Abstract. Let $R$ be a ring, and let $M$ be a left $R$-module. If $M$ is Rad-supplementing, then every direct summand of $M$ is Rad-supplementing, but not each factor module of $M$. Any finite direct sum of Rad-supplementing modules is Rad-supplementing. Every module with composition series is (Rad-)supplementing. $M$ has a Rad-supplement in its injective envelope if and only if $M$ has a Rad-supplement in every essential extension. $R$ is left perfect if and only if $R$ is semilocal, reduced and the free left $R$-module $(\mathbb{R}R)N$ is ample Rad-supplementing. $M$ is ample Rad-supplementing if and only if every submodule of $M$ is Rad-supplementing. Every left $R$-module is (ample) Rad-supplementing if and only if $R/P(R)$ is left perfect, where $P(R)$ is the sum of all left ideals $I$ of $R$ such that $\text{Rad} I = I$.

1. Introduction

All rings consider in this paper will be associative with an identity element. Unless otherwise stated, $R$ denotes an arbitrary ring and all modules will be left unitary $R$-modules. For a module $M$, by $X \subseteq M$, we mean $X$ is a submodule of $M$ or $M$ is an extension of $X$. As usual, $\text{Rad} M$ denotes the radical of $M$ and $J$ denotes the Jacobson radical of the ring $R$. $E(M)$ will be the injective envelope of $M$. For an index set $I$, $M(I)$ denotes the direct sum $\bigoplus I M$. By $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ we denote as usual the set of natural numbers, the ring of integers and the field of rational numbers, respectively. A submodule $K \subseteq M$ is called small in $M$ (denoted by $K \ll M$) if $M \neq K + T$ for every proper submodule $T$ of $M$. Dually, a submodule $L \subseteq M$ is called essential in $M$ (denoted by $L \triangleleft M$) if $L \cap X \neq 0$ for every nonzero submodule $X$ of $M$.

The notion of a supplement submodule was introduced in [12] in order to characterize semiperfect modules, that is projective modules whose factor modules have projective cover. For submodules $U$ and $V$ of a module $M$, $V$ is said to be a supplement of $U$ in $M$ or $U$ is said to have a supplement $V$ in $M$ if $U + V = M$ and $U \cap V \ll V$. The module $M$ is called supplemented if every
submodule of $M$ has a supplement in $M$. See [19, §41] and [9] for results and the definitions related to supplements and supplemented modules. Recently, several authors have studied different generalizations of supplemented modules. In [1], $\tau$-supplemented modules were defined for an arbitrary preradical $\tau$ for the category of left $R$-modules. For submodules $U$ and $V$ of a module $M$, $V$ is said to be a $\tau$-supplement of $U$ in $M$ or $U$ is said to have a $\tau$-supplement $V$ in $M$ if $U + V = M$ and $U \cap V \subseteq \tau(V)$. $M$ is called a $\tau$-supplemented module if every submodule of $M$ has a $\tau$-supplement in $M$. For the particular case $\tau = \text{Rad}$, Rad-supplemented modules have been studied in [6]; rings over which all modules are Rad-supplemented were characterized. Also, in the recent paper [7], the relation between Rad-supplemented modules and local modules have been investigated. See [18]; these modules are called generalized supplemented modules. Note that Rad-supplements $V$ of a module $M$ are also called cocnet submodules which can be characterized by the fact that each module with zero radical is injective with respect to the inclusion $V \subseteq M$; see [1], [9, §10] and [15]. On the other hand, modules that have supplements in every module in which it is contained as a submodule have been studied in [22]; the structure of these modules, which are called modules with the property $(E)$, has been completely determined over Dedekind domains. Such modules are also called Moduln mit Ergänzungseigenschaft in [3] and supplementing modules in [9, p. 255]. We follow the terminology and notation as in [9]. We call a module $M$ supplementing if it has a supplement in each module in which it is contained as a submodule. By considering these modules we define and study (ample) Rad-supplementing modules as a proper generalization of supplementing modules. A module $M$ is called (ample) Rad-supplementing if it has a (an ample) Rad-supplement in each module in which it is contained as a submodule, where a submodule $U \subseteq M$ has ample Rad-supplements in $M$ if for every $L \subseteq M$ with $U + L = M$, there is a Rad-supplement $L'$ of $U$ with $L' \subseteq L$.

