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STUDYING ON A SKEW RULED SURFACE BY USING THE GEODESIC FRENET TRIHEDRON OF ITS GENERATOR

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Abstract. In this article, we study skew ruled surfaces by using the geodesic Frenet trihedron of its generator. We obtained some conditions on this surface to ensure that this ruled surface is flat, II-flat, minimal, II-minimal and Weingarten surface. Moreover, the parametric equations of asymptotic and geodesic lines on this ruled surface are determined and illustrated through example using the program of mathematica.

1. Introduction

In the surfaces theory, it is well-known that a surface is said to be "ruled" if it is generated by a continuously moving of a straight line in the space. Ruled surfaces are one of the simplest objects in geometric modeling. One important fact about ruled surfaces is that they can be generated by straight lines. One would never know this from looking at the surface or its usual equation in terms of x, y, and z coordinates, but ruled surfaces can be rewritten to highlights the generating lines. A practical application of ruled surfaces is that they are used in civil engineering. Since building materials such as wood are straight, they can be thought of as straight lines. The result is that if engineers are

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planning to construct something with curvature, they can use a ruled surface since all the lines are straight.

The study of some classes of ruled surfaces with special properties in Euclidean 3-space such as developable, skew (non-developable), minimal, II-minimal, and II-flat is one of the principal aims of the classical differential geometry. This kind of surfaces has an important role and many applications in different fields, such as Physics, Computer Aided Geometric Design and the study of design problems in spatial mechanism, etc [12,13]. There are many studies that interested with many properties of these surfaces in Euclidean space and some characterizations [5, 11]. Furthermore, many geometers have studied some of the differential geometric concepts of the ruled surfaces in Minkowski space [1–3, 7–9].

In this paper, we consider a ruled surface for which the tangent of its base curve given as a linear combination of the Geodesic Frenet trihedron. The condition for skew ruled surfaces to be flat, minimal, II-flat, II-minimal and Weingarten surfaces are obtained. Moreover, the asymptotic and geodesic lines are determined and solved under some conditions with respect to skew ruled surfaces. Finally, example has been given related to the subject.

2. Preliminaries

In this section, we briefly review differential geometry of surfaces in so much as it is related to the developments in the subsequent section. More details can be found for example in [6, 10, 14].

A ruled surface is generated by a one-parameter family of straight lines and it possesses a parametric representation,

$$\psi(s, v) = \vec{\alpha}(s) + v\vec{e}(s), \tag{2.1}$$

where $\vec{\alpha}(s)$ represents a space curve which is called the base curve and $\vec{e}(s)$ is a unit vector representing the direction of a straight line and s is the arc-length along the base curve $\vec{\alpha}(s)$. Let $\vec{n}$ denotes the unit normal vector field on the surface (2.1) which is given by

$$\vec{n} = \frac{\vec{\psi}_s \wedge \vec{\psi}_v}{|\vec{\psi}_s \wedge \vec{\psi}_v|}, \qquad \vec{\psi}_s = \frac{\partial \psi}{\partial s}, \qquad \vec{\psi}_v = \frac{\partial \psi}{\partial v}, \tag{2.2}$$

where $\wedge$ stand the cross product in Euclidean 3-space. Then the metric I of the surface (2.1) is defined as

$$I = g_{11}ds^2 + 2g_{12}dsdv + g_{22}dv^2, \tag{2.3}$$

with differentiable coefficients

$$g_{11} = \langle \vec{\psi}_s, \vec{\psi}_s \rangle, \quad g_{12} = \langle \vec{\psi}_s, \vec{\psi}_v \rangle, \quad g_{22} = \langle \vec{\psi}_v, \vec{\psi}_v \rangle, \tag{2.4}$$

where $\langle , \rangle$ is the Euclidean inner product in Euclidean 3-space. The discriminate g of the first fundamental form I is given as

$$g = Det(g_{ij}) = g_{11}g_{22} - (g_{12})^2. \tag{2.5}$$

The second fundamental form II of the surface (2.1) is given by

$$II = h_{11}ds^2 + 2h_{12}dsdv + h_{22}dv^2, \tag{2.6}$$

with differentiable coefficients

$$h_{11} = \langle \vec{\psi}_{ss}, \vec{n} \rangle, \quad h_{12} = \langle \vec{\psi}_{sv}, \vec{n} \rangle, \quad h_{22} = \langle \vec{\psi}_{vv}, \vec{n} \rangle. \tag{2.7}$$

