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A FIXED POINT APPROACH TO THE STABILITY OF QUARTIC LIE $*$-DERIVATIONS

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Abstract. We obtain the general solution of the functional equation $f(ax+y) - f(x-ay) + \frac{1}{2}a(a^2+1)f(x-y) + (a^4-1)f(y) = \frac{1}{2}a(a^2+1)f(x+y) + (a^4-1)f(x)$ and prove the stability problem of the quartic Lie $*$-derivation by using a directed method and an alternative fixed point method.

1. Introduction

A mapping is said to be *stable* if a mapping is an almost-homomorphism, there exists a true homomorphism near the almost-homomorphism. Ulam introduced the stability problem for functional equations which concerned the stability of group homomorphisms, thai is, given two groups G and H , is every almost-homomorphism $G \to H$ close to a true homomorphism $G \to H$?; see [17]. Hyers [7] investigated stability problems related to the question of Ulam on Banach spaces. Subsequently, the result of Hyers was generalized by a number of authors. In particular, Aoki [1] studied the stability problem for additive mapping and Rassias [14] proved the problem for linear mappings by considering a unbounded Cauchy difference operator. Afterwards, the result of Rassias has provided a lot of influence in the development of what we call Hyers-Ulam stability or Hyers-Ulam-Rassias stability. The stability problems of this

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topic have been investigated by a number of authors; see [10], [8], [2] and [3]. In fact, the stability problems have been extensively investigated to the various points of views such as various functional equations, various spaces and so on. Especially, Jang and Park [9] introduced the concepts of $*$-derivations and investigated the stability problems of quadratic $*$-derivations on Banach C^*-algebra. Also, Park and Bodaghi and Yang et al. studied the stability properties of $*$-derivations by using an alternative fixed point method; see [12] and [19]. Also, Fošner and Fošner introduced the basic concepts of cubic Lie derivations and investigated the stability problem of cubic Lie derivations; see [6].

Rassias introduced the quartic functional equation in [13] which was the oldest quartic functional equation and investigated the stability problems of the following functional equation:

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y)\,. \tag{1.1}$$

Chung and Sahoo [4] obtained the general solution of (1.1) by using the properties of a certain mapping of the form $A(x,x,x,x)$, where the function $A:\mathbb{R}^4\to\mathbb{R}$ is symmetric and additive in each variable.

In this paper, we will consider the following functional equation which is generalized and different from the equation (1.1):

$$\begin{aligned} f(ax+y)-f(x-ay)+\frac{1}{2}a(a^2+1)f(x-y)+(a^4-1)f(y) \\ = \frac{1}{2}a(a^2+1)f(x+y)+(a^4-1)f(x) \end{aligned} \tag{1.2}$$

for all $x\,,y\in X$ and an integer $a(a\neq 0\,,\pm 1)$. We will show that the equation (1.2) is a general solution of quartic functional equation and introduced a quartic Lie $*$-derivation. Finally, we will prove the Hyers-Ulam stability problem of the quartic Lie $*$-derivations by using directed and fixed point methods.

2. A general solution of a quartic functional equation

Let X and Y be real vector spaces. In this section we will obtain the result that the functional equation (1.2) is a general solution of a quartic functional equation by using 4-additive symmetric mapping. Before we proceed, we will introduce some basic concepts concerning 4-additive symmetric mappings. A mapping $A_4:X^4\to Y$ is called 4-*additive* if it is additive in each variable. A mapping A_4 is said to be *symmetric* if $A_4(x_1, x_2, x_3, x_4) = A_4(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$ for every permutation $\{\sigma(1), \sigma(2), \sigma(3), \sigma(4)\}$ of $\{1, 2, 3, 4\}$. If $A_4(x_1, x_2, x_3, x_4)$ is a 4-additive symmetric mapping, then $A^4(x)$ will denote the diagonal $A_4(x, x, x, x)$ and $A^4(qx) = q^4 A^4(x)$ for all $x \in X$ and all $q \in \mathbb{Q}$. A mapping $A^4(x)$ is called a *monomial function* of degree 4 (assuming $A^4 \not\equiv 0$). On taking $x_1 = x_2 = \cdots = x_s = x$ and $x_{s+1} = x_{s+2} = \cdots = x_4 = y$ in $A_4(x_1, x_2, x_3, x_4)$, it is denoted by $A^{s,4-s}(x, y)$. We note that the generalized concepts of n-additive symmetric mappings are found in [16] and [18].

