, \quad n = 1, 2.$$

(A_4) The initial data satisfy

$$u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \ u_1 \in H_0^1(\Omega).$$

(B_1) $M(x, t, \lambda)$ is a real-valued function of class C^2 on $x \in \overline{\Omega}$, $t \geq 0$, $\lambda \leq 0$.

(B_2) $0 < m_0 \leq M(x, t, \lambda) \leq C_0 f(\lambda)$ with $M(x, t, \lambda) = M_1(x, t) + M_2(x, t, \lambda)$. And also, the following smallness condition satisfy

$$f(\lambda) < \sqrt{\frac{\frac{a_0 h(t)}{2} - C_p \widetilde{C_1} + \epsilon_2 \left(m_0 - \frac{1}{2}\right)}{\epsilon_3 \epsilon_8}}.$$

(B_3) $\frac{\partial M_1}{\partial t} \leq 0$, $\left|\frac{\partial M_2}{\partial t}\right| \leq C_1 g_1(\lambda)$, $\left|\frac{\partial M}{\partial \lambda}\right| \leq C_2 g_2(\lambda)$, $0 < m_1 \leq M_x(x, t, \lambda)$.

(B_4) $f, g_1, g_2 \in C^1([0, +\infty); \mathbb{R}_+)$ are strictly increasing. Furthermore, C_i ($i = 0, 1, 2$) is a positive constant.

In the following, we give a lemma which will be useful in this paper.

Lemma 2.1. *Denote* $(h \diamond u)(t) = \int_0^t h(t - \tau)\|\sqrt{a(x)}(u(t) - u(\tau))\|^2 d\tau$. *Then we have*

$$
\begin{aligned}
\int_0^t h(t - \tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau = &-\frac{1}{2}\frac{d}{dt}\left[(h \diamond u)(t)\right] + \frac{1}{2}(h' \diamond u)(t) \\
&+ \frac{1}{2}\frac{d}{dt}\left[\|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s)ds\right] \\
&- \frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2.
\end{aligned}
$$

(4)

Proof. A direct computation shows that

$$\int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau = \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau) - a(x)\nabla u(t), \nabla u'(t)\rangle d\tau$$

$$+ \int_0^t h(t-\tau)\langle a(x)\nabla u(t), \nabla u'(t)\rangle d\tau$$

$$= -\frac{1}{2}\int_0^t h(t-\tau)\left[\frac{d}{dt}\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2\right]d\tau$$

$$+ \frac{1}{2}\int_0^t h(t-\tau)\left[\frac{d}{dt}\|\sqrt{a(x)}\nabla u(t)\|^2\right]d\tau$$

$$= -\frac{1}{2}\frac{d}{dt}\left[\int_0^t h(t-\tau)\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2 d\tau\right]$$

$$+ \frac{1}{2}\int_0^t h'(t-\tau)\|\sqrt{a(x)}(\nabla u(\tau) - \nabla u(t))\|^2 d\tau$$

$$+ \frac{1}{2}\frac{d}{dt}\int_0^t h(t-\tau)\|\sqrt{a(x)}\nabla u(t)\|^2 d\tau$$

$$- \frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2.$$

$\square$

Then, we can state our result as follows.

Theorem 2.2. *Let the assumptions* $(A_1), (A_4)$ *and* (B_1)-(B_4) *hold. Then there exists a unique solution* u *of the problem* (1)-(3) *satisfying*

$$u \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)), \ u' \in L^\infty(0, T; H_0^1(\Omega)), \ u'' \in L^\infty(0, T; L^2(\Omega)),$$

and

$$u(x,t) \to u_0(x) \ in \ H_0^1(\Omega) \cap H^2(\Omega); \qquad u'(x,t) \to u_1(x) \ in \ H_0^1(\Omega),$$

as $t \to 0$.

