Korean J. Math. **24** (2016), No. 4, pp. 723–736
https://doi.org/10.11568/kjm.2016.24.4.723

BOUNDEDNESS IN NONLINEAR PERTURBED DIFFERENTIAL SYSTEMS VIA t_∞-SIMILARITY

Dong Man Im and Yoon Hoe Goo

Abstract. This paper shows that the solutions to nonlinear perturbed differential system

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s))ds + h(t, y(t), Ty(t))$$

have bounded properties. To show the bounded properties, we impose conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s))ds$, $h(t, y(t), Ty(t))$, and on the fundamental matrix of the unperturbed system $y' = f(t, y)$ using the notion of h-stability.

1. Introduction and preliminaries

We consider the nonautonomous differential system

$$x' = f(t, x), \quad x(t_0) = x_0, \tag{1.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t, 0) = 0$. Also, we consider the perturbed differential systems of (1.1)

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s))ds + h(t, y(t), Ty(t)),\ y(t_0) = y_0, \tag{1.2}$$

Received September 9, 2015. Revised November 17, 2016. Accepted December 20, 2016.

2010 Mathematics Subject Classification: 34D10, 34D20.

Key words and phrases: h-stability, t_∞-similarity, bounded, nonlinear nonautonomous system.

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $h \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t,0) = 0$, $h(t,0,0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator.

The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x|\leq 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (1.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (1.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t,0)v(t), \;\; v(t_0) = v_0 \tag{1.3}$$

and

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t), \;\; z(t_0) = z_0. \tag{1.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (1.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (1.3).

We recall some notions of h-stability [16].

Definition 1.1. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called an *h-system* if there exist a constant $c \geq 1$, and a positive continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq c\, |x_0|\, h(t)\, h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $|x_0|$ small enough (here $h(t)^{-1} = \frac{1}{h(t)}$).

Definition 1.2. The system (1.1) (the zero solution $x = 0$ of (1.1)) is called
(hS)*h-stable* if there exists $\delta > 0$ such that (1.1) is an h-system for $|x_0| \leq \delta$ and h is bounded.

Pinto [15, 16] introduced the notion of h-stability (hS) with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. That is, Pinto extended the study of exponential asymptotic stability to a variety of reasonable systems called h-systems. Choi, Ryu [5] and Choi et al. [6] investigated bounds of solutions for nonlinear perturbed systems. Also, Goo [8,9,10,11] and Choi and Goo [3,4] investigated boundedness of solutions for nonlinear perturbed systems.

In this paper, we investigate bounds for solutions of the nonlinear differential systems using the notion of t_∞-similarity.

Let $\mathcal{M}$ denote the set of all $n \times n$ continuous matrices $A(t)$ defined on $\mathbb{R}^+$ and $\mathcal{N}$ be the subset of $\mathcal{M}$ consisting of those nonsingular matrices $S(t)$ that are of class C^1 with the property that $S(t)$ and $S^{-1}(t)$ are bounded. The notion of t_∞-similarity in $\mathcal{M}$ was introduced by Conti [7].

Definition 1.3. A matrix $A(t) \in \mathcal{M}$ is *t_∞-similar* to a matrix $B(t) \in \mathcal{M}$ if there exists an $n \times n$ matrix $F(t)$ absolutely integrable over $\mathbb{R}^+$, i.e.,

$$\int_0^\infty |F(t)|dt < \infty$$

such that

$$\dot{S}(t) + S(t)B(t) - A(t)S(t) = F(t) \tag{1.5}$$

for some $S(t) \in \mathcal{N}$.

The notion of t_∞-similarity is an equivalence relation in the set of all $n \times n$ continuous matrices on $\mathbb{R}^+$, and it preserves some stability concepts [7, 12].

For the proof we prepare some related properties.

