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SOME PROPERTIES OF THE GENERALIZED FIBONACCI SEQUENCE $\{q_n\}$ BY MATRIX METHODS

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Abstract. In this note, we consider a generalized Fibonacci sequence $\{q_n\}$. We give a generating matrix for $\{q_n\}$. With the aid of this matrix, we derive and re-prove some properties involving terms of this sequence

1. Introduction

For any integer $n \geq 0$, the well-known Fibonacci sequence $\{F_n\}$ is defined by the second order linear recurrence relation $F_{n+2} = F_{n+1} + F_n$, where $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has been generalized in many ways, for example, by changing the recurrence relation (see [9]), by changing the initial values (see [6, 7]), by combining of these two techniques (see [5]), and so on.

In [4], Edson and Yayenie defined a further generalized Fibonacci sequence $\{q_n\}$ depending on two real parameters used in a non-linear (piecewise linear) recurrence relation, namely,

$$q_n = a^{1-\xi(n)} b^{\xi(n)} q_{n-1} + q_{n-2} \ (n \geq 2) \tag{1}$$

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with initial values $q_0 = 0$ and $q_1 = 1$, where a and b are positive real numbers and

$$\xi(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \tag{2}$$

is the parity function. Also, the authors showed that the terms of the sequence $\{q_n\}$ are given by the extended Binet's formula

$$q_n = \left(\frac{a^{1-\xi(n)}}{(ab)^{\frac{n-\xi(n)}{2}}} \right) \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{3}$$

where α and β are roots of the quadratic equation $x^2 - abx - ab = 0$ and $\alpha > \beta$.

These sequences arise in a natural way in the study of continued fractions of quadratic irrationals (see [3]) and combinatorics on words or dynamical system theory. Some well-known sequences are special cases of this generalization. The Fibonacci sequence is a special case of $\{q_n\}$ with $a = b = 1$. Pell sequence is $\{q_n\}$ with $a = b = 2$ and the k-Fibonacci sequence is $\{q_n\}$ with $a = b = k$.

Using the extended Binet's formula (3), Edson and Yayenie [4] derived a number of mathematical properties including generalizations of Cassini's, Catalan's and d'Ocagne's identities for the Fibonacci sequence, Yayenie [12] obtained numerous new identities of $\{q_n\}$, and Zhang and Wu [13] studied the partial infinite sums of reciprocal of $\{q_n\}$.

Matrix methods are very convenient for deriving certain properties of linear recurrence sequences. Some authors have used matrix methods or other methods to derive some identities, combinatorial representations of linear recurrence relations etc. In [11], the author formulated the nth power of an arbitrary 2×2 matrix. In [10], the author considered a 2×2 companion matrix and he derived some known relations involving Fibonacci numbers. In [2], the author gave the matrix method for generating the Pell sequence. Also in [1] the matrix method is used for the case of the k-Fibonacci and k-Lucas sequences.

In this paper, we investigate some properties of the sequence $\{q_n\}$ by matrix methods. In section 2, we give a generating matrix for the terms of the sequence $\{q_n\}$. With the aid of this matrix, in section 3, we derive or re-prove some properties involving the terms of this sequence.

2. Matrix representations for the sequence $\{q_n\}$

In this section, we define a 2×2 matrix M and we give the nth power M^n for any integer n.

We need the following lemma.

