Honam Mathematical J. **38** (2016), No. 1, pp. 17–23
http://dx.doi.org/10.5831/HMJ.2016.38.1.17

INTEGRAL REPRESENTATIONS OF THE k-BESSEL'S FUNCTION

Kuldeep Singh Gehlot and Sunil Dutt Purohit*

Abstract. This paper deals with the study of newly defined special function known as k-Bessel's function due to Gehlot [2]. Certain integral representations of k-Bessel's function are investigated. Known integrals of classical Bessel's function are seen to follow as special cases of our main results.

1. Introduction

Bessel functions are important in studying solutions of differential equations, and they are associated with a wide range of problems in many areas of mathematical physics, like problems of acoustics, radio physics, hydrodynamics, and atomic and nuclear physics. These considerations have led various workers in the field of special functions for exploring the possible extensions and applications for the Bessel function. A useful generalization of the Bessel function called as k-Bessel function has been introduced and studied in [2]. Here we aim at presenting certain integral representations for the k-Bessel functions.

Throughout this paper, let $\mathbb{C}, \mathbb{R}, \mathbb{R}^+, \mathbb{Z}, \mathbb{Z}^-, \mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers, integers, negative integers, positive integers respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Diaz and Pariguan [1] introduced the generalized k-Gamma function $\Gamma_k(x)$ as follows:

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \tag{1}$$

Received April 28, 2015. Accepted December 23, 2015.
2010 Mathematics Subject Classification. 33B15, 33C20, 34B30.
Key words and phrases. k-Gamma function, k-Pochhammer symbols, k-Bessel function
*Corresponding author

$$(k > 0; x \in \mathbb{C}\backslash k\mathbb{Z}^-, k\mathbb{Z}^- = \left\{ kn | \, n \in \mathbb{Z}^- \right\}),$$

where $(x)_{n,k}$ is the k-Pochhammer symbol defined by

(2) $(x)_{n,k} = x(x+k)(x+2k)\cdots(x+(n-1)k)$ $(x \in \mathbb{C}, k \in \mathbb{R}, n \in \mathbb{N})$.

The integral form of the generalized k-Gamma function is given by,

$$\varGamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt \ (x \in \mathbb{C}; k \in \mathbb{R}; \Re(x) > 0). \tag{3}$$

It is easy to find the following relations:

$$\varGamma_k(x) = k^{\frac{x}{k}-1}\varGamma(\frac{x}{k}) \tag{4}$$

and

$$\varGamma_k(x+k) = x\varGamma_k(x), \tag{5}$$

where $\varGamma$ is the familiar Gamma function.
We recall the Legender's duplication formula (see, e.g. ([5], p.24)):

$$\sqrt{\pi}\varGamma(2s) = 2^{2s-1}\varGamma(s)\varGamma(s+\frac{1}{2}) \tag{6}$$

and k-Bessel function ([2] and [3]):

$$J^k_{\pm\vartheta}(z) = \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{z}{2})^{2r \pm \frac{\vartheta}{k}}}{\varGamma_k(rk \pm \vartheta + k)(r!)}, \tag{7}$$

where $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ and $\vartheta > -k$.

2. Main results

Five different integral representations of k-Bessel function are presented.

Theorem 2.1. *For $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ and $\vartheta > -k$, then*

$$\pi J^k_{\vartheta k}(z) = \int_0^\pi e^{z(\frac{1-k}{2k})\cos\theta} \cos[\vartheta\theta - z(\frac{1+k}{2k})\sin\theta] d\theta. \tag{8}$$

Proof. Consider the generating function of k-Bessel function (cf., [2, Eq. (21)]):

$$e^{\frac{z}{2\sqrt{k}}(\frac{x}{\sqrt{k}} - \frac{\sqrt{k}}{x})} = \sum_{\vartheta=-\infty}^{\infty} x^\vartheta J^k_{\vartheta k}(z),$$

$$e^{\frac{z}{2\sqrt{k}}(\frac{x}{\sqrt{k}} - \frac{\sqrt{k}}{x})} = J^k_0(z) + xJ^k_k(z) + x^{-1}J^k_{-k}(z) + x^2 J^k_{2k}(z) + x^{-2}J^k_{-2k}(z) + \cdots.$$