Theorem 2.1. *Let $A^4(x)$ be the diagonal of the 4-additive symmetric mapping $A_4 : X^4 \to Y$. A mapping $f : X \to Y$ is a solution of the functional equation (1.2) if and only if f is of the form $f(x) = A^4(x)$ for all $x \in X$.*

Proof. Assume that f satisfies the functional equation (1.2). We will show that $f(x) = A^4(x)$ for all $x \in X$. On letting $y = 0$ in the equation (1.2), we have

$$f(ax) = a^4 f(x) - (a^4 - 1) f(0) \tag{2.1}$$

for all $x \in X$ and an integer number $a \neq 0, \pm 1$. Also, we have

$$\begin{aligned} &f(y) - f(-ay) + \frac{1}{2}a(a^2+1)f(-y) + (a^4-1)f(y) \\ &= \frac{1}{2}a(a^2+1)f(y) + (a^4-1)f(0) \end{aligned}$$

by letting $x = 0$ in the equation (1.2). Replacing y by x in the previous equation, we get

$$\begin{aligned} &f(x) - f(-ax) + \frac{1}{2}a(a^2+1)f(-x) + (a^4-1)f(x) \\ &= \frac{1}{2}a(a^2+1)f(x) + (a^4-1)f(0) \end{aligned}$$

for all $x \in X$ and $a \neq 0, \pm 1$. Hence the equation (2.1) implies that f is an odd mapping. On taking $x = y$ in the equation (1.2) and using the equation (2.1), we have

$$\begin{aligned} &(a+1)^4 f(x) - [(a+1)^4 - 1]f(0) - (a-1)^4 f(x) + [(a-1)^4 - 1]f(0) \\ &+\frac{1}{2}a(a^2+1)f(0) = 8a(a^2+1)f(x) - \frac{15}{2}a(a^2+1)f(0) \end{aligned}$$

for all $x \in X$ and an integer $a\,(a \neq 0\,, \pm 1)$. Then we have $a(a^2-1)f(0) = 0$ for an integer $a\,(a \neq 0\,, \pm 1)$. This means that $f(0) = 0$. Also, the equation (2.1) implies that

$$f(ax) = a^4 f(x) \tag{2.2}$$

for all $x \in X$. We can rewrite the functional equation (1.2) in the following form

$$\begin{aligned} &f(x) - \frac{1}{a^4-1} f(ax+y) + \frac{1}{a^4-1} f(x-ay) - \frac{a}{2(a^2-1)} f(x-y) \\ &+ \frac{a}{2(a^2-1)} f(x+y) - f(y) = 0\,, \end{aligned}$$

for all $x\,, y \in X$ and an integer $a\,(a \neq 0\,, \pm 1)$. By Theorems 3.5 and 3.6 in [18], f is a generalized polynomial function of degree at most 4, that is, f is of the form

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x) \tag{2.3}$$

for all $x \in X$, where $A^0(x) = A^0$ is an arbitrary element of Y and $A^i(x)$ is the diagonal i-additive symmetric mapping $A_i : X^i \to Y\,(i = 1, 2, 3, 4)$. Since $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in X$, $A^0(x) = A^0 = 0$ and $A^1(x) = A^3(x) = 0$. Hence we have

$$f(x) = A^4(x) + A^2(x)\,,$$

for all $x \in X$. The equation (2.3) and $A^n(qx) = q^n A^n(x)$ for all $x \in X$ and all $q \in \mathbb{Q}$ imply that $a^2(a^2-1)A^2(x) = 0$ for an integer $a\,(a \neq 0\,, \pm 1)$. Hence $A^2(x) = 0$, that is, $f(x) = A^4(x)$ for all $x \in X$, as desired.