LEMMA 2.1. *Let $\{q_n\}, a, b, \xi(n), \alpha, \beta$ be as in (1), (2) and (3). For any integer $n \geq 1$, we have*

$$\alpha^n = a^{\frac{n-2+\xi(n)}{2}} b^{\frac{n-\xi(n)}{2}} q_n \alpha + a^{\frac{n-\xi(n)}{2}} b^{\frac{n+\xi(n)}{2}} q_{n-1}. \tag{4}$$

Proof. The identity (4) holds for $n = 1$ from the conditions $\xi(1) = 1$, $q_0 = 0$ and $q_1 = 1$. Let $n \geq 2$. Using the extended Binet's formula (3), we have

$$\begin{aligned}
& q_n - \frac{\beta^2}{ab} q_{n-2} \\
= & \left(\frac{a^{1-\xi(n)}}{(ab)^{\frac{n-\xi(n)}{2}}} \right) \frac{\alpha^n - \beta^n}{\alpha - \beta} - \frac{\beta^2}{ab} \left(\frac{a^{1-\xi(n-2)}}{(ab)^{\frac{n-2-\xi(n-2)}{2}}} \right) \frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta} \\
= & \left(\frac{a^{1-\xi(n)}}{(ab)^{\frac{n-\xi(n)}{2}}} \right) \left\{ \frac{\alpha^n - \beta^n}{\alpha - \beta} - \frac{\beta^2 \left(\alpha^{n-2} - \beta^{n-2} \right)}{\alpha - \beta} \right\} \ (\because \ \xi(n-2) = \xi(n)) \\
= & \left(\frac{a^{1-\xi(n)}}{(ab)^{\frac{n-\xi(n)}{2}}} \right) \left\{ \frac{\alpha^{n-2}(\alpha^2 - \beta^2)}{\alpha - \beta} \right\} \\
= & \frac{a^{1-\xi(n)}}{(ab)^{\frac{n-\xi(n)}{2}-1}} \alpha^{n-2} \ (\because \ \alpha + \beta = ab).
\end{aligned} \tag{5}$$

Since $\alpha^2 - ab\alpha - ab = 0$, multiplying (5) by $\frac{\alpha^2}{ab} = \alpha + 1$ and using $\alpha\beta = -ab$, we have

$$q_n \alpha + (q_n - q_{n-2}) = \frac{a^{1-\xi(n)}}{(ab)^{\frac{n-\xi(n)}{2}}} \alpha^n. \tag{6}$$

Using (1), multiplying (6) by $\frac{(ab)^{\frac{n-\xi(n)}{2}}}{a^{1-\xi(n)}}$, we obtain the identity (4). □

For any positive real number k, if $a = b = k$, then $\{q_n\}$ is the k-Fibonacci sequence $\{f_{k,n}\}$. The Fibonacci sequence is a special case of $\{f_{k,n}\}$ with $k = 1$. Pell sequence is $\{f_{k,n}\}$ with $k = 2$. In this case, let

$Q = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}$ be a companion matrix of $\{f_{k,n}\}$ and

$$\phi = \frac{1}{2}(k + \sqrt{k^2+4}),\ \varphi = \frac{1}{2}(k - \sqrt{k^2+4})$$

be the roots of the quadratic equation $x^2 - kx - 1 = 0$ provided $k^2 + 4 \neq 0$. Then

$$M = \begin{pmatrix} k^2 & k \\ k & 0 \end{pmatrix} = kQ,\ \alpha = \frac{1}{2}(k^2 + \sqrt{k^4+4k^2}) = k\phi,\ \beta = \frac{1}{2}(k^2 - \sqrt{k^4+4k^2}) = k\varphi,$$

$$a^{\frac{n-2+\xi(n)}{2}} b^{\frac{n-\xi(n)}{2}} = k^{n-1},\ a^{\frac{n-\xi(n)}{2}} b^{\frac{n+\xi(n)}{2}} = k^n.$$

Thus, the result in Lemma 2.1 reduces to a known identity of k-Fibonacci numbers

$$\phi^n = f_{k,n}\phi + f_{k,n-1}\ (n \geq 1). \tag{7}$$

Let I be the 2×2 identity matrix. In (7), if we change ϕ into the matrix Q and change $f_{k,n-1}$ into the matrix $f_{k,n-1}I$, then the matrix form of the nth power Q^n is given by

$$Q^n = f_{k,n}Q + f_{k,n-1}I = \begin{pmatrix} f_{k,n+1} & f_{k,n} \\ f_{k,n} & f_{k,n-1} \end{pmatrix} \text{ (see Proposition 2 in [1])},$$

which is proved by an inductive argument (see [1, 8–10]).