The discriminate h of the second fundamental form II is

$$h = Det(h_{ij}) = h_{11}h_{22} - (h_{12})^2. \tag{2.8}$$

For the parametrization of the ruled surface (2.1), we have the mean curvature H and the Gaussian curvature K as follows

$$H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2g}, \quad K = \frac{h}{g}. \tag{2.9}$$

Using classical notation, we define the second Gaussian curvature K_{II} by [5]

$$K_{II} = \frac{1}{h^2}\left(\begin{vmatrix} \frac{-h_{11,22}}{2} + h_{12,12} - \frac{h_{22,11}}{2} & \frac{h_{11,1}}{2} & h_{12.1} - \frac{h_{11,2}}{2} \\ h_{12,2} - \frac{h_{22,1}}{2} & h_{11} & h_{12} \\ \frac{h_{22,2}}{2} & h_{12} & h_{22} \end{vmatrix} - \begin{vmatrix} 0 & \frac{h_{11,2}}{2} & \frac{h_{22,1}}{2} \\ \frac{h_{11,2}}{2} & h_{11} & h_{12} \\ \frac{h_{22,1}}{2} & h_{12} & h_{22} \end{vmatrix} \right), \tag{2.10}$$

where, $h_{ij,l} = \frac{\partial hij}{\partial s^l}$, and $h_{ij,lm} = \frac{\partial^2 hij}{\partial u^l \partial u^m}$, the indices i, j belong to $\{1, 2\}$ and the parameters u^1, u^2 are s, v, respectively.

Since Brioschis formulas in Euclidean 3-spaces, we are able to define the second mean curvature H_{II} of the surface (2.1) by replacing the components of the first fundamental form g_{ij} by the components of the second fundamental form h_{ij} respectively in Brioschis formula. Consequently, the second mean curvature H_{II} is given by [4]

$$H_{II} = H - \frac{1}{2}\Delta\big(\ln\sqrt{|K|}\,\big), \tag{2.11}$$

where, Δ is the Laplacian with respect to the second fundamental form of the surface (2.1), expressed as

$$\Delta = \frac{1}{\sqrt{|h|}} \sum_{i,j} \frac{\partial}{\partial u^i} \Big[\sqrt{|h|}\, h^{ij} \frac{\partial}{\partial u^j} \Big], \tag{2.12}$$

where, h^{ij} denotes the associated matrix with its inverse h_{ij}.

3. Geodesic Frenet trihedron of the ruled surface

For the ruled surface defined by the representation (2.1), the vector $\vec{e}(s)$ traces a general space curve (as s varies) on the surface of unit sphere $s^2(1)$ called *spherical indicatrix* of the ruled surface. If we denote the arc length of $\vec{e}(s)$ as s^*, then

$$s^* = \int_a^b \Big| \frac{d\vec{e}(s)}{ds} \Big| ds \tag{3.1}$$

The unit surface normal, $\vec{N}$, to a ruled surface represented by Eq.(2.1) is then

$$\vec{N}(s,v) = \frac{(\frac{d\vec{\alpha}}{ds} + v\frac{d\vec{e}}{ds}) \wedge \vec{e}}{[(\frac{d\vec{\alpha}}{ds} + v\frac{d\vec{e}}{ds})^2 - \langle \frac{d\vec{\alpha}}{ds}, \vec{e} \rangle^2]^{\frac{1}{2}}}, \tag{3.2}$$

The unit normal along a general generator $\vec{l} = \vec{\psi}(s_0, v)$ of the ruled surface approaches a limiting direction as v infinitely decreases. This direction is called *the asymptotic normal* direction and is defined as

$$\vec{g}(s)\Big|_{s=s_0} = \vec{N}(s,v)\Big|_{\substack{s=s_0 \\ v\to-\infty}} = \frac{-\frac{d\vec{e}}{ds} \wedge \vec{e}}{|\frac{d\vec{e}}{ds}|}\Big|_{s=s_0}. \tag{3.3}$$