Proof. By using Galerkin's approximation and a routine procedure similar to that of cite [12, 1], we can the global existence result for the solution subject to (1)-(3) under the assumptions (A_1)-(A_4) and (B_1)-(B_4). $\square$

Theorem 2.3. *Let* u *be the global solution of the problem* (1)-(3) *with the above all conditions. We define the Kirchhoff type energy functional* $E(t)$ *as*

$$E(t) = \frac{1}{2}\left[\|u'(t)\|^2 + \int_\Omega M(x, t, \|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx + \frac{2}{\gamma+2}\|u'(t)\|_{\gamma+2}^{\gamma+2}\right].$$

Then the energy functional decays exponentially to zero as the time goes to infinity, that is,

$$E(t) \le \kappa e^{-\vartheta t}, \ \forall t \ge 0$$

where κ, ϑ *are positive constants.*

3. Proof of Theorem 2.3 (Energy decay)

Proof. Multiplying u' on both sides of Eq.(1), integrating the resulting equations over Ω, and using the Green formula and (2), we have

$$
\begin{aligned}
&\langle u''(t), u'(t)\rangle + \langle M(x, t, \|\nabla u(t)\|^2)\nabla u(t), \nabla u'(t)\rangle \\
&\quad + \langle M_x(x, t, \|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
&\quad - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau + \langle |u|^\gamma u, u'\rangle = 0,
\end{aligned}
\tag{5}
$$

that is

$$
\begin{aligned}
\frac{d}{dt}E(t) =\ & \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}M_1(x, t)|\nabla u(x, t)|^2 dx \\
& + \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}M_2(x, t, \|\nabla u(t)\|^2)|\nabla u(x, t)|^2 dx \\
& + \left[\int_\Omega \frac{\partial}{\partial \lambda}M_2(x, t, \|\nabla u(t)\|^2)|\nabla u(x, t)|^2 dx\right]\langle\nabla u'(t), \nabla u(t)\rangle \\
& - \langle M_x(x, t, \|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
& - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau,
\end{aligned}
\tag{6}
$$

where

$$
E(t) = \frac{1}{2}\left[\|u'(t)\|^2 + \int_\Omega M(x, t, \|\nabla u(t)\|^2)|\nabla u(x, t)|^2 dx + \frac{1}{\gamma + 2}\|u'(t)\|_{\gamma+2}^{\gamma+2}\right].
\tag{7}
$$

From (B_3) and Hölder inequality, we obtain

$$
\begin{aligned}
E'(t) \leq\ & \|u(t)\|^2 \left\{\frac{C_1}{2}g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2)\|\nabla u'(t)\|\|u(t)\|\right\} \\
& - \langle M_x(x, t, \|\nabla u(t)\|^2)\nabla u(t), u'(t)\rangle \\
& - \int_0^t h(t-\tau)\langle a(x)\nabla u(\tau), \nabla u'(t)\rangle d\tau.
\end{aligned}
\tag{8}
$$

By (B_3), (4) and Young's inequality, we have

$$
\begin{aligned}
E'(t) \leq\ & \|u(t)\|^2 \widetilde{C_1} + \epsilon_1 m_1 \|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 \\
& - \frac{1}{2}\frac{d}{dt}[(h \diamond u)(t)] + \frac{1}{2}(h' \diamond \nabla u)(t) \\
& + \frac{1}{2}\frac{d}{dt}\left[\|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s)ds\right] \\
& - \frac{1}{2}h(t)\|\sqrt{a(x)}\nabla u(t)\|^2,
\end{aligned}
\tag{9}
$$

where

$$(10) \qquad \widetilde{C_1} = \frac{C_1}{2} g_1(\|\nabla u(t)\|^2) + C_2 g_2(\|\nabla u(t)\|^2)\|\nabla u'(t)\|\|u(t)\|$$

is a positive constant. And ϵ_1 is also a positive constant.
Define the new energy functional $E_1(t)$ as follows

$$E_1(t) = E(t) + \frac{1}{2}(h \diamond \nabla u)(t) - \frac{1}{2}\|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s)ds. \qquad (11)$$