In a similar way, the sequence $\{q_n\}$ can also be generated by matrix multiplication as follows.

THEOREM 2.2. *Let* $M = \begin{pmatrix} ab & b \\ a & 0 \end{pmatrix}$. *Then*

$$M^n = a^{\frac{n-2+\xi(n)}{2}} b^{\frac{n-\xi(n)}{2}} \begin{pmatrix} a^{1-\xi(n)}b^{\xi(n)}q_{n+1} & bq_n \\ aq_n & a^{1-\xi(n)}b^{\xi(n)}q_{n-1} \end{pmatrix} \tag{8}$$

for any integer $n \geq 1$.

Proof. First, in Lemma 2.1, if we change α into the matrix M and change q_{n-1} into the matrix $q_{n-1}I$, then the matrix form (8) is obtained. Next, we show that the matrix form (8) holds by induction on n. For $n = 1$, from the conditions $\xi(1) = 1, q_0 = 0, q_1 = 1$, we have

$$\begin{aligned} \text{RHS of (8)} &= \begin{pmatrix} bq_2 & bq_1 \\ aq_1 & bq_0 \end{pmatrix} = \begin{pmatrix} b(aq_1 + q_0) & bq_1 \\ aq_1 & bq_0 \end{pmatrix} = \begin{pmatrix} ab & b \\ a & 0 \end{pmatrix} = M^1 \\ &= \text{LHS of (8)}, \end{aligned}$$

and then, from the conditions $\xi(n+1) = 1-\xi(n), \xi(n+2) = \xi(n)$ and (1), we have

$$
\begin{aligned}
& M^{n+1} \\
= \;& M^n M \\
= \;& a^{\frac{n-2+\xi(n)}{2}} b^{\frac{n-\xi(n)}{2}} \begin{pmatrix} a^{1-\xi(n)} b^{\xi(n)} q_{n+1} & b q_n \\ a q_n & a^{1-\xi(n)} b^{\xi(n)} q_{n-1} \end{pmatrix} \begin{pmatrix} ab & b \\ a & 0 \end{pmatrix} \\
= \;& a^{\frac{n-2+\xi(n)}{2}} b^{\frac{n-\xi(n)}{2}} \begin{pmatrix} a^{2-\xi(n)} b^{1+\xi(n)} q_{n+1} + ab q_n & a^{1-\xi(n)} b^{1+\xi(n)} q_{n+1} \\ a^2 b q_n + a^{2-\xi(n)} b^{\xi(n)} q_{n-1} & ab q_n \end{pmatrix} \\
= \;& a^{\frac{n-\xi(n)}{2}} b^{\frac{n+\xi(n)}{2}} \begin{pmatrix} a^{\xi(n)} b^{1-\xi(n)} \left\{ a^{1-\xi(n)} b^{\xi(n)} q_{n+1} + q_n \right\} & b q_{n+1} \\ a \left\{ a^{\xi(n)} b^{1-\xi(n)} q_n + q_{n-1} \right\} & a^{\xi(n)} b^{1-\xi(n)} q_n \end{pmatrix} \\
= \;& a^{\frac{n-\xi(n)}{2}} b^{\frac{n+\xi(n)}{2}} \begin{pmatrix} a^{\xi(n)} b^{1-\xi(n)} \left\{ a^{1-\xi(n+2)} b^{\xi(n+2)} q_{n+1} + q_n \right\} & b q_{n+1} \\ a \left\{ a^{1-\xi(n+1)} b^{\xi(n+1)} q_n + q_{n-1} \right\} & a^{\xi(n)} b^{1-\xi(n)} q_n \end{pmatrix} \\
= \;& a^{\frac{n-\xi(n)}{2}} b^{\frac{n+\xi(n)}{2}} \begin{pmatrix} a^{\xi(n)} b^{1-\xi(n)} q_{n+2} & b q_{n+1} \\ a q_{n+1} & a^{\xi(n)} b^{1-\xi(n)} q_n \end{pmatrix} \\
= \;& a^{\frac{(n+1)-2+\xi(n+1)}{2}} b^{\frac{(n+1)-\xi(n+1)}{2}} \\
& \times \begin{pmatrix} a^{1-\xi(n+1)} b^{\xi(n+1)} q_{(n+1)+1} & b q_{n+1} \\ a q_{n+1} & a^{1-\xi(n+1)} b^{\xi(n+1)} q_{(n+1)-1} \end{pmatrix}.
\end{aligned}
$$