Using the equation (20) in [2], we have

$$e^{\frac{z}{2\sqrt{k}}(\frac{x}{\sqrt{k}}-\frac{\sqrt{k}}{x})} = J_0^k(z) + xJ_k^k(z) + x^{-1}(-k)^1 J_k^k(z) + x^2 J_{2k}^k(z)$$
$$+x^{-2}(-k)^2 J_{2k}^k(z) + \cdots$$
$$e^{\frac{z}{2\sqrt{k}}(\frac{x}{\sqrt{k}}-\frac{\sqrt{k}}{x})} = J_0^k(z) + (x - kx^{-1})J_k^k(z) + (x^2 + x^{-2}k^2)J_{2k}^k(z)$$
$$+(x^3 - x^{-3}k^3)J_{3k}^k(z) + \cdots .$$

Putting $x = e^{i\theta}$ and equating the real and imaginary parts, we have

$$e^{\frac{z(1-k)}{2k}\cos\theta}\cos[\frac{z(1+k)}{2k}\sin\theta] = J_0^k(z) + (1-k)\cos\theta J_k^k(z) \tag{9}$$
$$+(1+k^2)\cos 2\theta J_{2k}^k(z)(1-k^3)\cos 3\theta J_{3k}^k(z) + \cdots$$
$$+(1+k^{2m})\cos 2m\theta J_{2mk}^k(z) + (1-k^{2m+1})\cos(2m+1)\theta J_{(2m+1)k}^k(z) + \cdots$$

and

$$e^{\frac{z(1-k)}{2k}\cos\theta}\sin[\frac{z(1+k)}{2k}\sin\theta] = (1+k)\sin\theta J_k^k(z) \tag{10}$$
$$+(1-k^2)\sin 2\theta J_{2k}^k(z) + (1+k^3)\sin 3\theta J_{3k}^k(z) + \cdots$$

$$+(1-k^{2m})\sin 2m\theta J_{2mk}^k(z) + (1+k^{2m+1})\sin(2m+1)\theta J_{(2m+1)k}^k(z) + \cdots .$$

Multiplying (9) by $\cos 2m\theta$ and integrating between the limit 0 to π, we obtain

$$\int_0^\pi e^{\frac{z(1-k)}{2k}\cos\theta}\cos[\frac{z(1+k)}{2k}\sin\theta]\cos 2m\theta d\theta = \frac{\pi}{2}(1+k^{2m})J_{2mk}^k(z). \tag{11}$$

Multiplying (9) by $\cos(2m+1)\theta$ and integrating between the limit 0 to π, we obtain

$$\int_0^\pi e^{\frac{z(1-k)}{2k}\cos\theta}\cos[\frac{z(1+k)}{2k}\sin\theta]\cos(2m+1)\theta d\theta \tag{12}$$
$$= \frac{\pi}{2}(1-k^{(2m+1)})J_{(2m+1)k}^k(z).$$

Multiplying (10) by $\sin 2m\theta$ and integrating between the limit 0 to π, we obtain

$$\int_0^\pi e^{\frac{z(1-k)}{2k}\cos\theta}\sin[\frac{z(1+k)}{2k}\sin\theta]\sin 2m\theta d\theta = \frac{\pi}{2}(1-k^{2m})J_{2mk}^k(z). \tag{13}$$

Again, multiplying (10) by $\sin(2m+1)\theta$ and integrating between the limit 0 to π, we obtain

$$\int_0^\pi e^{\frac{z(1-k)}{2k}\cos\theta}\sin[\frac{z(1+k)}{2k}\sin\theta]\sin(2m+1)\theta d\theta \tag{14}$$

$$= \frac{\pi}{2}(1 + k^{(2m+1)})J^k_{(2m+1)k}(z).$$

Now consider the integral

$$A := \int_0^\pi e^{\frac{z(1-k)}{2k}\cos\theta} \cos[2m\theta - \frac{z(1+k)}{2k}\sin\theta]d\theta. \tag{15}$$

Using equation (11) and (13) in (15), we obtain

$$A = \int_0^\pi e^{\frac{z(1-k)}{2k}\cos\theta} \cos[2m\theta - \frac{z(1+k)}{2k}\sin\theta]d\theta = \pi J^k_{2mk}(z). \tag{16}$$