At v increases to $+\infty$, the unit normal rotates through 180° about $\vec{l}$ and ultimately takes the direction $-\vec{g}$. The point at which $\vec{N}$ has rotated only 90° and is perpendicular to $\vec{g}$ is called the striction point (or central point) on $\vec{l}$. The direction of $\vec{N}$ at this point is denoted by $\vec{t}$ and called *the central normal* of the ruled surface and is given by

$$\vec{t}(s) = \frac{d\vec{e}/ds}{|d\vec{e}/ds|}. \tag{3.4}$$

The Frenet trihedron on a ruled surface can then be defined by the dexterous triplet of vectors $\{\vec{e}, \vec{t}, \vec{g}\}$, where

$$\begin{aligned} \vec{e} &= \textit{spherical indicatrix}, \\ \vec{t} &= \textit{central normal} = \vec{e}' = \frac{d\vec{e}}{ds^*} = \frac{\vec{e_s}}{|\vec{e_s}|}, \\ \vec{g} &= \textit{asymptotic normal} = \vec{e} \wedge \vec{e}' = \frac{\vec{e} \wedge \vec{e_s}}{|\vec{e_s}|}. \end{aligned} \tag{3.5}$$

where $\vec{e_s} = \frac{d\vec{e}}{ds}$ and $' \equiv \frac{d}{ds^*}$. Differentiating the last two vector fields with respect to s^*, we arrive at a set of formulas similar to the Frenet formulas of a space curve, namely

$$\begin{aligned} \vec{e}' &= \vec{t}, \\ \vec{t}' &= \mu\vec{g} - \vec{e}, \\ \vec{g}' &= \mu\vec{t}, \end{aligned} \tag{3.6}$$

where $\mu = \frac{\langle \vec{e}, \vec{e}_s \wedge \vec{e}_{ss} \rangle}{|\vec{e}_s|^3}$ is the geodesic curvature of spherical indicatrix $\vec{e}$. These differentials are called *the geodesic Frenet trihedron formulas* of the indicatrix $\vec{e}$ for a ruled surface.

Definition 3.1. The striction point on a ruled surface ϕ is the foot of the common normal between two consecutive generators. The set of striction points defines *the striction curve* which is given by

$$\vec{c}(s) = \vec{\alpha}(s) - \frac{\langle \vec{\alpha}_s, \vec{e}_s \rangle}{\|\vec{e}_s\|^2}\, \vec{e}(s) \tag{3.7}$$

If consecutive generators of a ruled surface intersect, then the surface is said to be developable, otherwise the surface is said to be skew.

In this paper, the striction curve of the ruled surface ψ will be taken as the base curve. In this case, the parametric equation of the surface (2.1), can be written as

$$M : \phi(s, v) = \vec{c}(s) + v\vec{e}(s), \tag{3.8}$$

where the tangent of the base curve of M is given by

$$\vec{c}' = \lambda_1\vec{e} + \lambda_2\vec{t} + \lambda_3\vec{g} \ \in span\{\vec{e}, \vec{t}, \vec{g}\}, \tag{3.9}$$

with $\langle \vec{c'}, \vec{c'} \rangle$=1, implies that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$, where λ_i are constants for i=1,2,3.
Since the base curve of M is a striction curve, then we get

$$\langle \vec{c'}, \vec{e'} \rangle = \langle \vec{c'}, \vec{t} \rangle = 0 \tag{3.10}$$

Hence, Eq.(3.9) becomes

$$\vec{c'} = \lambda_1 \vec{e} + \lambda_3 \vec{g}. \tag{3.11}$$

The distribution parameter of the ruled surface M is defined by

$$P_e = \frac{\det(\vec{c'}, \vec{e}, \vec{e'})}{\langle \vec{e'}, \vec{e'} \rangle} = \lambda_3. \tag{3.12}$$

Since the ruled surface M defined by Eq. (3.8) is a skew ruled surface, then $\lambda_3 \neq 0$