Thus, the given formula in (8) is true for any positive integer n. □

Since M is invertible, M^{-n} may be interpreted as $(M^n)^{-1}$ for $n \geq 0$, where $M^0 = I$. From $\det(M^n) = \det(M)^n = (-ab)^n$ and the matrix form (8), we have

$$
\begin{aligned}
& (M^n)^{-1} \\
= \;& (-1)^n a^{\frac{-n-2+\xi(n)}{2}} b^{\frac{-n-\xi(n)}{2}} \begin{pmatrix} a^{1-\xi(n)} b^{\xi(n)} q_{n-1} & -b q_n \\ -a q_n & a^{1-\xi(n)} b^{\xi(n)} q_{n+1} \end{pmatrix} \\
= \;& a^{\frac{-n-2+\xi(n)}{2}} b^{\frac{-n-\xi(n)}{2}} \begin{pmatrix} a^{1-\xi(n)} b^{\xi(n)} (-1)^n q_{n-1} & b(-1)^{n+1} q_n \\ a(-1)^{n+1} q_n & a^{1-\xi(n)} b^{\xi(n)} (-1)^{n+2} q_{n+1} \end{pmatrix}.
\end{aligned}
$$

From this it is clear that if we define

$$q_{-n} = (-1)^{n+1} q_n \tag{9}$$

for $n \geq 1$, then the sequence $\{q_n\}$ can be extended to negative values of n and also Theorem 2.2 remains true for all integers n. Some of the terms of the extended sequence $\{q_n\}_{n=-\infty}^{\infty}$ are

$a^2b^2+3ab+1,\ -a^2b-2a,\ ab+1,\ -a,\ 1,\ 0,\ 1,\ a,\ ab+1,\ a^2b+2a,\ a^2b^2+3ab+1.$

PROPOSITION 2.3. *The recurrence relation of the sequence given in (1) holds for all integers n.*

Proof. If $n = 0$ and $n = 1$, then (1) is satisfied since both sides of (1) are equal to 0 and 1, respectively. If $n < 0$, then let $n = -m$, where $m > 0$. From the conditions (9), $\xi(m+2) = \xi(m)$ and (1), we have

$$\begin{aligned} & q_n - a^{1-\xi(n)}b^{\xi(n)}q_{n-1} - q_{n-2} \\ = \;& q_{-m} - a^{1-\xi(-m)}b^{\xi(-m)}q_{-m-1} - q_{-m-2} \\ = \;& (-1)^{m+1}q_m - a^{1-\xi(m)}b^{\xi(m)}(-1)^{m+2}q_{m+1} - (-1)^{m+3}q_{m+2} \\ = \;& (-1)^{m+1}\left((a^{1-\xi(m)}b^{\xi(m)}q_{m+1} + q_m) - q_{m+2}\right) \\ = \;& 0. \end{aligned}$$

□

3. Some properties of $\{q_n\}$ by matrix methods

In this section, with the aid of the matrix M^n in Theorem 2.2, we derive or re-prove some properties involving the terms of the sequence $\{q_n\}$ parallel to the result in [4]. The most notable side of this section is our proof method. Although the identities we proved are known, our proofs are not encountered in the generalized Fibonacci sequence $\{q_n\}$ literature.