Conversely, suppose $f(x) = A^4(x)$ for all $x \in X$, where $A^4(x)$ is a diagonal 4-additive symmetric mapping $A_4 : X^4 \to Y$. Note that

$$\begin{aligned} &A^4(qx+py) \\ &= q^4 A^4(x) + 4q^3 p A^{3,1}(x,y) + 6q^2p^2 A^{2,2}(x,y) + 4qp^3 A^{1,3}(x,y) + p^4 A^4(y) \\ &r^s A^{s,t}(x,y) = A^{s,t}(rx,y)\,, \quad r^t A^{s,t}(x,y) = A^{s,t}(x,ry) \end{aligned}$$

where $1 \le s, t \le 3$ and $p, q, r \in \mathbb{Q}$. Thus f satisfies the equation (1.2). □

For this reason, we call the mapping f a *generalized quartic mapping* if f satisfies the equation (1.2).

3. Quartic Lie $*$-Derivations

In this section, we will investigate the Hyers-Ulam stability of the qyartic Lie $*$-derivation by using directed method and a fixed point method. Let A be a complex normed $*$-algebra and M be a Banach A-bimodule. For convenience, we will use $||\cdot||$ as norms on a normed algebra A and a normed A-bimodule M.

A mapping $f : A \to M$ is called a *quartic homogeneous mapping* if $f(\mu a) = \mu^4 f(a)$, for all $a \in A$ and $\mu \in \mathbb{C}$. A quartic homogeneous mapping $f : A \to M$ is called a *quartic derivation* if

$$f(xy) = f(x)y^4 + x^4 f(y)$$

for all $x, y \in A$. A quartic homogeneous mapping f is called a *quartic Lie derivation* if

$$f([x, y]) = [f(x), y^4] + [x^4, f(y)]$$

for all $x, y \in A$, where $[x, y] = xy - yx$. A quartic Lie derivation f is called a *quartic Lie $*$-derivation* if f satisfies $f(x^*) = f(x)^*$ for all $x \in A$.

EXAMPLE 3.1. Let $A = \mathbb{C}$ be a complex number field with the map $z \mapsto z^* = \bar{z}$ (where $\bar{z}$ is the complex conjugate of z). Suppose that $f : A \to A$ by $f(x) = x^4$ for all $x \in A$. Then f is quartic and

$$f([x, y]) = [f(x), y^4] + [x^4, f(y)] = 0$$

for all $x, y \in A$. Also,

$$f(x^*) = f(\bar{x}) = \bar{x}^4 = \overline{f(x)} = f(x)^*$$

for all $x \in A$. Hence we know that f is a quartic Lie $*$-derivation, as desired.

For this entire section,

$$\mathbb{T}^1 = \{\mu \in \mathbb{C} \,|\, |\mu| = 1\}.$$

For the given mapping $f : A \to M$, we consider

$$\begin{aligned} \Delta_\mu f(a, b) := {} & f(m\mu a + \mu b) - f(\mu a - m\mu b) + \frac{1}{2}\mu^4 m(m^2+1) f(a-b) \\ & + \mu^4(m^4-1) f(b) - \frac{1}{2}\mu^4 m(m^2+1) f(a+b) - \mu^4(m^4-1) f(a), \end{aligned} \tag{3.1}$$

$$\Delta f(a, b) := f([a, b]) - [f(a), b^4] - [a^4, f(b)]$$

for all $a, b \in A$, $\mu \in \mathbb{C}$ and $m \in \mathbb{Z}\,(m \neq 0, \pm 1)$.

THEOREM 3.2. *Let n_0 be a positive integer. Suppose that there is a mapping $f : A \to M$ with $f(0) = 0$ and there exists a function $\phi : A^5 \to [0, \infty)$ such that*

$$\widetilde{\phi}(a,b,x,y,z) := \sum_{j=0}^{\infty} \frac{1}{|m|^{4j}} \phi(m^j a, m^j b, m^j x, m^j y, m^j z) < \infty \tag{3.2}$$

$$||\Delta_\mu f(a,b)|| \le \phi(a,b,0,0,0) \tag{3.3}$$

$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \le \phi(0,0,x,y,z) \tag{3.4}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}} = \{e^{i\theta} \,|\, 0 \le \theta \le \frac{2\pi}{n_0}\}$ and all $a,b,x,y,z \in A$. For each fixed $a \in A$, if the mapping $r \mapsto f(ra)$ from $\mathbb{R}$ to M is continuous then there exists a unique quartic Lie $*$-derivation $L : A \to M$ such that

$$||f(a) - L(a)|| \le \frac{1}{|m|^4} \widetilde{\phi}(a,0,0,0,0)\,, \tag{3.5}$$

for all $a \in A$.