Again consider the integral

$$B := \int_0^\pi e^{\frac{z(1-k)}{2k}\cos\theta} \cos[(2m+1)\theta - \frac{z(1+k)}{2k}\sin\theta]d\theta. \tag{17}$$

using equation (12) and (14) in (16), we obtain

$$B = \int_0^\pi e^{\frac{z(1-k)}{2k}\cos\theta} \cos[2m\theta - \frac{z(1+k)}{2k}\sin\theta]d\theta = \pi J^k_{(2m+1)k}(z). \tag{18}$$

Equations (16) and (18) together gives the desired result for all positive integer ϑ. □

Theorem 2.2. *For $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ and $\vartheta > -1$, we have*

$$\int_0^z z^{-\frac{\vartheta}{k}} J^k_{\vartheta+k}(z)dz = \frac{1}{2^{\frac{\vartheta}{k}} \mathit{\Gamma}_k(\vartheta + k)} - z^{-\frac{\vartheta}{k}} J^k_\vartheta(z). \tag{19}$$

Proof. Considering the equation (2.1) in [4], we have

$$z^{-\frac{\vartheta}{k}} J^k_{\vartheta+k}(z) = -\frac{d}{dz}[z^{-\frac{\vartheta}{k}} J^k_\vartheta(z)].$$

Integrating between the limit 0 to z, we have

$$\int_0^z z^{-\frac{\vartheta}{k}} J^k_{\vartheta+k}(z)dz = -[z^{-\frac{\vartheta}{k}} J^k_\vartheta(z)]^z_0$$

and

$$\int_0^z z^{-\frac{\vartheta}{k}} J^k_{\vartheta+k}(z)dz = -z^{-\frac{\vartheta}{k}} J^k_\vartheta(z) + \lim_{z\to 0} \frac{J^k_\vartheta(z)}{z^{\frac{\vartheta}{k}}}.$$

Using equation (7) evaluating the limit yields the desire result. □

Theorem 2.3. *For $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ and $\vartheta > -\frac{k}{2}$, we have*

$$J^k_\vartheta(z) = \frac{(\frac{z}{2})^{\frac{\vartheta}{k}}}{\sqrt{\pi k}\, \mathit{\Gamma}_k(\vartheta + \frac{k}{2})} \int_{-1}^1 (1 - t^2)^{\frac{\vartheta}{k} - \frac{1}{2}} e^{\frac{izt}{\sqrt{k}}} dt. \tag{20}$$

Proof. Denote the right hand side integral by A. Then we find

$$A = \sum_{r=0}^{\infty} \frac{(\frac{iz}{\sqrt{k}})^r}{r!} \int_{-1}^{1} (1-t^2)^{\frac{\vartheta}{k}-\frac{1}{2}}\, t^r dt.$$

If r is odd $\Rightarrow A = 0$.
If r is even, we obtain

$$A = \sum_{s=0}^{\infty} \frac{(\frac{iz}{\sqrt{k}})^{2s}}{(2s)!} 2 \int_{0}^{1} (1-t^2)^{\frac{\vartheta}{k}-\frac{1}{2}}\, t^{2s} dt.$$

Putting $t^2 = u$ and using the definition of Beta function, we obtain

$$A = \sum_{s=0}^{\infty} \frac{(-1)^s (\frac{z}{\sqrt{k}})^{2s} \Gamma(\frac{2s+1}{2}) \Gamma(\frac{\vartheta}{k}+\frac{1}{2})}{(2s)! \Gamma(\frac{\vartheta}{k}+s+1)}.$$

Using definition (7), we have

$$A = \sum_{s=0}^{\infty} \frac{\sqrt{\pi}(-1)^s (\frac{z}{\sqrt{k}})^{2s} \Gamma(\frac{\vartheta}{k}+\frac{1}{2})}{2^{2s}(s!)\Gamma(\frac{\vartheta}{k}+s+1)}.$$

Again, using (4) and rearranging the terms, we obtain

$$A = \frac{\sqrt{\pi k}\, \Gamma_k(\vartheta+\frac{k}{2})}{(\frac{z}{2})^{\frac{\vartheta}{k}}} J_{\vartheta}^{k}(z).$$

which completes the proof. □

Theorem 2.4. *For $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ and $\vartheta > -k$, we have*

$$\int_0^z t[J_{\vartheta}^{k}(t)]^2 dt = \frac{z^2}{2}[[J_{\vartheta}^{k}(z)]^2 - J_{\vartheta-k}^{k}(z) J_{\vartheta+k}^{k}(z)]. \tag{21}$$