For any integer n, if we consider the fact that $\det(M^n) = \det(M)^n = (-ab)^n$, then from Theorem 2.2 we obtain the generalized Cassini's Identity (see Theorem 3 in [4]):

$$a^{1-\xi(n)}b^{\xi(n)}q_{n-1}q_{n+1} - a^{\xi(n)}b^{1-\xi(n)}q_n^2 = a(-1)^n. \tag{10}$$

When $a = b = k$ we obtain the Cassini's Identity for k-Fibonacci sequences $\{f_{k,n}\}$

$$f_{k,n-1}f_{k,n+1} - f_{k,n}^2 = (-1)^n \text{ (see Proposition 3 in [1]).}$$

PROPOSITION 3.1. **(d'Ocagne's Identity)** *For any two integers m and n, we have*

$$\begin{aligned} & a^{\xi(m+n)}q_{m+n} \\ & = a^{\xi(n)(1-\xi(m))}b^{\xi(m)(1-\xi(n))}q_{m+1}q_n + a^{\xi(m)(1-\xi(n))}b^{\xi(n)(1-\xi(m))}q_m q_{n-1} \end{aligned} \tag{11}$$

or

$$a^{\xi(m+n)}q_{m+n} = a^{\xi(mn+n)}b^{\xi(mn+m)}q_{m+1}q_n + a^{\xi(mn+m)}b^{\xi(mn+n)}q_m q_{n-1}. \quad (12)$$

Proof. First, note that

$$\begin{aligned}\xi(m+n) &= \xi(m) + \xi(n) - 2\xi(m)\xi(n),\\ \xi(n)(1-\xi(m)) &= \xi(nm+n),\\ \xi(m)(1-\xi(n)) &= \xi(nm+m).\end{aligned}$$

From Theorem 2.2, we have

$$\begin{aligned}M^{m+n} =& a^{\frac{m+n-2+\xi(m+n)}{2}} b^{\frac{m+n-\xi(m+n)}{2}}\\ &\times \begin{pmatrix} a^{1-\xi(m+n)}b^{\xi(m+n)}q_{m+n+1} & bq_{m+n}\\ aq_{m+n} & a^{1-\xi(m+n)}b^{\xi(m+n)}q_{m+n-1}\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}M^m M^n &= a^{\frac{m+n-4+\xi(m)+\xi(n)}{2}} b^{\frac{m+n-\xi(m)-\xi(n)}{2}}\\ &\times \begin{pmatrix} a^{1-\xi(m)}b^{\xi(m)}q_{m+1} & bq_m\\ aq_m & a^{1-\xi(m)}b^{\xi(m)}q_{m-1}\end{pmatrix}\\ &\times \begin{pmatrix} a^{1-\xi(n)}b^{\xi(n)}q_{n+1} & bq_n\\ aq_n & a^{1-\xi(n)}b^{\xi(n)}q_{n-1}\end{pmatrix}.\end{aligned}$$

Since $M^{m+n} = M^m M^n$, by equating (1,2)-entry of M^{m+n} and $M^m M^n$, respectively, which gives the conclusion. □

If we change n into $-n$ in (12) and using (9), then we obtain the d'Ocagne's Identity given in Theorem 5 of [4]:

$$(-1)^n a^{\xi(m-n)}q_{m-n} = a^{\xi(mn+m)}b^{\xi(mn+n)}q_m q_{n+1} - a^{\xi(mn+n)}b^{\xi(mn+m)}q_{m+1}q_n. \quad (13)$$

Also, if we change m into $m-1$ in (12), then we obtain the identity given in Theorem 3 of [12]:

$$q_{m+n-1} = a^{\xi(mn+n-m)-1}b^{1-\xi(mn+n-m)}q_m q_n + a^{-\xi(mn)}b^{\xi(mn)}q_{m-1}q_{n-1}. \quad (14)$$