Then from (A_1) and (9), we have

$$(12) \qquad \begin{aligned} E_1'(t) \leq & \|u(t)\|^2\widetilde{C_1} + \epsilon_1 m_1\|\nabla u(t)\|^2 + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 \\ & - \frac{\zeta_2}{2}(h \diamond \nabla u)(t) - \frac{1}{2}a_0 h(t)\|\nabla u(t)\|^2. \end{aligned}$$

and also, by (A_2), the energy $E_1(t)$ is a positive functional. Applying Poincarè inequality to (12), we deduce

$$(13) \qquad \begin{aligned} E'(t) \leq & \left(C_p\widetilde{C_1} + \epsilon_1 m_1 - \frac{1}{2}a_0 h(t) \right) \|\nabla u(t)\|^2 \\ & + \frac{m_1}{4\epsilon_1}\|u'(t)\|^2 - \frac{\zeta_2}{2}(h \diamond \nabla u)(t), \end{aligned}$$

where C_p is the Poincarè coefficient. Meanwhile, we note from (A_1) and (A_2) that

$$(14)$$
$$\begin{aligned} E_1(t) \geq & \frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx \\ & + \frac{1}{2}\left(1 - \|a\|_\infty \int_0^t h(s)ds \right)\|\nabla u(t)\|^2 + \frac{1}{2}(h \diamond u)(t) + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2} \\ \geq & l\left[\frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2 dx + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2} \right]. \end{aligned}$$

So, we deduce the relation $0 \leq E(t) \leq l^{-1}E_1(t)$. Therefore, the uniform decay of $E(t)$ is a result of the decay of $E_1(t)$. For positive constants ϵ_2 and ϵ_3, let us define the perturbed modified energy by

$$F(t) = E_1(t) + \epsilon_2\varphi(t) + \epsilon_3\psi(t), \qquad (15)$$

where

$$\varphi(t) = \langle u'(t), u(t) \rangle. \qquad (16)$$

and

$$\psi(t) = -\int_0^t h(t-\tau)\langle a(x)u'(t), u(t) - u(\tau)\rangle d\tau. \qquad (17)$$

By using the Cauchy's inequality, Hölder inequality and Poincarè inequality, there exist positive constants α_1, α_2 such that for each $t > 0$

$$\alpha_1 F(t) \leq E_1(t) \leq \alpha_2 F(t). \qquad (18)$$

Proposition 3.1. (Energy equivalence)

$$\alpha_1 F(t) \leq E_1(t) \leq \alpha_2 F(t) \quad \text{for all } t \geq 0,$$

where

$$\alpha_1 = \frac{1}{\max\{1, \epsilon_4 + \epsilon_5, (\epsilon_4 + \epsilon_5)C_p\}} > 0$$

and

$$\alpha_2 = \frac{1}{\min\{1 - (\epsilon_4 + \epsilon_5), m_0 - \epsilon_4 C_p, 1 - \epsilon_5 C_p(1 - l)\}} > 0.$$

Proof. By the Cauchy inequality and Hölder inequality, we get

$$
\begin{aligned}
F(t) \leq & E_1(t) + \frac{\epsilon_4}{2}\|u'(t)\|^2 + \frac{\epsilon_4}{2}\|u(t)\|^2 \\
& + \frac{\epsilon_5}{2}\left\|\int_0^t h(t - \tau)a(x)(u(t) - u(\tau))d\tau\right\| + \frac{\epsilon_5}{2}\|u'(t)\|,
\end{aligned}
$$

(19)

where ϵ_4, ϵ_5 are positive constants.

By using the Poincarè inequality, we have

$$
\begin{aligned}
F(t) \leq & E_1(t) + \frac{\epsilon_4 + \epsilon_5}{2}\|u'(t)\|^2 + \frac{\epsilon_4}{2}\|u(t)\|^2 \\
& + \frac{\epsilon_5}{2}\|a\|_\infty \int_0^\infty h(s)ds \int_0^t h(t - \tau)\left\|\sqrt{a(x)}(u(t) - u(\tau))\right\|^2 d\tau \\
\leq & E_1(t) + \frac{\epsilon_4 + \epsilon_5}{2}\|u'(t)\|^2 + \frac{\epsilon_4}{2}\|u(t)\|^2 \\
& + \frac{\epsilon_5}{2}C_p(1 - l)\int_0^\infty h(s)ds \int_0^t h(t - \tau)\left\|\sqrt{a(x)}(u(t) - u(\tau))\right\|^2 d\tau \\
\leq & E_1(t) + \frac{\epsilon_4 + \epsilon_5}{2}\|u'(t)\|^2 + \frac{\epsilon_4}{2}C_p\|\nabla u(t)\|^2 \\
& + \frac{\epsilon_5}{2}C_p(1 - l)(h \diamond \nabla u)(t) \\
\leq & \max\{1, \epsilon_4 + \epsilon_5, (\epsilon_4 + \epsilon_5)C_p\}E_1(t),
\end{aligned}
$$