Lemma 1.4. *[16] The linear system*

$$x' = A(t)x, \; x(t_0) = x_0, \tag{1.6}$$

where $A(t)$ is an $n \times n$ continuous matrix, is an h-system (respectively h-stable) if and only if there exist $c \geq 1$ and a positive and continuous (respectively bounded) function h defined on $\mathbb{R}^+$ such that

$$|\phi(t, t_0)| \leq c\, h(t)\, h(t_0)^{-1} \tag{1.7}$$

for $t \geq t_0 \geq 0$, where $\phi(t, t_0)$ is a fundamental matrix of (1.6).

We need Alekseev formula to compare between the solutions of (1.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y), \; y(t_0) = y_0, \tag{1.8}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (1.8) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 1.5. [2] *Let x and y be a solution of (1.1) and (1.8), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

THEOREM 1.6. [5] *If the zero solution of (1.1) is hS, then the zero solution of (1.3) is hS.*

THEOREM 1.7. [6] *Suppose that $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \geq t_0 \geq 0$ and $|x_0| \leq \delta$ for some constant $\delta > 0$. If the solution $v = 0$ of (1.3) is hS, then the solution $z = 0$ of (1.4) is hS.*

LEMMA 1.8. (*Bihari − type inequality*) *Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \leq c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \ t \geq t_0 \geq 0.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big],$$

where $t_0 \leq t < b_1$, $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \mathrm{domW}^{-1}\Big\}.$$

LEMMA 1.9. [3] *Let $u, \lambda_1, \lambda_2, \lambda_3\lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$ and $w(u)$ be nondecreasing in u, $u \leq w(u)$. Suppose that for some $c > 0$,*

$$\begin{aligned} u(t) \leq c &+ \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) w(u(s)) ds \\ &+ \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) w(u(\tau)) d\tau ds \\ &+ \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) w(u(\tau)) d\tau ds, \ \ 0 \leq t_0 \leq t. \end{aligned}$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau + \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} (\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)d\tau + \lambda_5(s)\int_{t_0}^{s} \lambda_6(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 1.10. [3] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)\int_{t_0}^{s} (\lambda_3(\tau)u(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau} \lambda_5(r)w(u(r))dr)d\tau ds + \int_{t_0}^{t} \lambda_6(s)\int_{t_0}^{s} \lambda_7(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau + \lambda_6(s)\int_{t_0}^{s}\lambda_7(\tau)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau + \lambda_6(s)\int_{t_0}^{s}\lambda_7(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

For the proof we need the following corollary from Lemma 1.10.

Corollary 1.11. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$

and $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau)u(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r)w(u(r))dr)d\tau ds$$
$$+ \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s)\int_{t_0}^{s}(\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^{\tau}\lambda_4(r)dr)d\tau + \lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s)\int_{t_0}^{s}(\lambda_2(\tau) + \lambda_3(\tau)\int_{t_0}^{\tau}\lambda_4(r)dr)d\tau + \lambda_5(s)\int_{t_0}^{s}\lambda_6(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

LEMMA 1.12. [8] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)w(u(s))ds + \int_{t_0}^{t} \lambda_2(s) \int_{t_0}^{s} (\lambda_3(\tau)u(\tau)$$
$$+\lambda_4(\tau)\int_{t_0}^{\tau} \lambda_5(r)u(r)dr)d\tau ds + \int_{t_0}^{t} \lambda_6(s) \int_{t_0}^{s} \lambda_7(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau + \lambda_6(s)\int_{t_0}^{s}\lambda_7(\tau)d\tau)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 1.8, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}(\lambda_1(s) + \lambda_2(s)\int_{t_0}^{s}(\lambda_3(\tau) + \lambda_4(\tau)\int_{t_0}^{\tau}\lambda_5(r)dr)d\tau + \lambda_6(s)\int_{t_0}^{s}\lambda_7(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

We prepare the following corollary from Lemma 1.12 that is used in proving the theorem.