When $m = n$ in the d'Ocagne Identity (11), since $\xi(n)(1-\xi(n)) = 0$, we obtain

$$q_{2n} = q_{n+1}q_n + q_n q_{n-1} = q_n(q_{n+1} + q_{n-1}), \quad (15)$$

and if $m = n-1$ in the d'Ocagne identity (11), then we get

$$q_{2n-1} = (a^{-1}b)^{1-\xi(n)}q_n^2 + (a^{-1}b)^{\xi(n)}q_{n-1}^2. \quad (16)$$

Proposition 3.2. **([4], Theorem 7 (Sum Involving Binomial Coefficients))** *For any integer $n \geq 0$, we have*

$$q_{2n+1} = \sum_{k=0}^{n} a^{\frac{k-\xi(k)}{2}} b^{\frac{k+\xi(k)}{2}} \binom{n}{k} q_{k+1}, \tag{17}$$

$$q_{2n} = \sum_{k=0}^{n} a^{\frac{k+\xi(k)}{2}} b^{\frac{k-\xi(k)}{2}} \binom{n}{k} q_{k}, \tag{18}$$

$$q_{-2n+1} = \sum_{k=0}^{n} (-1)^k a^{\frac{k-\xi(k)}{2}} b^{\frac{k+\xi(k)}{2}} \binom{n}{k} q_{-k+1}, \tag{19}$$

$$q_{-2n} = \sum_{k=0}^{n} (-1)^k a^{\frac{k+\xi(k)}{2}} b^{\frac{k-\xi(k)}{2}} \binom{n}{k} q_{-k}. \tag{20}$$

Proof. For $M = \begin{pmatrix} ab & b \\ a & 0 \end{pmatrix}$ and $M^{-1} = \begin{pmatrix} 0 & a^{-1} \\ b^{-1} & -1 \end{pmatrix}$, Cayley-Hamilton theorem gives $M^2 = abI + abM$ and $M^{-2} = \frac{1}{ab} I - M^{-1}$. Using the binomial theorem we have

$$M^{2n} = (ab)^n (I+M)^n = (ab)^n \sum_{k=0}^{n} \binom{n}{k} M^k$$

and

$$M^{-2n} = \left(\frac{1}{ab} I - M^{-1} \right)^n = \sum_{k=0}^{n} (-1)^k (ab)^{k-n} \binom{n}{k} M^{-k}.$$

Now, the matrix form (8) in Theorem 2.2 and (9) gives

$$\begin{pmatrix} aq_{2n+1} & bq_{2n} \\ aq_{2n} & aq_{2n-1} \end{pmatrix} = \sum_{k=0}^{n} a^{\frac{k+\xi(k)}{2}} b^{\frac{k-\xi(k)}{2}} \binom{n}{k} \begin{pmatrix} a^{1-\xi(k)} b^{\xi(k)} q_{k+1} & bq_k \\ aq_k & a^{1-\xi(k)} b^{\xi(k)} q_{k-1} \end{pmatrix}$$

and

$$\begin{pmatrix} aq_{-2n+1} & bq_{-2n} \\ aq_{-2n} & aq_{-2n-1} \end{pmatrix} = \sum_{k=0}^{n} (-1)^k a^{\frac{k+\xi(k)}{2}} b^{\frac{k-\xi(k)}{2}} \binom{n}{k} \begin{pmatrix} a^{1-\xi(k)} b^{\xi(k)} q_{-k+1} & bq_{-k} \\ aq_{-k} & a^{1-\xi(k)} b^{\xi(k)} q_{-k-1} \end{pmatrix}.$$

Equating corresponding entries in the first row gives identities (17), (18), (19) and (20). □

Remark 3.3. *The above identities can be obtained directly from the extended Binet formula (3). However, the matrix method is noticeably simpler.*