where C_p is the Poincarè coefficient. Besides, choosing ϵ_4, ϵ_5 small enough, we have

$$
\begin{aligned}
F(t) \geq & E_1(t) - \frac{\epsilon_4 + \epsilon_5}{2}\|u'(t)\|^2 - \frac{\epsilon_4}{2}C_p\|\nabla u(t)\|^2 \\
& - \frac{\epsilon_5}{2}C_p(1 - l)(h \diamond \nabla u)(t) \\
\geq & \frac{1 - (\epsilon_4 + \epsilon_5)}{2}\|u'(t)\|^2 + \frac{m_0 - \epsilon_4 C_p}{2}\|\nabla u(t)\|^2 \\
& - \frac{1}{2}\|\sqrt{a(x)}\nabla u(t)\|^2 \int_0^t h(s)ds + \frac{1}{\gamma + 2}\|u(t)\|_{\gamma+2}^{\gamma+2} \\
& + \frac{1 - \epsilon_5 C_p(1 - l)}{2}(h \diamond \nabla u)(t) \\
\geq & \min\{1 - (\epsilon_4 + \epsilon_5), m_0 - \epsilon_4 C_p, 1 - \epsilon_5 C_p(1 - l)\}E_1(t).
\end{aligned}
$$

$\square$

In fact, using (1), we have

$$\varphi'(t) = \langle u''(t), u(t) \rangle + \|u'(t)\|^2.$$

$$= \|u'(t)\|^2 + \left\langle u(t), M(x,t,\|\nabla u(t)\|^2)\Delta u(x,t) \right.$$

(20)
$$\left. - \int_0^t h(t-\tau)div[a(x)\nabla u(\tau)]d\tau - |u(t)|^\gamma u(t) \right\rangle$$

$$= \|u'(t)\|^2 - \int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(t)|^2 dx$$

$$+ \int_0^t h(t-\tau) \langle a(x)\nabla u(\tau), \nabla u(t) \rangle]d\tau - |u(t)|^\gamma u(t).$$

By Cauchy inequality and Young's inequality, we have

$$\left| \int_0^t h(t-\tau) \langle a(x)\nabla u(\tau), \nabla u(t) \rangle]d\tau \right|$$

$$\leq \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2} \left\| \int_0^t h(t-\tau)(a(x)|\nabla u(\tau) - \nabla u(t)| + a(x)|\nabla u(t)|)d\tau \right\|^2$$

(21)
$$\leq \frac{1}{2}\|\nabla u(t)\|^2 + \left(\frac{1}{2} + \frac{1}{8\epsilon_6} \right) \left\| \int_0^t h(t-\tau)a(x)|\nabla u(\tau) - \nabla u(t)|d\tau \right\|^2$$

$$+ \left(\frac{1}{2} + \frac{\epsilon_6}{2} \right) \left\| \int_0^t h(t-\tau)a(x)|\nabla u(t)|d\tau \right\|^2,$$

where ϵ_6 with respect to Young's inequality is a positive constant. Using the assumption (A$_2$) and (21), we get

$$\left| \int_0^t h(t-\tau) \langle a(x)\nabla u(\tau), \nabla u(t) \rangle]d\tau \right|$$

$$\leq \left(\frac{1}{2} + \frac{1}{8\epsilon_6} \right) \|a\|_\infty \int_0^t h(s)ds \int_0^t h(t-\tau) \left\| \sqrt{a(x)}(\nabla u(\tau) - \nabla u(t)) \right\|^2 d\tau$$