4. Skew ruled surfaces

In this section, we study the skew ruled surface (3.8) and get the conditions for the surface M to be minimal, II-minimal, II-flat and Weingarten surface. For the skew ruled surface M, using Eqs.(3.6)and (3.11), the tangents of the parametric curves of the surface M are given by

$$\vec{\phi}_s = \lambda_1 \vec{e} + v\vec{t} + \lambda_3 \vec{g}, \quad \vec{\phi}_v = \vec{e}. \tag{4.1}$$

Thus, the first fundamental quantities are given as follows

$$g_{11} = \langle \phi_s, \phi_s \rangle = 1 + v^2, \quad g_{12} = \langle \phi_s, \phi_v \rangle = \lambda_1, \quad g_{22} = \langle \phi_v, \phi_v \rangle = 1. \tag{4.2}$$

Hence, the discriminate g of the first fundamental form of M is

$$g = v^2 + \lambda_3^2. \tag{4.3}$$

As an immediate result we have the following

Corollary 4.1. *The only singular points on the ruled surface M are along the points of striction curve(v=0) for which P_e=0.*

Using Eq. (4.1), one can get the the unit normal vector field of M in the form

$$\vec{n}(s,v) = \frac{\vec{N}}{\|\vec{N}\|} = \frac{\lambda_3 \vec{t} - v\vec{g}}{\sqrt{v^2 + \lambda_3^2}}. \tag{4.4}$$

Moreover, the principal normal vector $\vec{e_2}$ of M at the base curve(v=0) is

$$\vec{e_2} = \vec{n}(s,0) = \vec{t}. \tag{4.5}$$

From Eqs. (3.11) and (4.5), the binormal vector $\vec{e_3}$ of the curve $\vec{c}$ can be performed as

$$\vec{e_3} = \vec{e_1} \wedge \vec{e_2} = \lambda_1 \vec{g} - \lambda_3 \vec{e}. \tag{4.6}$$

From above, we can get the equations that describe the relation between Frenet-Frame $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ of the base curve $\vec{c}$ and the geodesic Frenet trihedron $\{\vec{e}, \vec{t}, \vec{g}\}$ of the indicatrix $\vec{e}$ of M in the form

$$\begin{pmatrix} \vec{e_1} \\ \vec{e_2} \\ \vec{e_3} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & 1 & 0 \\ -\lambda_3 & 0 & \lambda_1 \end{pmatrix} \begin{pmatrix} \vec{e} \\ \vec{t} \\ \vec{g} \end{pmatrix} \tag{4.7}$$

Using Eqs. (2.3) and (4.2), one can get the first fundamental form of M in the form

$$I = (1+v^2)ds^2 + 2\lambda_1 ds dv + dv^2, \tag{4.8}$$

The second derivatives for the function ϕ on M are given in the form

$$\vec{\phi}_{ss} = -v\vec{e} + (\lambda_1 - \mu\lambda_3)\vec{t} + \mu v \vec{g}, \quad \vec{\phi}_{sv} = \vec{t}, \quad \vec{\phi}_{vv} = 0. \tag{4.9}$$

Hence, we can reach the second fundamental quantities by using (2.7) as follows

$$h_{11} = \frac{\lambda_1\lambda_3 - \mu(\lambda_3^2 + v^2)}{\sqrt{\lambda_3^2 + v^2}}, \quad h_{12} = \frac{\lambda_3}{\sqrt{\lambda_3^2 + v^2}}, \quad h_{22} = 0, \quad h = \frac{-\lambda_3^2}{\lambda_3^2 + v^2}. \tag{4.10}$$

Using Eqs. (2.6) and (4.10), the second fundamental form of M is

$$II = \frac{\Big(\lambda_1\lambda_3 - \mu\left(\lambda_3^2 + v^2\right)\Big)ds^2 + 2\lambda_3 ds\, dv}{\sqrt{\lambda_3^2 + v^2}}. \tag{4.11}$$

Substituting from Eqs. (4.3) and (4.10) in Eq. (2.9), the Gaussian curvature K of the ruled surface M is

$$K = \frac{-\lambda_3^2}{(\lambda_3^2 + v^2)^2} \tag{4.12}$$

Indeed we have the following theorem:

THEOREM 4.1. *The Gaussian curvature K of the ruled surface M is non positive and K equal to zero only along the ruling which meet the striction curve at a sigular point*
($P_e = 0$, $v \neq 0$).

