Honam Mathematical J. **38** (2016), No. 1, pp. 71–84
http://dx.doi.org/10.5831/HMJ.2016.38.1.71

# VOLUME OF GRAPH POLYTOPES FOR THE PATH-STAR TYPE GRAPHS

Hyeong-Kwan Ju* and Soo-Jeong Seo

**Abstract.** The aim of this work is to compute the volume of the graph polytope associated with various type of finite simple graphs composed of paths and stars. Recurrence relations are obtained using the recursive volume formula (RVF) which was introduced in Lee and Ju ([3]). We also discussed the relationship between the volume of the graph polytopes and the number of linear extensions of the associated posets for given bipartite graphs.

## 1. Introduction

Let $G = (V, E)$ be a simple graph with $V = [k]$. Then the **graph polytope** $P(G)$ associated with the graph $G$ is defined as follows:

$$P(G) := \{(x_1, x_2, \ldots, x_k) \in [0,1]^k | ij \in E \Rightarrow x_i + x_j \leq 1\}.$$

Bóna et al.[1] discussed Ehrhart polynomials of the Ehrhart series for the polytope $P(G)$ of the bipartite graph $G$. We define the kernel function $K : [0,1]^2 \to \mathbb{R}$ by the following:

$$K(s,t) := \begin{cases} 1, & s + t \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Then the volume $vol(P(G))$ of the polytope $P(G)$ can be expressed in terms of the products of the kernel functions as follows:

$$vol(P(G)) = \int_{Q_k} H(x_1, x_2, \ldots, x_k) dx,$$

Received November 16, 2015. Accepted December 8, 2015.
2010 Mathematics Subject Classification. 05C22, 52B05.
Key words and phrases. graph polytope, volume, generating function, path-star type graph.
*Corresponding author

where $Q_k = [0,1]^k$ is the $k$-dimensional unit hypercube,

$$H(x_1, x_2, \ldots, x_k) = \prod_{ij \in E} K(x_i, x_j), \text{ and } dx = dx_1 dx_2 \cdots dx_k.$$

Our main interest is to obtain the volume formula of the finite simple graphs. With the help of the recursive volume formula (will be introduced next section) we can get the recurrence relations on the volume of the graph polytope for any finite simple graph (even including non-bipartite graphs). In Section 2 we discuss about the volumes for paths and star graphs briefly, and introduce the recursive volume formula. Volumes of polytopes for several path-star type graphs are obtained using the volume formula related with the kernel function in Section 3. We also obtain recursive volume formula for the corresponding graph polytopes there. In Sections 4 and 5 we introduce posets and linear extensions, and explain the relationship between volumes of graph polytopes and number of linear extensions for bipartite graphs. In the last section, concluding remarks and further open questions were suggested.

## 2. Volumes for Paths and Star Graphs

A star graph $S_n = (V, E)$ is a graph with $V = \{0\} \cup [n], E = \{0k \mid k \in [n]\}$. Then the volume of the cross section is

$$vol(P(S_n) \cap \{x_0 = t\}) = \int_0^1 \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} K(x_i, t) dx_1 \cdots dx_{n-1} dx_n = (1-t)^n.$$

Hence, we can get the volume of $P(S_n)$ immediately, and it is $\frac{1}{n+1}$. A path is the graph $P_n = ([n], E)$ where $E = \{i(i+1) | 1 \le i \le n-1\}$. In this case, we obtain the volume of the cross section as follows:

$$vol(P(L_n) \cap \{x_n = t\}) = \int_{x_{n-1}=0}^{1-t} \cdots \int_{x_2=0}^{1-x_3} \int_{x_1=0}^{1-x_2} dx_1 dx_2 \cdots dx_{n-1}.$$

Let $f_0(t) := 1, f_n(t) := vol(P(L_{n+1}) \cap \{x_{n+1} = t\})$. The following result is directly from [2].

**Lemma 2.1.** *Generating function of $f_n(t)$ is*

$$F(t, x) := \sum_{n \ge 0} f_n(t) x^n = (\cos(tx) + \sin(1 - tx)) \sec x,$$

*and the volume of $P(L_n)$ is*

$$vol(P(P_n)) = \frac{E_n}{n!},$$

*where $E_n$ is the Euler's up-down number(A000111 in OEIS*[4]*).*

**Lemma 2.2.** *$f_n(t)$ can be expressed in terms of Euler's up-down numbers as follows:*

$$f_n(t) = \begin{cases} \sum_{k\geq 0} \frac{(-1)^k t^{2k} E_{n-2k}}{(2k)!(n-2k)!} & n \text{ is even}, \\ \sum_{k\geq 0} \frac{(-1)^k (1-t)^{2k+1} E_{n-2k-1}}{(2k+1)!(n-2k-1)!} & n \text{ is odd}. \end{cases}$$