In Section 2, we investigate some properties of Rad-supplementing modules. It is clear that every supplementing module is Rad-supplementing, but the converse implication fails to be true; Example 2.3. If a module $M$ has a Rad-supplement in its injective envelope, $M$ need not be Rad-supplementing. However, we prove that $M$ has a Rad-supplement in its injective envelope if and only if $M$ has a Rad-supplement in every essential extension; Proposition 2.5. We prove that for modules $A \subseteq B$, if $A$ and $B/A$ are Rad-supplementing, then so is $B$. Using this fact we also prove that every module with composition series is Rad-supplementing; Theorem 2.12. A factor module of a Rad-supplementing module need not be Rad-supplementing; Example 2.15. For modules $A \subseteq B \subseteq C$ with $C/A$ injective, we prove that if $B$ is Rad-supplementing, then so is $B/A$. As one of the main results, we prove that $R$ is left perfect if and only if $R$ is semilocal, $R$ is reduced and $(R/(R)^{(N)})$ is Rad-supplementing; Theorem 2.20. Finally, using a result of [22], we show that
over a commutative ring $R$, a semisimple $R$-module $M$ is Rad-supplementing if and only if it is supplementing and that is equivalent the fact that $M$ is pure-injective; Theorem 2.21.

Section 3 contains some properties of ample Rad-supplementing modules. It starts by proving a useful property that a module $M$ is ample Rad-supplementing if and only if every submodule of $M$ is Rad-supplementing; Proposition 3.1. One of the main results of this part is that $R$ is left perfect if and only if $R$ is reduced and the free left $R$-module $(R(R)^{(x)})$ is ample Rad-supplementing; Theorem 3.3. In the proof of this result, Rad-supplemented modules plays an important role as, of course, every ample Rad-supplementing module is Rad-supplemented. Finally, using the characterization of Rad-supplemented modules given in [6], we characterize the rings over which every module is (ample) Rad-supplementing. We prove that every left $R$-module is (ample) Rad-supplementing if and only if every reduced left $R$-module is Rad-supplementing if and only if $R/P(R)$ is left perfect; Theorem 3.4.

2. Rad-supplementing modules

A module $M$ is called radical if $\text{Rad } M = M$, and $M$ is called reduced if it has no nonzero radical submodule. See [21, p. 47] for details for the notion of reduced and radical modules.

**Proposition 2.1.** Supplementing modules and radical modules are Rad-supplementing.

**Proof.** Let $M$ be a module and $N$ be any extension of $M$. If $M$ is supplementing, then it has a supplement, and so a Rad-supplement in $N$. Thus $M$ is Rad-supplementing. Now, if $\text{Rad } M = M$, then $N$ is a Rad-supplement of $M$ in $N$. □

By $P(M)$ we denote the sum of all radical submodules of the module $M$, that is,

$$P(M) = \sum\{U \subseteq M \mid \text{Rad } U = U\}.$$ 

Clearly $M$ is reduced if $P(M) = 0$.

Since $P(M)$ is a radical submodule of $M$ we have the following corollary.

**Corollary 2.2.** For a module $M$, $P(M)$ is Rad-supplementing.

A subset $I$ of a ring $R$ is said to be left $T$-nilpotent in case, for every sequence $\{a_k\}_{k=1}^{\infty}$ in $I$, there is a positive integer $n$ such that $a_1 \cdots a_n = 0$.

In general, Rad-supplementing modules need not be supplementing as the following example shows.