(22)
$$+ \left(\frac{1}{2} + \frac{\epsilon_6}{2} \right) \|\nabla u(t)\|^2 \left(\|a\|_\infty \int_0^t h(s)a(x)ds \right)^2 + \frac{1}{2}\|\nabla u(t)\|^2$$

$$\leq \frac{1}{2}(1 + (1 + \epsilon_6)(1-l)^2)\|\nabla u(t)\|^2 + \frac{(4\epsilon_6 + 1)(1-l)}{8\epsilon_6}(h \diamond \nabla u)(t).$$

By combining (20) and (22), we conclude

(23)
$$\varphi'(t) \leq \|u'(t)\|^2 + \frac{1}{2}(1 - 2m_0 + (1 + \epsilon_6)(1-l)^2)\|\nabla u(t)\|^2$$

$$+ \frac{(4\epsilon_6 + 1)(1-l)}{8\epsilon_6}(h \diamond \nabla u)(t) - \|u(t)\|_{\gamma+2}^{\gamma+2}.$$

Next, we estimate $\psi'(t)$ as follows. In fact, using (1), we have

(24)

$$\psi'(t) = -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t) - u(\tau)\rangle d\tau.$$
$$-\int_0^t h(t-\tau)\langle a(x)u''(t), u(t) - u(\tau)\rangle d\tau - \|\sqrt{a(x)}u'(t)\|^2 \int_0^t h(s)ds$$
$$= -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t) - u(\tau)\rangle d\tau.$$
$$-\int_0^t h(t-\tau)\langle M(x,t,\|\nabla u(t)\|^2)a(x)\nabla u(t), \nabla u(t) - \nabla u(\tau)\rangle d\tau$$
$$-\left\langle \int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau, \int_0^t h(t-\tau)a(x)(\nabla u(t) - \nabla u(\tau))d\tau \right\rangle$$
$$+\int_0^t h(t-\tau)\langle a(x)|u|^\gamma u, u(t) - u(\tau)\rangle d\tau$$
$$-\|\sqrt{a(x)}u'(t)\|^2 \int_0^t h(s)ds.$$

Using Cauchy inequality, Poincarè inequality and (A_1), we have

(25)

$$\left| -\int_0^t h'(t-\tau)\langle a(x)u'(t), u(t) - u(\tau)\rangle d\tau \right|$$
$$\leq \epsilon_7 \|\nabla u(t)\|^2 + \frac{\zeta_1}{4\epsilon_7} \left\| \int_0^t h(t-\tau)a(x)|u(t) - u(\tau)|d\tau \right\|^2$$
$$\leq \epsilon_7 \|\nabla u(t)\|^2 + \frac{\zeta_1}{4\epsilon_7}(1-l)C_p^2(h \diamond \nabla u)(t),$$

where ϵ_7 is a positive constant with respect to Cauchy inequality and C_p is the Poincarè coefficient. Similarly, using Cauchy inequality and (B_2), we get

(26)

$$\left| -\int_0^t h(t-\tau)\langle M(x,t,\|\nabla u(t)\|^2)a(x)\nabla u(t), \nabla u(t) - \nabla u(\tau)\rangle d\tau \right|$$
$$\leq \epsilon_8 f^2(\|\nabla u(t)\|^2)\|u'(t)\|^2 + \frac{C_0(1-l)}{4\epsilon_8}(h \diamond \nabla u)(t)$$

and

(27)
$$\left| -\left\langle \int_0^t h(t-\tau)a(x)\nabla u(\tau)d\tau, \int_0^t h(t-\tau)a(x)(\nabla u(t)-\nabla u(\tau))d\tau \right\rangle \right|$$

$$\leq \epsilon_9 \left\| \int_0^t h(t-\tau)(a(x)|\nabla u(t)-\nabla u(\tau)| + a(x)|\nabla u(t)|)d\tau \right\|^2$$