Proposition 3.4. *For any two integers $n \geq 1$ and $k \geq 1$, we have*

$$\begin{aligned}(a^{-1}b)^{\xi(kn)+k\xi(n)}q_{kn+1}q_{kn-1} - (a^{-1}b)^{1-\xi(kn)+k\xi(n)}q_{kn}^2 \\ = \left((a^{-1}b)^{2\xi(n)}q_{n+1}q_{n-1} - (a^{-1}b)q_n^2\right)^k\end{aligned} \tag{21}$$

Proof. Matrix identity $M^{kn} = (M^n)^k$ gives

$$a^{\frac{kn-2+\xi(kn)}{2}}b^{\frac{kn-\xi(kn)}{2}}\begin{pmatrix} a^{1-\xi(kn)}b^{\xi(kn)}q_{kn+1} & bq_{kn} \\ aq_{kn} & a^{1-\xi(kn)}b^{\xi(kn)}q_{kn-1}\end{pmatrix}$$

$$= a^{\frac{k(n-2+\xi(n))}{2}}b^{\frac{k(n-\xi(n))}{2}}\begin{pmatrix} a^{1-\xi(n)}b^{\xi(n)}q_{n+1} & bq_n \\ aq_n & a^{1-\xi(n)}b^{\xi(n)}q_{n-1}\end{pmatrix}^k.$$

Taking the determinant of the above matrices yields the identity (21). In fact, since $\det(M) = -ab$, both sides of (21) are equal to $(-ab)^{kn}$. □

Proposition 3.5. *For any integer $n \geq 0$ and for any integer m,*

$$q_{m+n} + (-1)^n q_{m-n} = (a^{-1}b)^{\xi(n)(1-\xi(m))}(q_{n-1} + q_{n+1})q_m. \tag{22}$$

Proof. Consider

$$M^n + (-ab)^n M^{-n} = a^{\frac{n-\xi(n)}{2}}b^{\frac{n+\xi(n)}{2}}(q_{n-1} + q_{n+1})I.$$

Multiplying through by M^m, for any m, and taking the off-diagonal entries yields

$$\begin{aligned}a^{\frac{\xi(m+n)}{2}}b^{-\frac{\xi(m+n)}{2}}q_{m+n} + (-1)^n a^{\frac{\xi(m-n)}{2}}b^{-\frac{\xi(m-n)}{2}}q_{m-n} \\ = a^{\frac{\xi(m)-\xi(n)}{2}}b^{\frac{\xi(n)-\xi(m)}{2}}(q_{n-1} + q_{n+1})q_m.\end{aligned}$$

From $\xi(n+m) = \xi(n-m)$ and $\xi(n+m) = \xi(n) + \xi(m) - 2\xi(n)\xi(m)$, we obtain the identity (22). □

Proposition 3.6. ([4], Theorem 6 (Additional Identities)) q_n *divides* q_{nm} *for any nonzero integers n and m.*

Proof. First, $q_1 = 1$ divides q_m for any integer m. Also, $q_2 = a$ divides q_{2m} for any integer m. Let n be an integer such that $n \geq 3$. We show that q_n divides q_{nm} for any integer m. From (15), q_n divides q_{2n}. More generally, from Theorem 2.2, M^n is a diagonal matrix modulo q_n, and so $(M^n)^m$ is also diagonal matrix modulo q_n for any integer m; but

$$(M^n)^m = M^{nm} = a^{\frac{mn-2+\xi(nm)}{2}} b^{\frac{mn-\xi(nm)}{2}} \times \begin{pmatrix} a^{1-\xi(nm)} b^{\xi(nm)} q_{nm+1} & b q_{nm} \\ a q_{nm} & a^{1-\xi(nm)} b^{\xi(nm)} q_{nm-1} \end{pmatrix}$$

has off-diagonal entries aq_{nm} and bq_{nm}, from which it follows that q_n divides q_{nm} for all n, m since a, b is not divided by q_n. Since $q_{-n} = (-1)^{n+1} q_n$ for any integer $n \geq 1$, our proof is complete. □