**Example 2.3.** Let $k$ be a field. In the polynomial ring $k[x_1, x_2, \ldots]$ with countably many indeterminates $x_n$, $n \in \mathbb{N}$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \ldots)$ generated by $x_1^2$ and $x_n^2 - x_n$ for each $n \in \mathbb{N}$. Then the quotient ring $R = k[x_1, x_2, \ldots]/I$ is a local ring with the unique maximal ideal
\[ J = J^2 \] (see [6, Example 6.2] for details). Now let \( M = J^{(N)} \). Then we have \( \text{Rad} \, M = M \), and so \( M \) is Rad-supplementing by Proposition 2.1. However, \( M \) does not have a supplement in \( R^{(N)} \). Because, otherwise, by [5, Theorem 1], \( J \) would be a left \( T \)-nilpotent as \( R \) is semilocal, but this is impossible. Thus \( M \) is not supplementing.

For instance, over a left max ring, supplementing modules and Rad-supplementing modules coincide, where \( R \) is called a left max ring if every left \( R \)-module has a maximal submodule or equivalently, \( \text{Rad} \, M \ll M \) for every left \( R \)-module \( M \).

**Proposition 2.4.** Every direct summand of a Rad-supplementing module is Rad-supplementing.

**Proof.** Let \( U \) be a direct summand of a Rad-supplementing module \( M \), and let \( N \) be any extension of \( U \). Then \( M = A \oplus U \) for some submodule \( A \subseteq M \). By hypothesis \( M \) has a Rad-supplement in the module \( A \oplus N \) containing \( M \), that is, there exists a submodule \( V \) of \( A \oplus N \) such that
\[
(A \oplus U) + V = A \oplus N \quad \text{and} \quad (A \oplus U) \cap V \subseteq \text{Rad} \, V.
\]
Now, let \( g : A \oplus N \to N \) be the projection onto \( N \). Then
\[
U + g(V) = g(A \oplus U) + g(V) = g((A \oplus U) + V) = g(A \oplus N) = N, \quad \text{and}
\]
\[
U \cap g(V) = g((A \oplus U) \cap V) \subseteq g(\text{Rad} \, V) \subseteq \text{Rad}(g(V)).
\]
Hence \( g(V) \) is a Rad-supplement of \( U \) in \( N \). \( \square \)

If a module \( M \) has a Rad-supplement in its injective envelope \( E(M) \), \( M \) need not be Rad-supplementing. For example, for \( R = \mathbb{Z} \), the \( R \)-module \( M = 2\mathbb{Z} \) has a Rad-supplement in \( E(M) = \mathbb{Q} \) since \( \text{Rad} \, \mathbb{Q} = \mathbb{Q} \) (and so \( \mathbb{Q} \) is Rad-supplemented). But, \( M \) does not have a Rad-supplement in \( \mathbb{Z} \), and thus \( M \) is not Rad-supplementing. However, we have the following result.

**Proposition 2.5.** Let \( M \) be a module. Then the following are equivalent.

(i) \( M \) has a Rad-supplement in every essential extension;
(ii) \( M \) has a Rad-supplement in its injective envelope \( E(M) \).

**Proof.** (i)\( \Rightarrow \) (ii) is clear.

(ii)\( \Rightarrow \) (i) Let \( M \subseteq N \) with \( M \triangleleft N \), and let \( f : M \to N \) and \( g : M \to E(M) \) be inclusion maps. Then we have the following commutative diagram with \( h \) necessarily monic:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow{g} & & \downarrow{h} \\
E(M) & & \\
\end{array}
\]

By hypothesis, \( M \) has a Rad-supplement in \( E(M) \), say \( K \). That is, \( M + K = E(M) \) and \( M \cap K \subseteq \text{Rad} \, K \). Since \( M \subseteq h(N) \), we obtain that \( h(N) = \)
$h(N) \cap E(M) = h(N) \cap (M + K) = M + h(N) \cap K$. Now, taking any $n \in N$, we have $h(n) = m + h(n_1) = h(m + n_1)$ where $m \in M$ and $h(n_1) \in h(N) \cap K$. So, $n = m + n_1 \in M + h^{-1}(K)$ since $h$ is monic, and so $M + h^{-1}(K) = N$. Moreover, $M \cap h^{-1}(K) = h^{-1}(M \cap K) \subseteq h^{-1}(\text{Rad}(K)) \subseteq \text{Rad}(h^{-1}(K))$ since $h^{-1}(M) = M$ as $h$ is monic. Hence $h^{-1}(K)$ is a Rad-supplement of $M$ in $N$.