$$+ \frac{1}{4\epsilon_9}\left(\|a\|_\infty \int_0^t h(s)ds \right)\int_0^t h(t-\tau)\|\sqrt{a(x)}(\nabla u(t)-\nabla u(\tau))\|^2 d\tau$$

$$\leq 2\epsilon_9 \left(\left\| \int_0^t h(t-\tau)a(x)|\nabla u(t)-\nabla u(\tau)|d\tau \right\|^2 + \left\| \int_0^t h(t-\tau)a(x)|\nabla u(t)|d\tau \right\|^2 \right)$$

$$+ \frac{1-l}{4\epsilon_9}(h\diamond \nabla u)(t)$$

$$\leq \left(2\epsilon_9 + \frac{1}{4\epsilon_9} \right)(1-l)(h\diamond \nabla u)(t) + 2\epsilon_9(1-l)^2\|\nabla u(t)\|^2,$$

where ϵ_8, ϵ_9 are positive constants with respect to Cauchy inequality. And also, using Cauchy inequality and Poincarè inequality, we have

(28)
$$\left| \int_0^t h(t-\tau)\langle a(x)|u(t)|^\gamma u, u(t)-u(\tau)\rangle d\tau \right|$$

$$\leq \epsilon_{10}\|u(t)\|_{2(\gamma+1)}^{2(\gamma+1)} + \frac{C_p(1-l)}{4\epsilon_{10}}(h\diamond \nabla u)(t),$$

where ϵ_{10} is a positive constant with respect to Cauchy inequality and C_p is the Poincarè coefficient. Noting $H^1(\Omega) \hookrightarrow L^{2(\gamma+1)}(\Omega)$ and using Poincarè inequality, (11), (12) and (28), we get

(29)
$$\left| \int_0^t h(t-\tau)\langle a(x)|u(t)|^\gamma u, u(t)-u(\tau)\rangle d\tau \right|$$

$$\leq \epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l} \right)^\gamma \|\nabla u(t)\|^2 + \frac{C_p(1-l)}{4\epsilon_{10}}(h\diamond \nabla u)(t),$$

where C_p is the Poincarè coefficient. Combining (22)-(27) and (29) and also using (A_2), we deduce

(30)
$$\psi'(t) \leq \left(\epsilon_7 - a_0^2 \int_0^t h(s)ds \right)\|u'(t)\|^2$$

$$+ \left(\epsilon_8 f^2(\|\nabla u(t)\|^2) + 2\epsilon_9(1-l)^2 + \epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l} \right)^\gamma \right)\|\nabla u(t)\|^2$$

$$+ \left(\frac{\zeta_1}{4\epsilon_7}C_p^2 + \frac{C_0}{4\epsilon_8} + 2\epsilon_9 + \frac{1}{4\epsilon_9} + \frac{C_p}{4\epsilon_{10}} \right)(1-l)(h\diamond \nabla u)(t).$$

Combining (13), (15), (23) and (30), we deduce

(31)
$$F(t) = E_1(t) + \epsilon_2\varphi(t) + \epsilon_3\psi(t)$$

$$\leq w_1\|u'(t)\|^2 + w_2\int_\Omega M(x,t,\|\nabla u(t)\|^2)|\nabla u(x,t)|^2dx + w_3(h \diamond \nabla u(t)) - \|u(t)\|_{\gamma+2}^{\gamma+2},$$

where

$$w_1 = \frac{m_1}{4\epsilon_1} + \epsilon_2 + \epsilon_3\left(\epsilon_7 - a_0^2\int_0^t h(s)ds\right),$$

$$w_2 = f(\|\nabla u(t)\|^2)C_0\left[C_p\widetilde{C_1} + \epsilon_1 m_1 - \frac{1}{2}a_0 h(t)\right]$$

$$+ \frac{\epsilon_2 f(\|\nabla u(t)\|^2)C_0}{2}(1 - 2m_0 + (1 + \epsilon_6)(1 - l)^2)$$

$$+ \epsilon_3 f(\|\nabla u(t)\|^2)C_0\left(\epsilon_8 f^2(\|\nabla u(t)\|^2) + 2\epsilon_9(1 - l)^2 + \epsilon_{10}C_p^{2(\gamma+1)}\left(\frac{2E_1(0)}{l}\right)^\gamma\right),$$

$$w_3 = -\frac{\zeta_2}{2} + \left[\frac{\epsilon_2(4\epsilon_6 + 1)}{8\epsilon_6} + \epsilon_3\left(\frac{\zeta_1}{4\epsilon_7}C_p^2 + \frac{C_0}{4\epsilon_8} + 2\epsilon_9 + \frac{1}{4\epsilon_9} + \frac{C_p}{4\epsilon_{10}}\right)\right](1 - l),$$