Corollary 1.13. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and $0 \le t_0 \le t$,*

$$u(t) \le c + \int_{t_0}^t \lambda_1(s) w(u(s)) ds + \int_{t_0}^t \lambda_2(s) \int_{t_0}^s \lambda_3(\tau) u(\tau) d\tau ds$$
$$+ \int_{t_0}^t \lambda_4(s) \int_{t_0}^s \lambda_5(\tau) w(u(\tau)) d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \textstyle\int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau) d\tau$$
$$+\lambda_4(s) \textstyle\int_{t_0}^s \lambda_5(\tau) d\tau) ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \textstyle\int_{t_0}^t (\lambda_1(s) + \lambda_2(s) \int_{t_0}^s \lambda_3(\tau) d\tau$$
$$+\lambda_4(s) \textstyle\int_{t_0}^s \lambda_5(\tau) d\tau) ds \in \mathrm{domW^{-1}}\Big\}.$$

2. Main Results

In this section, we investigate boundedness for solutions of nonlinear perturbed differential systems via t_∞-similarity.

To obtain the bounded result, the following assumptions are needed:

(H1) $f_x(t, 0)$ is t_∞-similar to $f_x(t, x(t, t_0, x_0))$ for $t \ge t_0 \ge 0$ and $|x_0| \le \delta$ for some constant $\delta > 0$.

(H2) The solution $x = 0$ of (1.1) is hS with the increasing function h.

(H3) $w(u)$ is nondecreasing in u such that $u \le w(u)$ and $\frac{1}{v} w(u) \le w(\frac{u}{v})$ for some $v > 0$.

Theorem 2.1. *Let $a, b, c, k, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$|g(t, y(t))| \le a(t) w(|y(t)|) + b(t) \int_{t_0}^t k(s) w(|y(s)|) ds \tag{2.1}$$

and

$$|h(t, y(t), Ty(t))| \le \int_{t_0}^t c(s) |y(s)| ds, \tag{2.2}$$

where $a, b, c, k \in L^1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}\int_{t_0}^{s}(a(\tau) + c(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau ds\Big],$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^{t}\int_{t_0}^{s}(a(\tau) + c(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By Theorem 1.6, since the solution $x = 0$ of (1.1) is hS, the solution $v = 0$ of (1.3) is hS. Using (H1), by Theorem 1.7, the solution $z = 0$ of (1.4) is hS. Using the nonlinear variation of constants formula Lemma 1.5, Lemma 1.4, together with (2.1), and (2.2), we have

$$\begin{aligned}|y(t)| &\le |x(t)| + \int_{t_0}^{t}|\Phi(t, s, y(s))|\Big(\int_{t_0}^{s}|g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|\Big)ds\\ &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)h(s)^{-1}\Big(\int_{t_0}^{s}(a(\tau)w(|y(\tau)|)\\ &\quad + b(\tau)\int_{t_0}^{\tau}k(r)w(|y(r)|)dr)d\tau + \int_{t_0}^{s}c(\tau)|y(\tau)|d\tau\Big)ds.\end{aligned}$$

Applying (H2) and (H3), we obtain

$$\begin{aligned}|y(t)| &\le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t}c_2h(t)\Big(\int_{t_0}^{s}c(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau\\ &\quad + \int_{t_0}^{s}(a(\tau)w(\frac{|y(\tau)|}{h(\tau)}) + b(\tau)\int_{t_0}^{\tau}k(r)w(\frac{|y(r)|}{h(r)})dr)d\tau\Big)ds.\end{aligned}$$

Set $u(t) = |y(t)||h(t)^{-1}|$. Then, by Corollary 1.11, we have

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}\int_{t_0}^{s}(a(\tau) + c(\tau) + b(\tau)\int_{t_0}^{\tau}k(r)dr)d\tau ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. From the above estimation, we obtain the desired result. Thus, the theorem is proved. □

REMARK 2.2. Letting $c(s) = 0$ in Theorem 2.1, we obtain the similar result as that of Theorem 3.6 in [9].