Using Eqs. (4.2) and (4.10), the mean curvature H of the ruled surface M is given by

$$H = \frac{-\lambda_1\lambda_3 + \mu(\lambda_3^2 - v^2)}{2(\lambda_3^2 + v^2)^{3/2}} \tag{4.13}$$

Hence, we have the following

Theorem 4.2. *The ruled surface M is minimal if the spherical indicatrix $\vec{e}$ is a geodesic and the base curve is parallel to asymptotic normal $\vec{g}$.*

From Eq.(4.13), if the spherical indicatrix $\vec{e}$ with constant geodesic curvature, then the mean curvature is a function of the variable v only, i.e., $H = H(v)$. Thus, using Eqs. (4.12) and (4.13), we have the following

Corollary 4.2. *If the spherical indicatrix $\vec{e}$ of M with constant geodesic curvature, then the surface M is Weingarten surface or, equivalently, the corresponding Jacobian determinant is identically zero, i.e.,*

$$\varphi(K, H) = \left|\frac{\partial(K, H)}{\partial(s, v)}\right| \equiv 0. \tag{4.14}$$

Using Eqs.(2.10) and (2.11) , it is easy to see that the second Gaussian curvature K_{II} and the second mean curvature H_{II} of the ruled surface M are, respectively

$$K_{II} = \frac{-\lambda_1\lambda_3(\lambda_3^2 - v^2) + \mu(\lambda_3^2 + v^2)}{2\lambda_3^2(\lambda_3^2 + v^2)^2}. \tag{4.15}$$

$$H_{II} = \frac{1}{2}\Big[\frac{-\lambda_1\lambda_3 - \mu(\lambda_3^2 + v^2)}{2(\lambda_3^2 + v^2)^{3/2}} + \frac{\mu^2(\lambda_3^2 + v^2)^3 + \lambda_3\mu'[-\lambda_1^2\lambda_3(\lambda_3^2 - 3v^2) + v(v^{2+\lambda_3^2})]}{\lambda_3^4(\lambda_3^2 + v^2)^2}\Big]. \tag{4.16}$$

Corollary 4.3. *The ruled surface M is II-flat and II-minimal if the spherical indicatrix $\vec{e}$ is a geodesic and the base curve is parallel to asymptotic normal $\vec{g}$.*

4.1. Orthogonal trajectory of the rulings. Let $\vec{\gamma}$ be a regular curve on M, then it can be expressed as

$$\vec{\gamma}(s) = \vec{c}(s) + v(s)\,\vec{e}(s). \tag{4.17}$$

If point P displaced orthogonally along the ruling $\vec{e}$ to a neighbouring point P_0, then we have an orthogonal trajectory

$$\vec{\gamma}\colon \mathrm{I} \to \phi(s, v)$$

The condition that the point P be displaced orthogonally to the ruling is

$$\langle \vec{\gamma}', \vec{e} \rangle = 0. \tag{4.18}$$

More explicitly,

$$\lambda_1 + v' = 0. \tag{4.19}$$

Integrating, we have

$$v = -\int \lambda_1 ds + c_1, \tag{4.20}$$

where c_1 is an arbitrary real constant. The points on the base curve $\vec{c}(s)$ from which the arc length is measured may be chosen to make $c_1 = 0$, then the distance v to a particular orthogonal trajectory is given by

$$v = -\lambda_1 s. \tag{4.21}$$

As an immediate result we have the following

Corollary 4.4. *The distance v along a ruling from the base curve curve $\vec{c}(s)$ to an orthogonal trajectory $\vec{\gamma}$ is proportional to the arc length of the base curve.*

5. Curves on skew ruled surfaces

In this section, we introduce the parametric equations which determine the asymptotic and the geodesic lines on the ruled surface M. Finally, example is given to show the asymptotic and the geodesic curves on the skew ruled surfaces under investigation.