By using the smallness condition in (A_2) and (B_2), for the fixed $\epsilon_i, i = 1, 4, \cdots, 10$, we choose $\epsilon_j > 0, j = 2, 3$ such that $w_k < 0, k = 1, 2, 3$. According to (11) and (31), there exist a positive constant s such that

$$F(t) \leq -sE_1(t) \tag{32}$$

for all t which is larger than the fixed time T_0. We conclude from (18) and (32) that

$$F(t) \leq -s\alpha_1 F(t)$$

for all t which is larger than the fixed time T_0. That is, for all t which is larger than the fixed time T_0,

$$F(t) \leq F(T_0)e^{s\alpha_1 T_0}e^{-s\alpha_1 t}. \tag{33}$$

Therefore, we deduce from (18), (14) and (33) that there are positive constants κ and ϑ such that

$$E(t) \leq \kappa \exp\{-\vartheta t\} \quad for \ all \ t \geq 0 \ and \ as \ t \to +\infty.$$

$\square$

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References

[1] F. Li, Z. Zhao and Y. Chen, *Global existence and uniqueness and decay estimates for nonlinear viscoelastic wave equation with boundary dissipation*, J Nonlinear Analysis: Real World Applications, **12** (2011), 1759–1773.

[2] F. Li and Z. Zhao, *Uniform energy decay rates for nonlinear viscoelastic wave equation with nonlocal boundary damping*, Nonlinear Analysis: Real World Applications, **7**4 (2011), 3468–3477.

[3] C. F. Carrier, *On the vibration problem of elastic string*, J. Appl. Math., **3** (1945), 151-165.

[4] R. W. Dickey, *The initial value problem for a nonlinear semi-infinite string*, Proc. Roy. Soc. Edinburgh Vol. **82** (1978), 19-26.

[5] S. Y. Lee and C. D. Mote, *Vibration control of an axially moving string by boundary control*, ASME J. Dyna. Syst., Meas., Control, **118** (1996), 66-74.

[6] Y. Li, D. Aron and C. D. Rahn, *Adaptive vibration isolation for axially moving strings: Theory and experiment*, Automatica, **38** (1996), 379-390.

[7] J. L. Lions, *On some question on boundary value problem of mathematical physics*, 1, in: G.M. de La Penha, L. A. Medeiros (Eds.), Contemporary Developments of Continuum Mechanics and Partial Differential Equations, North-Holland, Amsterdam, 1978.

[8] M. Aassila and D. Kaya, *On Local Solutions of a Mildly Degenerate Hyperbolic Equation*, Journal of Mathematical Analysis and Applications, **238** (1999), 418-428.

[9] F. Pellicano and F. Vestroni, *Complex dynamics of high-speed axially moving systems*, Journal of Sound and Vibration, **258** (2002), 31-44.

[10] G. Kirchhoff, *Vorlesungen über Mechanik*, Teubner, Leipzig, 1983.

[11] _____, *Asymptotic behavior of a nonlinear Kirchhoff type equation with spring boundary conditions*, Computers and Mathematics with Applications **6**2 (2011), 3004–3014.

[12] _____, *Stabilization for the Kirchhoff type equation from an axially moving heterogeneous string modeling with boundary feedback control*, Nonlinear Analysis: Theory, Methods and Applications **7**5 (2012), 3598–3617.

[13] J. Límaco, H. R. Clark, and L. A. Medeiros, *Vibrations of elastic string with non-homogeneous material*, Journal of Mathematical Analysis and Applications **3**44 (2008), 806–820.

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