THEOREM 2.3. *Let $a, b, c, k, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s, y(s))|ds \leq a(t)w(|y(t)|) + b(t)\int_{t_0}^{t} k(s)|y(s)|ds,\ t \geq t_0 \geq 0, \tag{2.3}$$

and

$$|h(t, y(t), Ty(t))| \leq \int_{t_0}^{t} c(s)w(|y(s)|)ds, \tag{2.4}$$

where $a, b, c, k \in L^1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$ and it satisfies

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^{t}(a(s) + \int_{t_0}^{s} c(\tau)d\tau + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds\Big],$$

where $t_0 \leq t < b_1$ and W, W^{-1} are the same functions as in Lemma 1.8 and

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + c_2\int_{t_0}^{t}(a(s) + \int_{t_0}^{s} c(\tau)d\tau + b(s)\int_{t_0}^{s} k(\tau)d\tau)ds \in \mathrm{domW}^{-1}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. Applying Lemma 1.4, lemma 1.5, together with (2.3) and (2.4) we have

$$\begin{aligned}|y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|(\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds \\ &\leq c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)h(s)^{-1}\Big(a(s)w(|y(s)|) \\ &\quad + b(s)\int_{t_0}^{s} k(\tau)|y(\tau)|d\tau + \int_{t_0}^{s} c(\tau)w(|y(\tau)|)d\tau\Big)ds.\end{aligned}$$

It follows from (H2) and (H3), we obtain

$$|y(t)| \le c_1|y_0|h(t)\,h(t_0)^{-1} + \int_{t_0}^t c_2 h(t)a(s)w(\frac{|y(s)|}{h(s)})ds$$
$$+ \int_{t_0}^t c_2 h(t)(\int_{t_0}^s c(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau + b(s)\int_{t_0}^s k(\tau)\frac{|y(\tau)|}{h(\tau)}d\tau)ds.$$

Set $u(t) = |y(t)||h(t)^{-1}|$. Now an application of Corollary 1.13 yields

$$|y(t)| \le h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t (a(s) + \int_{t_0}^s c(\tau)d\tau + b(s)\int_{t_0}^s k(\tau)d\tau)ds\Big],$$

where $c = c_1|y_0|\,h(t_0)^{-1}$. Thus, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$, and so the proof is complete. □

Remark 2.4. Letting $c(s) = 0$ in Theorem 2.3, we obtain the same result as that of Theorem 3.4 in [10].

Theorem 2.5. *Let $a, b, c, k, q, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$|g(t, y(t))| \le a(t)w(|y(t)|) + b(t)\int_{t_0}^t k(s)|y(s)|ds \tag{2.5}$$

and

$$|h(t, y(t), Ty(t))| \le c(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)| \le \int_{t_0}^t q(s)|y(s)|ds, \tag{2.6}$$

where $t \ge t_0 \ge 0$ and $a, b, c, k, q \in L^1(\mathbb{R}^+)$. Then, any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on on $[t_0, \infty)$ and it satisfies

$$|y(t)| \le \; h(t)W^{-1}\Big[W(c) + c_2\int_{t_0}^t [c(s) + \int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau$$
$$+c(s)\int_{t_0}^s q(\tau)d\tau]ds\Big],$$

where W, W^{-1} are the same functions as in Lemma 1.8, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + c_2\int_{t_0}^t [c(s) + \int_{t_0}^s (a(\tau) + b(\tau)\int_{t_0}^\tau k(r)dr)d\tau$$
$$+c(s)\int_{t_0}^s q(\tau)d\tau]ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. From Lemma 1.4, Lemma 1.5, together with (2.5), and (2.6), we have

$$|y(t)| \le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| (\int_{t_0}^{s} |g(\tau, y(\tau))| d\tau + |h(s, y(s), Ty(s))|) ds$$
$$\le c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) h(s)^{-1} \Big(\int_{t_0}^{s} (a(\tau) w(|y(\tau)|)$$
$$+ b(\tau) \int_{t_0}^{\tau} k(r)|y(r)| dr) d\tau + c(s)(w(|y(s)|) + \int_{t_0}^{s} q(\tau)|y(\tau)| d\tau) \Big) ds.$$