**Proposition 2.6.** Let $B$ be a module, and let $A$ be a submodule of $B$. If $A$ and $B/A$ are Rad-supplementing, then so is $B$.

**Proof.** Let $B \subseteq N$ be any extension of $B$. By hypothesis, there is a Rad-supplement $V/A$ of $B/A$ in $N/A$ and a Rad-supplement $W$ of $A$ in $V$. We claim that $W$ is a Rad-supplement of $B$ in $N$. We have epimorphisms $f : W \to V/A$ and $g : V/A \to N/B$ such that $\text{Ker} f = W \cap A \subseteq \text{Rad} W$ and $\text{Ker} g = V/A \cap B/A \subseteq \text{Rad}(V/A)$. Then $g \circ f : W \to N/B$ is an epimorphism such that $W \cap B = \text{Ker}(g \circ f) \subseteq \text{Rad} W$ by [20, Lemma 1.1]. Finally, $N = V + B = (W + A) + B = W + B$. □

**Remark 2.7.** The previous result holds for supplementing modules; see [22, Lemma 1.3-(c)].

**Corollary 2.8.** If $M_1$ and $M_2$ are Rad-supplementing modules, then so is $M_1 \oplus M_2$.

**Proof.** Consider the short exact sequence

$$0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0.$$ 

Thus the result follows by Proposition 2.6. □

$R$ is said to be a *left hereditary* ring if every left ideal of $R$ is projective.

**Corollary 2.9.** If $M/P(M)$ is Rad-supplementing, then $M$ is Rad-supplementing. For left hereditary rings, the converse is also true.

**Proof.** Since $P(M)$ is Rad-supplementing by Corollary 2.2, the result follows by Proposition 2.6. Over left hereditary rings, any factor module of a Rad-supplementing module is Rad-supplementing (see Corollary 2.18). □

We give the proof of the following known fact for completeness.

**Lemma 2.10.** Every simple submodule $S$ of a module $M$ is either a direct summand of $M$ or small in $M$.

**Proof.** Suppose that $S$ is not small in $M$, then there exists a proper submodule $K$ of $M$ such that $S + K = M$. Since $S$ is simple and $K \neq M$, $S \cap K = 0$. Thus $M = S \oplus K$. □

**Proposition 2.11.** Every simple module is (Rad-)supplementing.
Proof. Let \( S \) be a simple module and \( N \) any extension of \( S \). Then by Lemma 2.10, \( S \ll N \) or \( S \oplus S' = N \) for a submodule \( S' \subseteq N \). In the first case, \( N \) is a (Rad-)supplement of \( S \) in \( N \), and in the second case, \( S' \) is a (Rad-)supplement of \( S \) in \( N \). So, in each case \( S \) has a (Rad-)supplement in \( N \), that is, \( S \) is (Rad-)supplementing. \( \square \)

**Theorem 2.12.** Every module with composition series is (Rad-)supplementing.

**Proof.** Let \( 0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M \) be a composition series of a module \( M \). The proof is by induction on \( n \in \mathbb{N} \). If \( n = 1 \), then \( M = M_1 \) is simple, and so \( M \) is (Rad-)supplementing by Proposition 2.11. Suppose that this is true for each \( k \leq n - 1 \). Then \( M_{n-1} \) is (Rad-)supplementing.

Since \( M_n/M_{n-1} \) is also (Rad-)supplementing as a simple module, we obtain by Proposition 2.6 that \( M = M_n \) is (Rad-)supplementing. \( \square \)

**Corollary 2.13.** A finitely generated semisimple module is (Rad-)supplementing.

In general, a factor module of a Rad-supplementing module need not be Rad-supplementing. To give such a counterexample we need the following result.

\( R \) is called Von Neumann regular if every element \( a \in R \) can be written in the form \( axa \), for some \( x \in R \).

**Proposition 2.14.** Let \( R \) be a commutative Von Neumann regular ring. Then an \( R \)-module \( M \) is Rad-supplementing if and only if \( M \) is injective.