Proof. On letting $b = 0$ and $\mu = 1$ in the inequality (3.3), we have

$$||f(a) - \frac{1}{m^4} f(ma)|| \le \frac{1}{|m|^4} \phi(a,0,0,0,0) \tag{3.6}$$

for all $a \in A$. By using the induction steps with (3.6), we have the following inequality

$$||\frac{1}{m^{4t}} f(m^t a) - \frac{1}{m^{4k}} f(m^k a)|| \le \frac{1}{|m|^4} \sum_{j=k}^{t-1} \frac{\phi(m^j a,0,0,0,0)}{|m|^{4j}} \tag{3.7}$$

for $t > k \ge 0$ and $a \in A$. Both (3.2) and (3.7) imply that $\{\frac{1}{m^{4n}} f(m^n a)\}_{n=0}^{\infty}$ is a Cauchy sequence. By the completeness of M, we know that the sequence is convergent. Hence we can define a mapping $L : A \to M$ as

$$L(a) = \lim_{n\to\infty} \frac{1}{m^{4n}} f(m^n a) \tag{3.8}$$

for $a \in A$. On taking $t = n$ and $k = 0$ in the inequality (3.7), we get

$$||\frac{1}{m^{4n}} f(m^n a) - f(a)|| \le \frac{1}{|m|^4} \sum_{j=0}^{n-1} \frac{\phi(m^j a,0,0,0,0)}{|m|^{4j}} \tag{3.9}$$

for $n > 0$ and $a \in A$. On taking $n \to \infty$ in the inequality (3.9), the inequality (3.2) implies that the inequality (3.5) holds.

We know that

$$||\Delta_\mu L(a,b)|| = \lim_{n\to\infty} \frac{1}{|m|^{4n}} ||\Delta_\mu f(m^n a, m^n b)|| \le \lim_{n\to\infty} \frac{\phi(m^n a, m^n b, 0,0,0)}{|m|^{4n}} = 0\,, \tag{3.10}$$

for all $a, b \in A$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. On taking $\mu = 1$ in the inequality (3.10), we may conclude that the mapping L is a quartic mapping. Also, the inequality (3.10) implies that $\Delta_\mu L(a,0) = 0$. Then we have

$$L(\mu a) = \mu^4 L(a)$$

for all $a \in A$ and $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. Let $\nu \in \mathbb{T}^1$. Then we may let $\nu = e^{i\theta}$, where $0 \le \theta \le 2\pi$, and let $\nu_1 = \nu^{\frac{1}{n_0}} = e^{\frac{i\theta}{n_0}}$. Then $\nu_1 \in \mathbb{T}^1_{\frac{1}{n_0}}$. Hence we have

$$L(\nu a) = L(\nu_1^{n_0} a) = \nu_1^{4n_0} L(a) = \nu^4 L(a)$$

for all $\nu \in \mathbb{T}^1$ and $a \in A$. Suppose that ρ is any continuous linear functional on A and a is a fixed element in A. Then we may define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(r) = \rho(L(ra))$$

for all $r \in \mathbb{R}$. It is not hard to check that the mapping g is quartic. For all $k \in \mathbb{N}$ and $r \in \mathbb{R}$, we may let

$$g_k(r) = \rho\Big(\frac{f(m^k ra)}{m^{4k}}\Big)\,.$$

We note that g is measurable because g is the pointwise limit of the sequence of measurable functions g_k. In addition, the measurable quartic function g is continuous (see [5]) and we have

$$g(r) = r^4 g(1)$$

for all $r \in \mathbb{R}$. Thus

$$\rho(L(ra)) = g(r) = r^4 g(1) = r^4 \rho(L(a)) = \rho(r^4 L(a))$$

for all $r \in \mathbb{R}$. Since ρ was an arbitrary continuous linear functional on A,

$$L(ra) = r^4 L(a)$$

for all $r \in \mathbb{R}$. Let $\omega \in \mathbb{C}$ $(\omega \ne 0)$. Then $\frac{\omega}{|\omega|} \in \mathbb{T}^1$. Hence

$$L(\omega a) = L\Big(\frac{\omega}{|\omega|}|\omega| a\Big) = \Big(\frac{\omega}{|\omega|}\Big)^4 L(|\omega| a) = \Big(\frac{\omega}{|\omega|}\Big)^4 |\omega|^4 L(a) = \omega^4 L(a)$$

for all $a \in A$. Since a was an arbitrary element in A, we may conclude that L is quartic homogeneous.