Using the assumptions (H2) and (H3), we obtain

$$|y(t)| \le c_1 |y_0| h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2 h(t) \Big(c(s) w(\frac{|y(s)|}{h(s)})$$
$$+ \int_{t_0}^{s} (a(\tau) w(\frac{|y(\tau)|}{h(\tau)}) + b(\tau) \int_{t_0}^{\tau} k(r) \frac{|y(r)|}{h(r)} dr) d\tau + c(s) \int_{t_0}^{s} q(\tau) \frac{|y(\tau)|}{h(\tau)} d\tau \Big) ds.$$

Set $u(t) = |y(t)||h(t)|^{-1}$. Then, by Lemma 1.12, we have

$$|y(t)| \le h(t) W^{-1} \Big[W(c) + c_2 \int_{t_0}^{t} [c(s) + \int_{t_0}^{s} (a(\tau) + b(\tau) \int_{t_0}^{\tau} k(r) dr) d\tau$$
$$+ c(s) \int_{t_0}^{s} q(\tau) d\tau] ds \Big],$$

where $c = c_1 |y_0|\, h(t_0)^{-1}$. The above estimation yields the desired result since the function h is bounded. Hence the proof is complete. □

REMARK 2.6. Letting $a(t) = b(t)$ and $c(t) = 0$ in Theorem 2.5, we obtain the similar result as that of Theorem 3.7 in [10].

THEOREM 2.7. *Let $a, b, k, q, w \in C(\mathbb{R}^+)$. Suppose that (H1), (H2), (H3), and g in (1.2) satisfies*

$$\int_{t_0}^{t} |g(s), y(s))| ds \le a(t) \Big(|y(t)| + \int_{t_0}^{t} k(s) w(|y(s)|) ds \Big) \qquad (2.7)$$

and

$$|h(t, y(t), Ty(t))| \le b(t)(w(|y(t)|) + |Ty(t)|), |Ty(t)| \le \int_{t_0}^{t} q(s) w(|y(s)|) ds, \qquad (2.8)$$

where $t \geq t_0 \geq 0$ *and* $a, b, k, q \in L^1(\mathbb{R}^+)$. *Then, any solution* $y(t) = y(t, t_0, y_0)$ *of (1.2) is bounded on* $[t_0, \infty)$ *and it satisfies*

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) + a(s) \int_{t_0}^{s} k(\tau)d\tau + b(s) \int_{t_0}^{s} q(\tau)d\tau)ds\Big],$$

where $t_0 \leq t < b_1$ *and* W, W^{-1} *are the same functions as in Lemma 1.8 and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) + a(s) \int_{t_0}^{s} k(\tau)d\tau + b(s) \int_{t_0}^{s} q(\tau)d\tau)ds \in \mathrm{domW^{-1}}\Big\}.$$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (1.1) and (1.2), respectively. By the same argument as in the proof in Theorem 2.1, the solution $z = 0$ of (1.4) is hS. By Lemma 1.4, Lemma 1.5, together with (2.7), and (2.8), we have

$$\begin{aligned}|y(t)| &\leq |x(t)| + \int_{t_0}^{t} |\Phi(t, s, y(s))|(\int_{t_0}^{s} |g(\tau, y(\tau))|d\tau + |h(s, y(s), Ty(s))|)ds \\ &\leq c_1|y_0|h(t)\, h(t_0)^{-1} + \int_{t_0}^{t} c_2h(t)h(s)^{-1}\Big(a(s)|y(s)| + b(s)w(|y(s)|) \\ &\quad + a(s) \int_{t_0}^{s} k(\tau)w(|y(\tau)|)d\tau + b(s) \int_{t_0}^{s} q(\tau))w(|y(\tau)|)d\tau\Big)ds.\end{aligned}$$

