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HARMONIC HOMOMORPHISMS BETWEEN TWO LIE GROUPS

Heui-Sang Son*, Hyun Woong Kim and Joon-Sik Park

Abstract. In this paper, we get a complete condition for a group homomorphism of a compact Lie group with an arbitrarily given left invariant Riemannian metric into another Lie group with a left invariant metric to be a harmonic map, and then obtain a necessary and sufficient condition for a group homomorphism of $(SU(2), g)$ with a left invariant metric g into the Heisenberg group (H, h_0) to be a harmonic map.

1. Introduction

Harmonic maps of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h) are the extrema of the energy functional (cf. [7])

$$E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2 v_g,$$

where $\|d\phi\|$ is the norm of the differential $d\phi$ of a mapping $\phi \in C^\infty(M, N)$ with respect to the metrics g, h.

In this paper, we construct group homomorphisms of a closed (compact and connected) Lie group G with a left invariant metric g into another Lie group with a left invariant metric h which are harmonic.

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*Corresponding author

It is well known that every inner automorphism of a Lie group G into itself is both isomorphic and harmonic with respect to a bi-invariant Riemannian metric g_0 on G.

However, we here deal with a group homomorphism between two Lie groups with arbitrarily given left invariant metrics.

First of all, we get a necessary and sufficient condition (cf. Proposition 2.1) for a group homomorphism ϕ of a compact Lie group G with a left invariant metric g into another Lie group H with a left invariant metric h to be a harmonic map.

And then, using this complete condition, we obtain a necessary and sufficient condition for a group homomorphism ϕ of $SU(2)$ $(= G)$ with a left invariant metric g into the Heisenberg Lie group (H, h_0) (cf. [4], [5]) to be a harmonic map.

2. Harmonic group homomorphisms

2.1. Harmonic maps

Let (M, g), (N, h) be two Riemannian manifolds of dimension n, m, respectively. Let $\phi : M \to N$ be a smooth map and let $E := \phi^{-1}TN$ be the induced bundle by ϕ over M of the tangent bundle TN of N. We denote by $\Gamma(E)$, the space of all sections V of E, that is, $V \in \Gamma(E)$ implies that V is a map of M into E such that $V_x \in T_{\phi(x)}N$ for all $x \in M$. For $X \in \Gamma(TM)$, we define $\phi_* X \in \Gamma(E)$ by $(\phi_* X)_x := \phi_{*x} X_x \in T_{\phi(x)}N$ $(x \in M)$, where ϕ_{*x} is the differential of ϕ at x. For $Y \in \Gamma(TN)$, we also define $\tilde{Y} \in \Gamma(E)$ by $\tilde{Y}_x := Y_{\phi(x)}$ $(x \in M)$.

We denote ∇, ${}^N\nabla$ the Levi-Civita connections of (M, g), (N, h), respectively. Then we give the induced connection $\tilde{\nabla}$ on E (cf. [1], [2]) by

$$(2.1) \quad (\tilde{\nabla}_X V)_x := \frac{d}{dt}\, {}^N P_{\phi(\gamma(t))}^{\;-1}\, V_{\gamma(t)}|_{t=o} \quad (X \in \Gamma(TM),\ V \in \Gamma(E)),$$

where $x \in M$, $\gamma(t)$ is a curve through x at $t = 0$ whose tangent vector at x is X_x, and ${}^N P_{\phi(\gamma(t))} : T_{\phi(x)}N \to T_{\phi(\gamma(t))}N$ is the parallel displacement along a curve $\phi(\gamma(s))$ $(0 \leq s \leq t)$ given by the Levi-Civita connection ${}^N\nabla$ of (N, h)).

We define a tension field $\tau(\phi) \in \Gamma(E)$ of ϕ by

$$\tau(\phi) := \sum_{i=1}^{n} \left(\tilde{\nabla}_{\mathbf{e}_i} \phi_* \mathbf{e}_i - \phi_* \nabla_{\mathbf{e}_i} \mathbf{e}_i \right), \tag{2.2}$$

where $\{\mathbf{e}_i\}_{i=1}^n$ is a (locally defined) orthonormal frame field on M. We call ϕ to be a *harmonic map* if $\tau(\phi) = 0$ on M.