Next, replacing x by $m^k x$ and y by $m^k y$ and $z = 0$ in the inequality (3.4), we have

$$\begin{aligned} ||\Delta L(x,y)|| &= \lim_{n\to\infty} ||\frac{\Delta f(m^n x, m^n y)}{m^{4n}}|| \\ &\le \lim_{n\to\infty} \frac{1}{|m|^{4n}} \phi(0,0,m^n x, m^n y, 0) = 0 \end{aligned}$$

for all $x, y \in A$. Then we get $\Delta L(x,y) = 0$ for all $x, y \in A$. This means that L is a quartic Lie derivation. On letting $x = y = 0$ and $z = m^k z$ in the inequality (3.4), we have

$$\Big|\Big|\frac{f(m^n z^*)}{m^{4n}} - \frac{f(m^n z)^*}{m^{4n}}\Big|\Big| \le \frac{\phi(0,0,0,0,m^n z)}{|m|^{4n}} \tag{3.11}$$

for all $z \in A$. As $n \to \infty$ in the inequality (3.11), we have

$$L(z^*) = L(z)^*$$

for all $z \in A$. This means that L is a quartic Lie $*$-derivation. Now, we will show that the quartic Lie $*$-derivation is unique. Hence we assume $L' : A \to A$ is another quartic $*$-derivation satisfying the inequality (3.5). Then

$$\begin{aligned} ||L(a) - L'(a)|| &= \frac{1}{|m|^{4n}} ||L(m^n a) - L'(m^n a)|| \\ &\le \frac{1}{|m|^{4n}} \Big(||L(m^n a) - f(m^n a)|| + ||f(m^n a) - L'(m^n a)|| \Big) \\ &\le \frac{1}{|m|^{4n+1}} \sum_{j=0}^{\infty} \frac{1}{|m|^{4j}} \phi(m^{j+n} a, 0,0,0,0) \\ &= \frac{1}{|m|^4} \sum_{j=n}^{\infty} \frac{1}{|m|^{4j}} \phi(m^j a, 0,0,0,0)\,, \end{aligned}$$

which tends to zero as $k \to \infty$, for all $a \in A$. Thus $L(a) = L'(a)$ for all $a \in A$. Hence the uniqueness of L was proved, as claimed. □

COROLLARY 3.3. *Let θ, r be positive real number with $r < 4$. Suppose that $f : A \to M$ is an even mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a,b)|| \le \theta(||a||^r + ||b||^r)$$
$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \le \theta(||x||^r + ||y||^r + ||z||^r)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ *and* $a, b, x, y, z \in A$. *Then there exists a unique quartic Lie* $*$*-derivation* $L : A \to M$ *satisfying*

$$||f(a) - L(a)|| \leq \frac{\theta||a||^r}{(|m|^4 - |m|^r)}$$

for all $a \in A$.

Proof. On taking $\phi(a, b, x, y, z) = \theta(||a||^r + ||b||^r + ||x||^r + ||y||^r + ||z||^r)$ in Theorem 3.2 for all $a, b, x, y, z \in A$, we have the desired results. □

In the following corollaries, we will investigate the hyperstability for the quartic Lie $*$-derivations.

COROLLARY 3.4. *Let* r *be positive real number with* $r < 4$. *Suppose that* $f : A \to M$ *is an even mapping with* $f(0) = 0$ *such that*

$$||\Delta_\mu f(a, b)|| \leq ||a||^r ||b||^r$$

$$||\Delta f(x, y) + f(z^*) - f(z)^*|| \leq ||x||^r ||y||^r ||z||^r$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ *and* $a, b, x, y, z \in A$. *Then* f *is a quartic Lie* $*$*-derivation on* A.