5.1. The geodesic torsion. Consider a regular curve $\vec{\gamma}$ on the the surface M defined by Eq. (4.17). Since $\vec{n}$ is the unit normal vector field of the surface M, then the geodesic torsion τ_g for the curve $\vec{\gamma}$ is given by

$$\tau_g = \frac{[\vec{\gamma}', \vec{n}, \vec{n}']}{|\vec{\gamma}'|}. \tag{5.1}$$

Taking the derivative of the curve $\vec{\gamma} = \vec{\gamma}(s)$ and the unit normal vector $\vec{n} = \vec{n}(s, v(s))$ with respect to the arc length s^*, respectively, we obtain

$$\vec{\gamma}' = (\lambda_1 + v')\vec{e} + v\vec{t} + \lambda_3 \vec{g}. \tag{5.2}$$

$$\vec{n}' = \frac{(-\lambda_3^2 - \lambda_3 v^2)\vec{e} + (\lambda_3^2 \mu v + \mu v^3 - \lambda_3 v v')\vec{t} + (\lambda_3^2 \mu + \lambda_3 \mu v^2 + v^2 v')\vec{g}}{(\lambda_3^2 + v^2)^{3/2}}. \tag{5.3}$$

Thus, we have

$$\tau_g = \frac{(\lambda_1 + v^{'})(\lambda_3^4\mu + \lambda_3^2\mu v^2 + \lambda_3^2\mu v^2 + \mu v^4) + (\lambda_3^2 + \lambda_3^2 v^2)(\lambda_3^2 + v^2)}{(\lambda_3^2 + v^2)^2\big[(\lambda_1 + v^{'})^2 + \lambda_3^2 + v^2\big]^{1/2}} \tag{5.4}$$

REMARK 5.1. The geodesic torsion for the base curve $\vec{c}$ ($v = 0$) on M is

$$\tau_g\Big|_{v=0} = \lambda_1\mu + \lambda_3 \tag{5.5}$$

5.2. The normal curvature. For the curve $\vec{\gamma}$ on the the surface M defined by Eq. (4.17), the normal curvature k_n is given by

$$k_n = \frac{\langle \vec{\gamma}^{''}, \vec{n}\rangle}{|\vec{\gamma}^{'}|^2}. \tag{5.6}$$

Taking the derivative of Eq. (5.2), we have

$$\vec{\gamma}^{''} = (v^{''} - v)\vec{e} + (\lambda_1 + 2v^{'} - \lambda_3\mu)\vec{t} + \mu v\vec{g}. \tag{5.7}$$

Hence, we get

$$k_n = \frac{\lambda_1\lambda_3 + 2\lambda_3 v^{'} - \mu(\lambda_3^2 + v^2)}{(\lambda_3^2 + v^2)^{1/2}\big[(\lambda_1 + v^{'})^2 + \lambda_3^2 + v^2\big]} \tag{5.8}$$

From Eq. (5.8), the differential equation of the asymptotic lines ($k_n = 0$) on the surface M is given by

$$2\lambda_3 v^{'} - \mu(\lambda_3^2 + v^2) + \lambda_1\lambda_3 = 0. \tag{5.9}$$

In general, the linear differential equation (5.9) can not be solved analytically. Hence, let us assume that λ_1=0 and λ_3=1.
Under this assumption, the surface M is a skew and hence, Eq. (5.9) takes the form

$$2v^{'} - \mu(1 + v^2) = 0 \tag{5.10}$$

It is well known that the solution of the Eq. (5.10) is

$$v = \tan(\frac{1}{2}\int \mu ds + c_2), \tag{5.11}$$

where c_2 is an arbitrary constant. From Eqs. (4.17) and (5.11), the equation of asymptotic line on M is

$$\vec{\gamma}_1(s) = \vec{c}(s) + \tan(\frac{1}{2}\int \mu ds + c_2)\,\vec{e}(s), \tag{5.12}$$

as shown in figure (1). Then we have the following

THEOREM 5.1. *For the skew ruled surface M, if the base curve is parallel to the asymptotic normal of the spherical indicatrix $\vec{e}$, then the asymptotic lines on the ruled surface M is given by Eq. (5.12).*