By the assumptions (H2) and (H3), we obtain

$$\begin{aligned}|y(t)| \leq c_1|y_0|h(t)\, h(t_0)^{-1} &+ \int_{t_0}^{t} c_2h(t)\Big(a(s)\frac{|y(s)|}{h(s)} + b(s)w(\frac{|y(s)|}{h(s)}) \\ &+b(s) \int_{t_0}^{s} q(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau + a(s) \int_{t_0}^{s} k(\tau)w(\frac{|y(\tau)|}{h(\tau)})d\tau\Big)ds.\end{aligned}$$

Set $u(t) = |y(t)||h(t)^{-1}|$. Now an application of Lemma 1.9 yields

$$|y(t)| \leq h(t)W^{-1}\Big[W(c) + c_2 \int_{t_0}^{t} (a(s) + b(s) + a(s) \int_{t_0}^{s} k(\tau)d\tau + b(s) \int_{t_0}^{s} q(\tau)d\tau)ds\Big].$$

Thus any solution $y(t) = y(t, t_0, y_0)$ of (1.2) is bounded on $[t_0, \infty)$. This completes the proof. □

REMARK 2.8. Letting $b(t) = 0$ in Theorem 2.7, we obtain the similar result as that of Theorem 3.3 in [9].

Acknowledgement. The authors are very grateful for the referee's valuable comments.

References

[1] V. M. Alekseev, *An estimate for the perturbations of the solutions of ordinary differential equations*, Vestn. Mosk. Univ. Ser. I. Math. Mekh. **2** (1961), 28–36(Russian).

[2] F. Brauer, *Perturbations of nonlinear systems of differential equations*, J. Math. Anal. Appl. **14** (1966), 198–206.

[3] S. I. Choi and Y. H. Goo, *$h-$stability and boundedness in perturbed functional differential systems*, Far East J. Math. Sci(FJMS) **97**(2015), 69–93.

[4] S. I. Choi and Y . H. Goo, *Boundedness in perturbed functional differential systems via t_∞-similarity*, Korean J. Math. **23** (2015), 269–282.

[5] S. K. Choi and H. S. Ryu, *$h-$stability in differential systems*, Bull. Inst. Math. Acad. Sinica **21** (1993), 245–262.

[6] S. K. Choi, N. J. Koo and H.S. Ryu, *h-stability of differential systems via t_∞-similarity*, Bull. Korean. Math. Soc. **34** (1997), 371–383.

[7] R. Conti, Sulla t_∞-similitudine tra matricie l'equivalenza asintotica dei sistemi differenziali lineari, Rivista di Mat. Univ. Parma **8** (1957), 43–47.

[8] Y. H. Goo, Boundedness in the perturbed functional differential systems via t_∞-similarity, Far East J. Math. Sci(FJMS) **97**(2015),763-780.

[9] Y. H. Goo, Boundedness in the perturbed nonlinear differential systems, Far East J. Math. Sci(FJMS) **79**(2013),205-217.

[10] Y . H. Goo, Boundedness in the perturbed differential systems, J. Korean Soc. Math. Edu. Ser.B: Pure Appl. Math. **20** (2013), 223-232.

[11] Y. H. Goo, Boundedness in perturbed nonlinear differential systems, J. Chungcheong Math. Soc. **26**(2013), 605-613.

[12] G. A. Hewer, *Stability properties of the equation by t_∞-similarity*, J. Math. Anal. Appl. **41** (1973), 336–344.

[13] V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities: Theory and Applications*, Academic Press, New York and London, 1969.

[14] B.G. Pachpatte, *On some retarded inequalities and applications*, J. Ineq. Pure Appl. Math. **3** (2002) 1–7.

[15] M. Pinto, *Perturbations of asymptotically stable differential systems*, Analysis **4** (1984), 161–175.

[16] M. Pinto, *Stability of nonlinear differential systems*, Applicable Analysis **43** (1992), 1–20.

Dong Man Im
Department of Mathematics Education
Cheongju University
Cheongju 360-764, Republic of Korea
E-mail: dmim@cheongju.ac.kr

Yoon Hoe Goo
Department of Mathematics
Hanseo University
Seosan 356-706, Republic of Korea
E-mail: yhgoo@hanseo.ac.kr