Proof. If we take $\phi(a, b, x, y, z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r ||z||^r)$ in Theorem 3.2 for all $a, b, x, y, z \in A$, then we have $\widetilde{\phi}(a, 0, 0, 0, 0) = 0$. Hence (3.5) implies that f is a quartic Lie $*$-derivation on A. □

COROLLARY 3.5. *Let* r *be positive real number with* $r < 4$. *Suppose that* $f : A \to M$ *is an even mapping with* $f(0) = 0$ *such that*

$$||\Delta_\mu f(a, b)|| \leq ||a||^r ||b||^r$$

$$||\Delta f(x, y) + f(z^*) - f(z)^*|| \leq ||x||^r (||y||^r + ||z||^r)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ *and* $a, b, x, y, z \in A$. *Then* f *is a quartic Lie* $*$*-derivation on* A.

Proof. Assume that $\phi(a, b, x, y, z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r + ||z||^r)$ in Theorem 3.2 for all $a, b, x, y, z \in A$. Then $\widetilde{\phi}(a, 0, 0, 0, 0) = 0$. Hence the inequality (3.5) implies that f is a quartic Lie $*$-derivation on A. □

The following statements are relative to the alternative of fixed point; see [11] and [15]. By using this method, we will prove the Hyers-Ulam stability.

THEOREM 3.6 (The alternative of fixed point [11], [15]). *Suppose that we are given a complete generalized metric space* (Ω, d) *and a strictly contractive mapping* $T : \Omega \to \Omega$ *with Lipschitz constant* l . *Then for each given* $x \in \Omega$, *either*

$$d(T^n x,\, T^{n+1}x) = \infty \text{ for all } n \geq 0\,,$$

or there exists a natural number n_0 *such that*

1. $d(T^n x,\, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;
2. *The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T ;
3. y^* *is the unique fixed point of* T *in the set*

$$\triangle = \{y \in \Omega | d(T^{n_0}x, y) < \infty\}\,;$$

4. $d(y, y^*) \leq \frac{1}{1-l}\, d(y, Ty)$ *for all* $y \in \triangle$.

THEOREM 3.7. *Let* n_0 *be a positive integer. Suppose that* $f : A \to M$ *is a continuous even mapping with* $f(0) = 0$. *Assume that* $\phi : A^5 \to [0, \infty)$ *is a continuous mapping such that*

$$||\Delta_\mu f(a, b)|| \leq \phi(a, b, 0, 0, 0) \tag{3.12}$$

$$||\Delta f(x, y) + f(z^*) - f(z)^*|| \leq \phi(0, 0, x, y, z) \tag{3.13}$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ *and* $a, b, x, y, z \in A$. *If there is a constant* $l \in (0, 1)$ *such that*

$$\phi(ma, mb, mx, my, mz) \leq |m|^4 l \phi(a, b, x, y, z) \tag{3.14}$$

then there exists a quartic Lie $*$*-derivation* $L : A \to M$ *such that*

$$||f(a) - L(a)|| \leq \frac{1}{|m|^4(1 - l)}\phi(a, 0, 0, 0, 0) \tag{3.15}$$

for all $a, b, x, y, z \in A$.

Proof. We will consider the following set

$$\Omega = \{g \,|\, g : A \to A\,, g(0) = 0\}\,.$$

Then there is the generalized metric on Ω ,

$$d(g,\, h) = \inf\{\lambda \in (0, \infty) \,|\; \| \, g(a) - h(a) \,\| \leq \lambda\phi(a, 0, 0, 0, 0)\,, \text{for all } a \in A\}\,.$$

It is not hard to prove that (Ω, d) is a complete space. A function $T : \Omega \to \Omega$ is defined by

$$T(g)(a) = \frac{1}{m^4} g(ma) \tag{3.16}$$

for all $a \in A$. We know that there is an arbitrary constant with $d(g,h) \leq \lambda$, for all $g, h \in \Omega$, where $\lambda \in (0, \infty)$. Then

$$||g(a) - h(a)|| \leq \lambda\phi(a,0,0,0,0) \tag{3.17}$$

for all $a \in A$. On taking $a = ma$ in the inequality (3.17) and using the inequality (3.14) and the equation (3.16), we get

$$\begin{aligned} ||T(g)(a) - T(h)(a)|| &= \frac{1}{|m|^4}||g(ma) - h(ma)|| \\ &\leq \frac{1}{|m|^4}\lambda\,\phi(ma,0,0,0,0) \leq c\,l\,\phi(a,0,0,0,0)\,. \end{aligned}$$