2.2. Harmonic group homomorphisms

Let G be an n-dimensional closed (compact and connected) Lie group with an arbitrarily given left invariant metric g, and H an m-dimensional Lie group with a left invariant metric h. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of all left invariant vector fields on G (resp. H). Let $\phi : G \to H$ be a group homomorphism, $\{\mathbf{e}_i\}_{i=1}^n$ (resp. $\{\mathbf{d}_a\}_{a=1}^m$) an orthonormal basis of $(\mathfrak{g}, g)$ (resp. $(\mathfrak{h}, h)$). We use the following notations:

$$\begin{aligned}
(d\phi)(\mathbf{e}_i) &=: \sum_{a=1}^{m} {\phi_i}^a \; \mathbf{d}_a, \\
{}^g\nabla_{\mathbf{e}_i} \mathbf{e}_j &=: D_{\mathbf{e}_i} \mathbf{e}_j =: \sum_{k=1}^{n} {\alpha_{ij}}^k \; \mathbf{e}_k, \\
{}^h\nabla_{\mathbf{d}_a} \mathbf{d}_b &=: \nabla_{\mathbf{d}_a} \mathbf{d}_b =: \sum_{c=1}^{m} {\beta_{ab}}^c \; \mathbf{d}_c.
\end{aligned} \tag{2.3}$$

Here D (resp. ∇) is the Levi-Civita connection on (G, g) (resp. (H, h)), and $d\phi$ $(= \phi_*)$ is the differential of the group homomorphism ϕ. From (2.3) we obtain

$$\begin{aligned}
\tilde{\nabla}_{\mathbf{e}_i} \phi_* \mathbf{e}_i &= \sum_{a,b,c=1}^{m} {\phi_i}^a \; {\phi_i}^b \; {\beta_{ab}}^c \; \mathbf{d}_c \\
\phi_*(D_{\mathbf{e}_i} \mathbf{e}_i) &= \sum_{j=1}^{n} \sum_{a=1}^{m} {\alpha_{ii}}^j \; {\phi_j}^a \; \mathbf{d}_a
\end{aligned} \tag{2.4}$$

since ${\alpha_{ij}}^k$ and ${\beta_{ab}}^c$ are constants. By the help of (2.2), (2.4) and the definition of harmonic map, we obtain the following proposition.

Proposition 2.1. *Let (G, g) be an n-dimensional closed Lie group with an arbitrarily given left invariant metric g on G, (H, h) an m-dimensional Lie group with an arbitrarily given left invariant metric h on H. Then a group homomorphism $\phi : (G, g) \to (H, h)$ is a harmonic map if and only if*

$$\sum_{i=1}^{n}\left(\sum_{a,b=1}^{m} {\phi_i}^a\ {\phi_i}^b\ {\beta_{ab}}^c - \sum_{j=1}^{n} {\alpha_{ii}}^j\ {\phi_j}^c\right) = 0 \tag{2.5}$$

for all $c = 1, 2, \ldots, m$.

2.3. Left invariant Riemannian metric on $SU(2)$

Let $\mathfrak{su}(2)$ be the Lie algebra of $SU(2)$. The Killing form B of $\mathfrak{su}(2)$ satisfies

$$B(X, Y) = 4\ Trace(XY) \quad (X, Y \in \mathfrak{s}u(2)). \tag{2.6}$$

We define an inner product $<\ ,\ >_0$ on $\mathfrak{su}(2)$ by

$$< X, Y >_0 := -B(X, Y) \quad (X, Y \in \mathfrak{su}(2)). \tag{2.7}$$

Here and from now on, let g be an arbitrarily given left invariant Riemannian metric on $SU(2)$. The following lemma is known (cf. [1], [3], [6])

Lemma 2.2. *Let g be a left invariant Riemannian metric on $SU(2)$. Let $<\ ,\ >$ be an inner product on $\mathfrak{su}(2)$ defined by $< X, Y > := g_e(X_e, Y_e)$, where $X, Y \in \mathfrak{su}(2)$ and e is the identity matrix of $SU(2)$. Then there exists an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $<\ ,\ >_0$ $(= -B)$ such that*

$$\begin{aligned} &[X_1, X_2] = (1/\sqrt{2})X_3, \qquad [X_2, X_3] = (1/\sqrt{2})X_1, \\ &[X_3, X_1] = (1/\sqrt{2})X_2, \qquad < X_i, X_j >= \delta_{ij}{a_i}^2, \end{aligned} \tag{2.8}$$

where a_i $(i = 1, 2, 3)$ are positive constants determined by the given left invariant Riemannian metric g on $SU(2)$.