**Proof.** Suppose that \( M \) is a Rad-supplementing module. Let \( M \subseteq N \) be any extension of \( M \). Then there is a Rad-supplement \( V \) of \( M \) in \( N \), that is, \( V + M = N \) and \( V \cap M \subseteq \text{Rad} V \). Since all \( R \)-modules have zero radical by [13, 3.73 and 3.75], we have \( \text{Rad} V = 0 \), and so \( N = V \oplus M \). Conversely, if \( M \) is injective and \( M \subseteq N \) is any extension of \( M \), then \( N = M \oplus K \) for some submodule \( K \subseteq N \). Thus \( K \) is a Rad-supplement of \( M \) in \( N \). \( \square \)

It is known that a ring \( R \) is lefty hereditary if and only if every quotient of an injective \( R \)-module is injective (see [8, Ch.I, Theorem 5.4]).

**Example 2.15.** Let \( R = \prod_{i \in I} F_i \) be a ring, where each \( F_i \) is a field for an infinite index set \( I \). Then \( R \) is a commutative Von Neumann regular ring. Indeed, let \( a = (a_i)_{i \in I} \in R \) where \( a_i \in F_i \) for all \( i \in I \). Taking \( b = (b_i)_{i \in I} \in R \) where \( b_i \in F_i \) such that

\[
    b_i = \begin{cases} 
        a_i^{-1} & \text{if } a_i \neq 0, \\ 
        0 & \text{if } a_i = 0.
    \end{cases}
\]

Then we obtain that

\[
    aba = (a_i)(b_i)(a_i) = (a_i b_i a_i)_{i \in I} = (a_i)_{i \in I} = a.
\]

Now, by Proposition 2.14, \( R \) is a Rad-supplementing module over itself since it is injective (see [13, Corollary 3.11B]). Since \( R \) is not noetherian, it cannot be
semisimple (by [14, Corollary 2.6]). Thus \( R \) is not hereditary by [16, Corollary]. Hence, there is a factor module of \( R \) which is not injective.

The following technical lemma will be useful to show that Rad-supplementing modules are closed under factor modules, under a special condition.

**Lemma 2.16.** Let \( A \subseteq B \subseteq C \) be modules with \( C/A \) injective. Let \( N \) be a module containing \( B/A \). Then there exists a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & A & \to & B & \to & B/A & \to & 0 \\
& & id & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & P & \to & N & \to & 0 \\
\end{array}
\]

**Proof.** By pushout we have the following commutative diagram, where \( \varphi \) exists since \( C/A \) is injective:

\[
\begin{array}{ccccccccc}
0 & \to & B/A & \to & N & \to & N/(B/A) & \to & 0 \\
& & \downarrow(1) & & \downarrow g & & \downarrow(2) & & id \\
0 & \to & C/A & \to & N' & \to & N/(B/A) & \to & 0 \\
\end{array}
\]

In the diagram, since the triangle-(1) is commutative, there exists a homomorphism \( \alpha : N/(B/A) \to N' \) making the triangle-(2) is commutative by [11, Lemma I.8.4]. So, the second row splits. Then we can take \( N' = (C/A) \oplus (N/(B/A)) \), and so we may assume that \( \beta : C/A \to N' \) is an inclusion. Therefore, we have the following commutative diagram since \( B/A = \beta(B/A) = g(B/A) \subseteq N' \):

\[
\begin{array}{ccccccccc}
0 & \to & A & \to & B & \to & B/A & \to & 0 \\
& & id & & \downarrow \phi & & \downarrow & & \\
0 & \to & A & \to & C \oplus (N/(B/A)) & \to & N' & \to & 0 \\
\end{array}
\]

where \( \gamma(a) = (a, 0) \) for every \( a \in A \), \( \phi(b) = (b, 0) \) for every \( b \in B \), and \( \sigma(c, \overline{\tau}) = (c + A, \overline{\tau}) \) for every \( c \in C \) and \( \overline{\tau} \in N/(B/A) \). Finally, taking \( P = \sigma^{-1}(g(N)) \) and defining a homomorphism \( \tilde{\sigma} : P \to g(N) \) by \( \tilde{\sigma}(x) = \sigma(x) \) for every \( x \in P \) (in fact, \( \tilde{\sigma} \) is an epimorphism as so is \( \sigma \)), we obtain the following desired commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \to & A & \to & B & \to & B/A & \to & 0 \\
& & id & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & P & \to & g(N) \cong N & \to & 0 \\
\end{array}
\]

\[ \Box \]