This implies that

$$d(Tg,\, Th) \leq \lambda\, l\,.$$

Hence we have that

$$d(Tg,\, Th) \leq l\, d(g,\, h)\,,$$

for all $g, h \in \Omega$. This means that T is a strictly self-mapping of Ω with the Lipschitz constant l. On taking $\mu = 1$, $b = 0$ in the inequality (3.12), we have

$$||\frac{1}{m^4}f(ma) - f(a)|| \leq \frac{1}{|m|^4}\phi(a,0,0,0,0)$$

for all $a \in A$. This means that

$$d(Tf, f) \leq \frac{1}{|m|^4}\,.$$

Now, We will apply to Theorem of the alternative of fixed point. Since $\lim_{n\to\infty} d(T^n f, L) = 0$, we know that there exists a fixed point L of T in Ω such that

$$L(a) = \lim_{n\to\infty}\frac{f(m^n a)}{m^{4n}}\,, \tag{3.18}$$

for all $a \in A$. Hence

$$d(f, L) \leq \frac{1}{1-l}d(Tf, f) \leq \frac{1}{|m|^4}\frac{1}{1-l}\,.$$

Hence we may conclude that the inequality (3.15) holds. Since $l \in (0,1)$, the inequality (3.14) implies that

$$\lim_{n\to\infty}\frac{\phi(m^n a, m^n b, m^n x, m^n y, m^n z)}{|m|^{4n}} = 0\,. \tag{3.19}$$

Replacing a by $m^n a$ and b by $m^n b$ in the inequality (3.12), we get

$$\frac{1}{|m|^{4n}}||\Delta_\mu f(m^n a, m^n b)|| \leq \frac{\phi(m^n a, m^n b, 0, 0, 0)}{|m|^{4n}}.$$

On taking the limit as $k \to \infty$, we get $\Delta_\mu f(a,b) = 0$ and all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$. The remains of this proof are analogous to the proof in Theorem 3.2. □

Corollary 3.8. *Let θ, r be real numbers with $0 < r < 4$. Suppose that $f : A \to M$ is a mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a,b)|| \leq \theta(||a||^r + ||b||^r)$$
$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \leq \theta(||x||^r + ||y||^r + ||z||^r)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. Then there exists a unique quartic Lie $*$-derivation $L : A \to M$ satisfying

$$||f(a) - L(a)|| \leq \frac{\theta||a||^r}{|m|^4(1-l)}$$

for all $a \in A$.

Proof. The proof follows from Theorem 3.7 by taking $\phi(a,b,x,y,z) = \theta(||a||^r + ||b||^r + ||x||^r + ||y||^r + ||z||^r)$ for all $a, b, x, y, z \in A$. □

Next, we will prove the hyperstability for the quartic Lie $*$-derivations.

Corollary 3.9. *Let r be a real number with $0 < r < 4$. Suppose that $f : A \to M$ is an even mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a,b)|| \leq ||a||^r ||b||^r$$
$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \leq ||x||^r ||y||^r ||z||^r$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ and $a, b, x, y, z \in A$. Then f is a quartic Lie $*$-derivation on A.

Proof. If $\phi(a,b,x,y,z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r ||z||^r)$ in Theorem 3.7, then we get $\widetilde{\phi}(a,0,0,0,0) = 0$. Thus we may conclude that f is a quartic Lie $*$-derivation on A because of the inequality (3.15). □

Corollary 3.10. *Let r be a real number with $0 < r < 4$. Suppose that $f : A \to M$ is an even mapping with $f(0) = 0$ such that*

$$||\Delta_\mu f(a,b)|| \leq ||a||^r ||b||^r$$
$$||\Delta f(x,y) + f(z^*) - f(z)^*|| \leq ||x||^r (||y||^r + ||z||^r)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}}$ *and* $a, b, x, y, z \in A$. *Then* f *is a quartic Lie* ∗*-derivation on* A.

Proof. On letting $\phi(a, b, x, y, z) = (||a||^r + ||x||^r)(||b||^r + ||y||^r + ||z||^r)$ in Theorem 3.7, we get $\widetilde{\phi}(a, 0, 0, 0, 0) = 0$. Thus f is a quartic Lie ∗-derivation because of the inequality (3.15). □

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