2.4. Heisenberg Riemannian Lie group (H, h_0)

Let H be the Heisenberg group (cf. [4], [5]), that is,

$$H = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \mid a_{12}, a_{23}, a_{13} \in \mathbb{R} \right\}. \tag{2.9}$$

Denote by $x,\ y,\ z$ coordinates on H, say for $A \in H$, $x(A) = a_{12}$, $y(A) = a_{23}$, $z(A) = a_{13}$. If L_B is the left translation by an element $B \in H$, we have

$$L_B^*\, dx = dx, \quad L_B^*\, dy = dy, \quad L_B^*\, (dz - xdy) = dz - xdy. \tag{2.10}$$

On H, the vector fields

$$\mathbf{d}_1 := \frac{\partial}{\partial x}, \quad \mathbf{d}_2 := \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad \mathbf{d}_3 := \frac{\partial}{\partial z} \tag{2.11}$$

are dual to $dx,\ dy,\ dz - xdy$, and are left invariant. Moreover, $\{\mathbf{d}_a\}_{a=1}^3$ is orthonormal with respect to the left invariant metric h_0 on H given by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2. \tag{2.12}$$

The Riemannian manifold (H, h_0) is referred to as the *Heisenberg Riemannian Lie group.*

2.5. Harmonic group homomorphisms of $(SU(2), g)$ into (H, h_0)

We retain the notations as in subsections 2.2, 2.3 and 2.4. In general, the Riemannian connection ∇ on a Riemannian manifold (M, g) is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &\quad + g([X,Y],Z) + g([Z,X],Y) \\ &\quad - g([Y,Z],X) \qquad (X, Y, Z \in \mathfrak{X}(M)). \end{aligned} \tag{2.13}$$

We fix an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $<\ ,\ >_0$ satisfying (2.8) in Lemma 2.2 and denote by $g_{(a_1,a_2,a_3)}$ the left invariant Riemannian metric on $SU(2)$ which is determined by positive real numbers a_1, a_2, a_3 in Lemma 2.2. Moreover, we normalize left invariant Riemannian metrics on $SU(2)$ by putting $a_3 = 1$. We denote by $g_{(a_1,a_2,1)}$, or simply by $g_{(a_1,a_2)}$, the left invariant Riemannian metric which is determined by positive real numbers $a_3 = 1, a_1, a_2$.

For the orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $-B =: <\ ,\ >_0$ in Lemma 2.2, if we put

$$\mathbf{e}_1 := \frac{1}{a_1} X_1, \quad \mathbf{e}_2 := \frac{1}{a_2} X_2, \quad \mathbf{e}_3 := X_3,$$

then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal frame basis of $(SU(2), g_{(a_1,a_2)})$. From (2.8), we have

$$[\mathbf{e}_1, \mathbf{e}_2] = \frac{1}{\sqrt{2}\ a_1 a_2}\mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = \frac{a_1}{\sqrt{2}\ a_2}\mathbf{e}_1, \quad [\mathbf{e}_3, \mathbf{e}_1] = \frac{a_2}{\sqrt{2}\ a_1}\mathbf{e}_2. \tag{2.14}$$

By virtue of (2.13) and (2.14), we get

$$\begin{aligned} D_{\mathbf{e}_1}\mathbf{e}_2 &= \frac{1 - (a_1)^2 + (a_2)^2}{2\sqrt{2}\ a_1 a_2}\mathbf{e}_3, \qquad D_{\mathbf{e}_2}\mathbf{e}_3 = \frac{1 + (a_1)^2 - (a_2)^2}{2\sqrt{2}\ a_1 a_2}\mathbf{e}_1, \\ D_{\mathbf{e}_3}\mathbf{e}_1 &= \frac{-1 + (a_1)^2 + (a_2)^2}{2\sqrt{2}\ a_1 a_2}\mathbf{e}_2, \quad D_{\mathbf{e}_i}\mathbf{e}_i = 0 \ \ (i = 1, 2, 3). \end{aligned} \tag{2.15}$$