**Proposition 2.17.** Let \( A \subseteq B \subseteq C \) with \( C/A \) injective. If \( B \) is Rad-supplementing, then so is \( B/A \).
Proof. Let $B/A \subseteq N$ be any extension of $B/A$. By Lemma 2.16, we have the following commutative diagram with exact rows since $C/A$ is injective:

$$
\begin{array}{cccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & B/A & \longrightarrow & 0 \\
0 & \downarrow{id} & \downarrow{h} & \downarrow{f} & & & & & \\
0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
\end{array}
$$

Since $h$ is monic and $B$ is Rad-supplementing, $B \cong \text{Im } h$ has a Rad-supplement in $P$, say $V$. That is, $\text{Im } h + V = P$ and $\text{Im } h \cap V \subseteq \text{Rad } V$. We claim that $g(V)$ is a Rad-supplement of $B/A$ in $N$.

$$
N = g(P) = g(h(B)) + g(V) = (f \sigma)(B) + g(V) = (B/A) + g(V), \quad \text{and}
$$

$$(B/A) \cap g(V) = f(\sigma(B)) \cap g(V) = g[h(B)] \cap V \subseteq g(\text{Rad } V) \subseteq \text{Rad}(g(V)). \Box
$$

Corollary 2.18. If $R$ is a left hereditary ring, then every factor module of Rad-supplementing module is Rad-supplementing.

Proposition 2.19. If $M$ is a reduced, projective and Rad-supplementing module, then $\text{Rad } M \ll M$.

Proof. Suppose $X + \text{Rad } M = M$ for a submodule $X$ of $M$. Then since $M$ is projective, there exists $f \in \text{End}(M)$ such that $\text{Im } f \subseteq X$ and $\text{Im } (1 - f) \subseteq \text{Rad } M = JM$ where $J$ is a Jacobson radical of $R$. Therefore $f$ is a monomorphism by [4, Theorem 3]. Since $M$ is Rad-supplementing and $\text{Im } f \cong M$, if $f$ has a Rad-supplement $V$ in $M$, that is, $\text{Im } f + V = M$ and $\text{Im } f \cap V \subseteq \text{Rad } V$. Now we have an epimorphism $g : V \rightarrow M/\text{Im } f$ such that $\text{Ker } g = V \cap \text{Im } f \subseteq \text{Rad } V$. Moreover, since $M = \text{Im } f + \text{Im } (1 - f) = \text{Im } f + \text{Rad } M$ we have $\text{Rad } (M/\text{Im } f) = M/\text{Im } f$. Thus $\text{Rad } V = V$, and so $V = 0$ since $M$ is reduced. Hence $M = \text{Im } f \subseteq X$ implies that $X = M$ as required. \Box

$R$ is said to be a semilocal ring if $R/J$ is a semisimple ring, that is a left (and right) semisimple $R$-module (see [14, §20]).

Theorem 2.20. A ring $R$ is left perfect if and only if $R$ is semilocal, $rR$ is reduced and the free left $R$-module $F = (rR)^{(0)}$ is Rad-supplementing.

Proof. If $R$ is left perfect, then $R$ is semilocal by [2, 28.4], and clearly $rR$ is reduced. Since all left $R$-modules are supplemented and so Rad-supplemented, $F$ is Rad-supplementing. Conversely, since $P(rR) = 0$ we have $P(F) = (P(rR))^{(0)} = 0$, that is, $F$ is reduced. Thus by Proposition 2.19, $JF = \text{Rad } F \ll F$, that is, $J$ is left $T$-nilpotent by, for example, [2, 28.3]. Hence $R$ is left perfect by [2, 28.4] since it is moreover semilocal. \Box

Supplementing modules over commutative noetherian rings have been studied in [3]; the author showed that if a module $M$ is supplementing, then it is cotorsion, that is, $\text{Ext}_1^R(F, M) = 0$ for every flat module $F$ (see [10] for cotorsion modules). So the question was raised When Rad-supplementing modules
are cotorsion? Since any pure-injective module is cotorsion, the following result gives an answer of the question for a semisimple module over a commutative ring. The relation between (Rad-)supplementing modules and cotorsion modules needs to be further investigated.