REMARK 5.2. The normal curvature for the base curve $\vec{c}$ $(v = 0)$ on M is

$$k_n\Big|_{v=0} = \lambda_1 - \mu\lambda_3 \tag{5.13}$$

5.3. The geodesic curvature. Consider the regular curve $\vec{\gamma}$ on the the surface M defined by Eq. (4.17). Since $\vec{n}$ is the unit normal vector field of the surface M, then the geodesic curvature k_g for the curve $\vec{\gamma}$ is given by

$$k_g = \frac{[\vec{\gamma}'', \vec{\gamma}', \vec{n}]}{|\vec{\gamma}'|^3}. \tag{5.14}$$

Using Eqs. (4.4), (5.2) and (5.7), we get

$$k_g = \frac{(v'' - v)(-v^2 - \lambda_3^2) - (\lambda_1 + v')(-\lambda_1 v - 2vv')}{(\lambda_3^2 + v^2)^{1/2}\left[(\lambda_1 + v')^2 + \lambda_3^2 + v^2\right]^{3/2}}. \tag{5.15}$$

From Eq. (5.15), the differential equation of the geodesic curves on the surface M is given by

$$(\lambda_3^2 + v^2)v'' - 2vv'^2 - 3\lambda_1 vv' - (v^2 + 1)v = 0. \tag{5.16}$$

In general, the non-linear differential equation (5.16) can not be solved analytically. Hence, let us assume that λ_1=0 and λ_3=1. Under this assumption, the surface M is a skew and hence, Eq. (5.16) takes the form

$$(1 + v^2)v'' - 2vv'^2 - (v^2 + 1)v = 0. \tag{5.17}$$

After some calculations, we can get the following solution for the differential equation (5.17)

$$v = \pm\sinh(s - c_4). \tag{5.18}$$

where c_4 is an arbitrary constant. From Eqs. (4.17) and (5.18), the equation of a geodesic curve on M is

$$\vec{\gamma}_2(s) = \vec{c}(s) + \sinh(s - c_4)\,\vec{e}(s), \tag{5.19}$$

as shown in figure (2). Thus, we have the following

THEOREM 5.2. *For the skew ruled surface M, if the base curve is parallel to the asymptotic normal of the spherical indicatrix $\vec{e}$, then the geodesic lines on the ruled surface M is given by Eq. (5.19).*

REMARK 5.3. The geodesic curvature for the base curve $\vec{c}\,(v=0)$ on the ruled surface M is given by

$$k_g\Big|_{v=0} = 0 \tag{5.20}$$

Since the geodesic curvature k_g and the normal curvature k_n of the base curve satisfy the relation $k^2 = k_g^2 + k_n^2$, then we have

COROLLARY 5.1. *The curvature k of the base curve $\vec{c}$ of the ruled surface M is equal the normal curvature k_n, i.e.,*

$$k = k_n\Big|_{v=0} = \lambda_1 - \mu\lambda_3. \tag{5.21}$$

EXAMPLE 5.3.1. Consider the elliptic hyperboloid of one sheet parameterized by

$$\phi(s,v) = \frac{\sqrt{2}}{2}\Big(\cos(s) - v\sin(s), \sin(s) + v\cos(s), s+v\Big), \tag{5.22}$$

with λ_1=0 and λ_3=1. Short calculations give us the following $\vec{e} = \frac{\sqrt{2}}{2}\big(-\sin(s), \cos(s), 1\big)$, $\vec{t} = \big(-\cos(s), -\sin(s), 0\big)$ and $\vec{g} = \frac{\sqrt{2}}{2}\big(-\sin(s), -\cos(s), 1\big)$, From Eq. (5.12), it follows that the equation of asymptotic lines on the surface M is

$$\vec{\gamma}_1(s) = \frac{1}{\sqrt{2}}\Big(-\sin(s)\tan\big(c+\frac{s}{2}\big)-\cos(s), \sin\big(c-\frac{s}{2}\big)\sec\big(c+\frac{s}{2}\big), \tan\big(c+\frac{s}{2}\big)+s\Big), \tag{5.23}$$

(see fig. 1). Also, from Eq. (5.18), it follows that the equation of a geodesic curve on the surface M is given as

$$\vec{\gamma}_2(s) = \frac{1}{\sqrt{2}}\Big(-\cos(s)-\sqrt{\sinh(s^2)}\sin(s), \sqrt{\sinh(s^2)}\cos(s)-\sin(s), \sqrt{\sinh(s^2)}+s\Big), \tag{5.24}$$

(see fig. 2).