Using (2.3), (2.14) and (2.15), we have

$$\begin{aligned} {\alpha_{12}}^3 &= \frac{1 - (a_1)^2 + (a_2)^2}{2\sqrt{2}\ a_1 a_2}, \\ {\alpha_{23}}^1 &= -{\alpha_{21}}^3 = \frac{1 + (a_1)^2 - (a_2)^2}{2\sqrt{2}\ a_1 a_2}, \\ {\alpha_{31}}^2 &= -{\alpha_{32}}^1 = \frac{-1 + (a_1)^2 + (a_2)^2}{2\sqrt{2}\ a_1 a_2}, \\ {\alpha_{13}}^2 &= \frac{-1 - (a_1)^2 - (a_2)^2}{2\sqrt{2}\ a_1 a_2}, \\ {\alpha_{ij}}^k &= 0 \ \ \text{otherwise}. \end{aligned} \tag{2.16}$$

Moreover, by virtue of (2.11) and (2.13), we get

$$\begin{aligned} &[\mathbf{d}_1, \mathbf{d}_2] = \mathbf{d}_3, \quad [\mathbf{d}_2, \mathbf{d}_3] = [\mathbf{d}_3, \mathbf{d}_1] = 0, \\ &\nabla_{\mathbf{d}_i}\mathbf{d}_i = 0 \ \ (i = 1, 2, 3), \quad \nabla_{\mathbf{d}_1}\mathbf{d}_2 = -\nabla_{\mathbf{d}_2}\mathbf{d}_1 = \frac{1}{2}\mathbf{d}_3, \\ &\nabla_{\mathbf{d}_2}\mathbf{d}_3 = \nabla_{\mathbf{d}_3}\mathbf{d}_2 = \frac{1}{2}\mathbf{d}_1, \quad \nabla_{\mathbf{d}_3}\mathbf{d}_1 = \nabla_{\mathbf{d}_1}\mathbf{d}_3 = -\frac{1}{2}\mathbf{d}_2. \end{aligned} \tag{2.17}$$

From (2.3) and (2.17), we have

$$\begin{aligned} {\beta_{12}}^3 &= -{\beta_{21}}^3 = \frac{1}{2}, \quad {\beta_{23}}^1 = {\beta_{32}}^1 = \frac{1}{2}, \\ {\beta_{31}}^2 &= {\beta_{13}}^2 = -\frac{1}{2}, \quad {\beta_{bc}}^a = 0 \text{ otherwise.} \end{aligned} \tag{2.18}$$

By virtue of Proposition 2.1, (2.16) and (2.18), we obtain a group homomorphism $\phi : (SU(2), g_{(a_1,a_2)}) \to (H, h_0)$ is a harmonic map if and only if

$$\sum_{i=1}^{3} {\phi_i}^2 {\phi_i}^3 = 0 \text{ and } \sum_{i=1}^{3} {\phi_i}^3 {\phi_i}^1 = 0.$$

Hence, we have the following theorem.

Theorem 2.3. *A group homomorphism ϕ of $(SU(2), g_{(a_1,a_2)})$ into the Heisenberg group (H, h_0) is a harmonic map if and only if*

$$\sum_{i=1}^{3} h_0(\phi_* \mathbf{e}_i, \mathbf{d}_2) \cdot h_0(\phi_* \mathbf{e}_i, \mathbf{d}_3) = 0$$

and

$$\sum_{i=1}^{3} h_0(\phi_* \mathbf{e}_i, \mathbf{d}_3) \cdot h_0(\phi_* \mathbf{e}_i, \mathbf{d}_1) = 0.$$

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Heui-Sang Son
Department of Applied Mathematics,
Pukyong National University,
Pusan 48513, Korea
E-mail: sonheuisang@hanmail.net

Hyun Woong Kim
Department of Applied Mathematics,
Pukyong National University,
Pusan 48513, Korea
E-mail: 0127woong@hanamil.net

Joon-Sik Park
Department of Mathematics,
Pusan University of Foreign Studies,
Busan 46234, Korea
E-mail: iohpark@pufs.ac.kr