Proof. Let

$$B := \frac{d}{dt}[\frac{t^2}{2}\{(J_{\vartheta}^{k}(t))^2 - J_{\vartheta-k}^{k}(t) J_{\vartheta+k}^{k}(t)\}].$$

Then we have

$$B = t\{(J_{\vartheta}^{k}(t))^2 - J_{\vartheta-k}^{k}(t) J_{\vartheta+k}^{k}(t)\}$$
$$+[\frac{t^2}{2}\{2J_{\vartheta}^{k}(t)\dot{J}_{\vartheta}^{k}(t) - \dot{J}_{\vartheta-k}^{k}(t) J_{\vartheta+k}^{k}(t) - J_{\vartheta-k}^{k}(t) \dot{J}_{\vartheta+k}^{k}(t)\}].$$

Setting the values of $\dot{J}_{\vartheta-k}^{k}(t), \dot{J}_{\vartheta+k}^{k}(t)$ and $\dot{J}_{\vartheta}^{k}(t)$ from Equation (2.1), (2.2) and (2.4) in [4], respectively, we obtain

$$B = t\{J_{\vartheta}^{k}(t)\}^2.$$

Integrating both sides from 0 to z, we get

$$\int_0^z t[J_\vartheta^k(t)]^2 dt = \frac{z^2}{2}[[J_\vartheta^k(z)]^2 - J_{\vartheta-k}^k(z)J_{\vartheta+k}^k(z)].$$

This completes the proof. □

Theorem 2.5. *For $k \in \mathbb{R}^+, \vartheta \in \mathbb{Z}$ and $\vartheta > -k$, we have*

$$\int_0^z z^{\frac{\vartheta}{k}} J_{\vartheta-k}^k(z)dz = kz^{\frac{\vartheta}{k}} J_\vartheta^k(z). \tag{22}$$

Proof. Let

$$C := \frac{d}{dz}[z^{\frac{\vartheta}{k}} J_\vartheta^k(z)].$$

Then we find

$$C = z^{\frac{\vartheta}{k}-1}[\frac{\vartheta}{k} J_\vartheta^k(z) + z\dot{J}_\vartheta^k(z)].$$

Setting the value of $\dot{J}_\vartheta^k(z)$ from Equation (2.2) in [4], we have

$$C = \frac{z^{\frac{\vartheta}{k}}}{k} J_{\vartheta-k}^k(z).$$

Now, on integrating both sides between the limit 0 to z, we obtain

$$\int_0^z z^{\frac{\vartheta}{k}} J_{\vartheta-k}^k(z)dz = kz^{\frac{\vartheta}{k}} J_\vartheta^k(z).$$

This completes the proof. □

It is noted that the special case of (8) when $k = 1$ yields the well-known Bessel's integral (see [5, p.114]).

3. Concluding remark

The k-Bessel function defined by (7), possess the advantage that the Bessel function, trigonometric functions and hyperbolic functions happen to be the particular cases of this function. Therefore, we conclude this paper with the remark that, the results deduced above can lead to have numerous other integral representations involving the Bessel function and trigonometric functions by the suitable specializations of arbitrary parameters in the main results.

Acknowledgments

The authors would like to express their appreciation to the referees for their valuable suggestions which helped to better presentation of this paper.

References

[1] R. Diaz and E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol*, Divulgaciones Mathemáticas **15(2)** (2007), 179-192.

[2] K. S. Gehlot, *Differential Equation of k-Bessel's Function and its Properties*, Nonl. Anal. Diff. Eq. **2(2)** (2014), 61-67.

[3] K. S. Gehlot, *Recurrence relations of k-Bessel's function*, Thai J. Math. (2015), Accepted.

[4] K. S. Gehlot and S. D. Purohit, *Fractional calculus of k-Bessel's function*, Acta Univ. Apulensis, Math. Inform. **38** (2014), 273-278.

[5] Earl D. Rainville, *Special Functions*, The Macmillan Company, New York, 1963.

Kuldeep Singh Gehlot
Government College Jodhpur, Jodhpur
First Pulia, Chopasni Housing Board,
Jodhpur 342008, India.
E-mail: drksgehlot@rediffmail.com

Sunil Dutt Purohit
Department of HEAS (Mathematics),
Rajasthan Technical University,
Kota 324010, India.
E-mail: sunil_a_purohit@yahoo.com