Proposition 3.7. *If q_n is prime then n is prime where n is any integer such that $n > 1$ and $n \neq 4$.*

Proof. If q_n is prime and $n = rs$ with $1 < r < n$, $1 < s < n$, then q_r and q_s both divide q_n by Proposition 3.6; hence $q_r = q_s = 1$, and so n is either itself prime or equal to 4, which is contradiction. Thus n is prime. □

Proposition 3.8. ([4], Theorem 6 (Additional Identities)) *For any two integers r and s, $\gcd(q_r, q_s) = q_{\gcd(r,s)}$, where $\gcd(r, s)$ is the greatest common divisor of two integers r and s.*

Proof. Let $d = \gcd(r, s)$. Then d divides r and s, so q_d divides q_r and q_s by Proposition 3.6; hence it divides $\gcd(q_r, q_s)$. And we also know that $d = rx + sy$ for some integers x and y, whence from the d'Ocagne identity (11) we have

$$q_d = q_{rx+sy} = (a^{-1}b)^{\frac{\xi(rx)-\xi(sy)+\xi(d)}{2}} q_{rx+1} q_{sy} + (a^{-1}b)^{\frac{\xi(sy)-\xi(rx)+\xi(d)}{2}} q_{rx} q_{sy-1}.$$

Since $\gcd(q_r, q_s)$ divides both q_r and q_s, it divides q_{rx} and q_{sy} by Proposition 3.6; hence it divides q_d. We have thus shown that $q_d = \gcd(q_r, q_s)$. □

References

[1] A. Borges, P. Catarino, A. P. Aires, P. Vasco and H. Campos, *Two-by-two matrices involving k-Fibonacci and k-Lucas sequences*, Applied Mathematical Sciences, **8** (34) (2014), 659–1666.

[2] J. Ercolano, *Matrix generators of Pell sequences*, Fibonacci Quart., **17** (1) (1979), 71–77.
[3] M. Edson, S. Lewis and O. Yayenie, *The k-periodic Fibonacci sequence and extended Binet's formula*, Integer **11** (2011), 1–12.
[4] M. Edson and O. Yayenie, *A new generalization of Fibonacci sequence and extended Binet's formula*, Integer **9** (2009), 639–654.
[5] Y. K. Gupta, Y. K. Panwar and O. Sikhwal, *Generalized Fibonacci Sequences*, Theoretical Mathematics and Applications **2** (2) (2012), 115–124.
[6] Y. K. Gupta, M. Singh and O. Sikhwal, *Generalized Fibonacci-Like Sequence Associated with Fibonacci and Lucas Sequences*, Turkish Journal of Analysis and Number Theory **2** (6) (2014), 233–238.
[7] A. F. Horadam, *A generalized Fibonacci sequences*, Amer. Math. Monthly **68** (1961), 455–459.
[8] E. Kilic, *Sums of the squares of terms of sequence* $\{u_n\}$, Proc. Indian Acad. Sci.(Math. Sci.) **118** (1), February 2008, 27–41.
[9] D. Kalman and R. Mena, *The Fibonacci numbers - Exposed*, The Mathematical Magazine **2** (2002).
[10] J. R. Silvester, *Fibonacci properties by matrix methods*, Mathematical Gazette **63** (1979), 188–191.
[11] K. S. Williams, *The nth power of a* 2×2 *matrix*, Math. Mag. **65** (5) (1992), 336.
[12] O. Yayenie, *A note on generalized Fibonacci sequences*, Applied Mathematics and Computation **217** (2011), 5603–5611.
[13] H. Zhang and Z. Wu, *On the reciprocal sums of the generalized Fibonacci sequences*, Adv. Differ. Equ. (2013), Article ID 377 (2013).

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