The part (iii)⇒(i) of the proof of the following theorem follows from [22, Theorem 1.6-(ii)⇒(i)], but we give it by explanation for completeness.

**Theorem 2.21.** Let \( R \) be a commutative ring. Then the following are equivalent for a semisimple \( R \)-module \( M \).

(i) \( M \) is supplementing;

(ii) \( M \) is Rad-supplementing;

(iii) \( M \) is pure-injective.

**Proof.** (i)⇒(ii) is clear.

(ii)⇒(iii) Let \( M \subseteq N \) be a pure extension of \( M \). By hypothesis \( M \) has a Rad-supplement \( V \) in \( N \), that is, \( M + V = N \) and \( M \cap V \subseteq \text{Rad } V \). Since \( M \) is pure in \( N \), we have \( \text{Rad } M = M \cap \text{Rad } N \) (as \( R \) is commutative). Thus \( M \cap V \subseteq M \cap \text{Rad } N = \text{Rad } M = 0 \) as \( M \) is semisimple. Hence \( N = M \oplus V \) as required.

(iii)⇒(i) Let \( M \subseteq N \) be any extension of \( M \). Then the factor module \( X = (M + \text{Rad } N)/\text{Rad } N \) of \( M \) is again semisimple and pure-injective. Since semisimple submodules are pure in every module with zero radical and \( \text{Rad } (N/\text{Rad } N) = 0 \), it follows that \( X \) is a direct summand of \( N/\text{Rad } N \). Now let

\[
(V/\text{Rad } N) \oplus X = N/\text{Rad } N
\]

for a submodule \( V \subseteq N \) such that \( \text{Rad } N \subseteq V \). So we have \( V + M = N \) with \( V \) minimal, and thus \( V \) is a supplement of \( M \) in \( N \). This is because, if \( T + M = N \) for a submodule \( T \) of \( N \) with \( T \subseteq V \), then from

\[
\text{Rad } (N/T) = \text{Rad } ((M + T)/T) = \text{Rad } (M/M \cap T) = 0
\]
as \( M/M \cap T \) is semisimple, we obtain that \( \text{Rad } N \subseteq T \). Moreover, since

\[
\text{Rad } N = V \cap (M + \text{Rad } N) = V \cap M + \text{Rad } N,
\]
we have \( V \cap M \subseteq \text{Rad } N \) and \( V = T + V \cap M \subseteq T + \text{Rad } N = T \), thus \( T = V \).

\( \square \)

3. Ample Rad-supplementing modules

The following useful result gives a relation between Rad-supplementing modules and ample Rad-supplementing modules.

**Proposition 3.1.** A module \( M \) is ample Rad-supplementing if and only if every submodule of \( M \) is Rad-supplementing.

**Proof.** \( (\Leftarrow) \) Let \( M \) be a module and \( N \) be any extension of \( M \). Suppose that for a submodule \( X \subseteq N \), \( X + M = N \). By hypothesis the submodule \( X \cap M \) of \( M \) has a Rad-supplement \( V \) in \( X \) containing \( X \cap M \), that is, \( (X \cap M) + V = X \) and
(X ∩ M) ∩ V ⊆ \text{Rad} V. Then \( N = M + X = M + (X ∩ M) + V = M + V \) and, 
\( M \cap V = M \cap (V \cap X) = (X ∩ M) \cap V \subseteq \text{Rad} V \). Hence \( V \) is a Rad-supplement of \( M \) in \( N \) such that \( V \subseteq X \).