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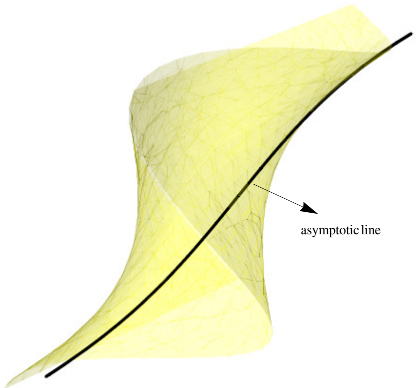

FIGURE 1. Skew ruled surface M and its asymptotic line.

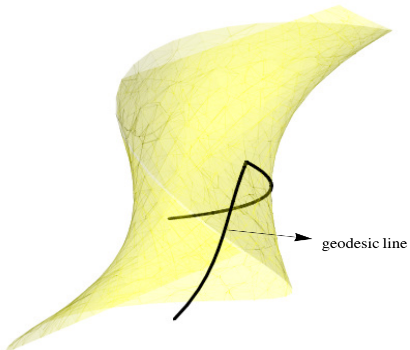

FIGURE 2. Skew ruled surface M and its geodesic line.

References

[1] Abdel-All, N. H. and Hamdoon, F. M., *Extension B-scroll surfaces in Lorentz 3-dimensional space*, Rivista Di Math., Parma **6**(3) (2000), 57–67.

[2] Abdel-All, N. H. and Hamdoon, F. M., *Timelike ruled surfaces immersed in Lorentz space*, International conference Centennial Vranceanu Bucharest, July (2000) (Vranceanu proceedings March 2001).

[3] Abdel-All, N. H., Abdel-Baky, R. A. and Hamdoon, F. M., *Ruled surfaces with timelike rullings*, Appl. Math. comput., **147** (2004), 241–253.

[4] Abd-Ellah, H. N., *Translation L/W-surfaces in Euclidean 3-space* E^3, Journal of the Egyptian Mathematical Society, **23** (2015) 513–517.

[5] Ali, A., Abdel Aziz, H. and Sorour, A., *Ruled surfaces generated by some special curves in Euclidean 3-Space*, Journal of the Egyptian Mathematical Society, **21** (2013), 285–294.

[6] Carmo, M. P., *Differential geometry of curves and surfaces*, Prentice- Hall, Englewood Cliffs, NJ, (1976).

[7] Dillen, F. and Sodsiri, W., *Ruled surfaces of Weingarten type in Minkowski 3-space*, J. Geom., **83** (2005) 10–21.

[8] Kim, Y. H. and Yoon, D. W., *On Non-Developable Ruled Surfaces in Lorentz-Minkoweski 3- Space*, Taiwanese Journal of mathematics, **1**(11) (2007), 197–214.

[9] Kim, Y. H. and Yoon, D. W., *Classification of ruled surfaces in Minkowski 3-space*, J. Geometry Phys., **49** (2004) 89–100.

[10] O'Neill, B., *Elementary Differential geometry*, Academic Press ,(1966).

[11] Onder, M and Huseyin, H., *Some Results and Characterizations for Mannheim Offsets of Ruled Surfaces*, Bol. Soc. Paran. Mat., **34** (2015), 85–95.

[12] Orbay, K. and Kasap, E., *Mannheim offsets of ruled surfaces*, Mathematical proplems in engineering, (2009).

[13] Ravani, B. and Ku, T., *Bertrand offsets of ruled and developable surfaces*, Computer-Aided Design, **2**(23) (1991), 145–152.

[14] Weatherburn, C. E., *Differential Geometry of Three Dimensions*, Cambridge, UK, (1930).

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