\((⇒)\) Let \( U \) be a submodule of \( M \) and \( N \) be any module containing \( U \). Thus we can draw the pushout for the inclusion homomorphisms \( i_1 : U \hookrightarrow N \) and 
\( i_2 : U \hookrightarrow M \):

\[
\begin{array}{c}
\text{M} \xrightarrow{i_1} F \xrightarrow{i_2} \text{N} \\
\text{U} \xrightarrow{\alpha} F \xrightarrow{\beta} \text{N}
\end{array}
\]

In the diagram, \( \alpha \) and \( \beta \) are also monomorphisms by the properties of pushout (see, for example, [17, Exercise 5.10]). Let \( M' = \text{Im} \alpha \) and \( N' = \text{Im} \beta \). Then \( F = M' + N' \) by the properties of pushout. So by hypothesis, \( M' \cong M \)
has a Rad-supplement \( V \) in \( F \) such that \( V \subseteq N' \), that is, \( M' + V = F \) and 
\( M' \cap V \subseteq \text{Rad} V \). Therefore \( V \) is a Rad-supplement of \( M' \cap N' \) in \( N' \), because 
\( N' = N' \cap F = N' \cap (M' + V) = (M' \cap N') + V \) and \( (M' \cap N') \cap V = M' \cap V \subseteq \text{Rad} V \). Now, we claim that \( \beta^{-1}(V) \) is a Rad-supplement of \( U \) in \( N \). 
Since \( \beta : N \to F \) is a monomorphism with \( N' = \text{Im} \beta \), we have an isomorphism 
\( \tilde{\beta} : N \to N' \) defined as \( \tilde{\beta}(x) = \beta(x) \) for all \( x \in N \). By this isomorphism, since \( V \) is a Rad-supplement of \( M' \cap N' \) in \( N' \), we obtain \( \tilde{\beta}^{-1}(V) \) is a Rad-supplement of 
\( \tilde{\beta}^{-1}(M' \cap N') \) in \( \tilde{\beta}^{-1}(N') \). Since it can be easily shown that \( \tilde{\beta}^{-1}(V) = \beta^{-1}(V) \), 
\( \tilde{\beta}^{-1}(N') = N \), and \( \tilde{\beta}^{-1}(M' \cap N') = U \) the result follows. \( \Box \)

Corollary 3.2. Every ample Rad-supplementing module is both Rad-supplementing and Rad-supplemented.

Theorem 3.3. A ring \( R \) is left perfect if and only if \( _RR \) is reduced and the free left \( R \)-module \( F = (_RR)^{[N]} \) is ample Rad-supplementing.

Proof. If \( R \) is left perfect, then \( _RR \) is reduced and all left \( R \)-modules are supplemented, and so Rad-supplemented. Thus every submodule of \( F \) is Rad-supplementing. Hence \( F \) is ample Rad-supplementing by Proposition 3.1. Conversely, if \( F \) is ample Rad-supplementing, then it is Rad-supplemented by Corollary 3.2, and so \( R \) is left perfect by [6, Theorem 5.3]. \( \Box \)

Finally, we give the characterization of the rings over which every module is (ample) Rad-supplementing.

Theorem 3.4. For a ring \( R \), the following are equivalent:

(i) Every left \( R \)-module is Rad-supplementing;
(ii) Every reduced left \( R \)-module is Rad-supplementing;
(iii) Every left \( R \)-module is ample Rad-supplementing;
(iv) Every left \( R \)-module is Rad-supplemented;
(v) \( R/P(R) \) is left perfect.
Proof. Let $M$ be a module. (i)$\Rightarrow$(ii) is clear.

(ii)$\Rightarrow$(i) Since $M/P(M)$ is reduced, it is Rad-supplementing by hypothesis. So $M$ is Rad-supplementing by Corollary 2.9.

(i)$\Rightarrow$(iii) Since every submodule of $M$ is Rad-supplementing, $M$ is ample Rad-supplementing by Proposition 3.1.

(iii)$\Rightarrow$(iv) by Corollary 3.2.

(iv)$\Rightarrow$(i) Let $M \subseteq N$ be any extension of $M$. By hypothesis, $N$ is Rad-supplemented, and so $M$ has a Rad-supplement in $N$.

(iv)$\Leftrightarrow$(v) by [6, Theorem 6.1].

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