*Proof.*

$$\begin{aligned} &\sum_{n\geq 0} f_n(t)x^n = (\cos tx + \sin(1-t)x)\sec x \\ &= \Big(\sum_{k\geq 0}(-1)^k \frac{(tx)^{2k}}{(2k)!} + \sum_{k\geq 0}(-1)^k \frac{((1-t)x)^{2k+1}}{(2k+1)!}\Big)\Big(\sum_{\ell\geq 0}\frac{E_{2\ell}}{(2\ell)!}x^{2\ell}\Big) \\ &= \sum_{k\geq 0}\sum_{\ell\geq 0}\Big(\frac{(-1)^k t^{2k}}{(2k)!}\cdot\frac{E_{2\ell}}{(2\ell)!}\Big)x^{2k+2\ell} \\ &+ \sum_{k\geq 0}\sum_{\ell\geq 0}\Big(\frac{(-1)^k (1-t)^{2k+1}}{(2k+1)!}\cdot\frac{E_{2\ell}}{(2\ell)!}\Big)x^{2k+2\ell+1} \end{aligned}$$

□

**Theorem 2.3.** *(**RVF**, [3]) Let $G = (V, E)$ be a graph with the vertex set $V = [n]$ and having no isolated vertex. Then*

$$vol(G) = \sum_{i=1}^{n} \frac{vol(G-i)}{2n}$$

*where $G - i$ is the graph with the vertex set $[n]\backslash\{i\}$ and, accordingly, with the inherited edge set in the original edge set $E$.*

## 3. Volumes for Path-Star Type Graphs

Path-Star type graphs are those graphs where star graphs are attached at some vertices of a path. Here we consider only three kinds of them given in the following subsections.

### 3.1. Path-Star Graphs

Let $PS(n,m) = (V,E)$ be a graph with $V = \{0,1,2,\dots,n,1',2',\dots,m'\}$ and $E = \{i(i+1)|0 \le i \le n-1\} \bigcup \{ni'|1 \le i \le m\}$. The graph $PS(m,n)$ is called a path-star graph of order $n$ and $m$ (see Figure 3.1) and the volume of $PS(m,n)$ is denoted by $a(n,m)$. Then,

$$a(n,m) := vol(P(PS(n,m))) = \int_0^1 f_n(t)(1-t)^m dt.$$

Remind that $f_n(t) := vol(P(L_{n+1}) \cap \{x_n = t\})$. In other words, $f_n(t)$ is the ($n$-dimensional) volume of the cross-section of the polytope $P(L_{n+1})$ with an hyperplane $x_n = t$. (Note that the subscript $i$ in the $x_i$ runs from 0, not 1.)

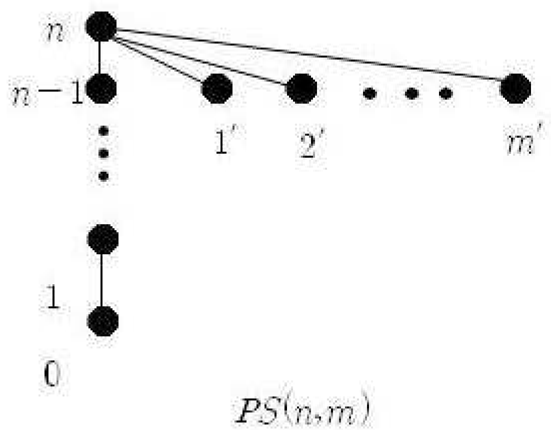


FIGURE 1

If we apply Theorem 2.3 to the path-star graph $PS(n,m)$, we obtain the recursive formula on $a(n,m)$ in the next corollary.

**Corollary 3.1.** *The volume sequence $a(n,m)$ for the path-star graphs satisfies the following recursive formula:*

$$a(n,m) = \frac{1}{2(n+m+1)}\Big(a(n,m-1)m + \frac{E_n}{n!} + \sum_{k=0}^{n-1} a(k,m)\frac{E_{n-k-1}}{(n-k-1)!}\Big).$$

*Proof.* By Theorem 2.3 we have the formula

$$a(n,m) = \frac{1}{2(n+m+1)}\Big(a(n,m-1)m + vol(P(L_n)) + \sum_{k=0}^{n-1} a(k,m) vol(P(L_{n-k-1}))\Big).$$

Since $vol(P(L_n)) = \frac{E_n}{n!}$ we get the desired formula. □

**Theorem 3.2.** *The volume sequence $a(n,m)$ for the path-star graphs also satisfies the following formula: If $n$ is odd, then*

$$a(n,m) = \sum_{k\geq 0}^{\frac{n-1}{2}} \frac{(-1)^k E_{n-2k-1}}{(2k+1)!(n-2k-1)!(2k+m+2)},$$

*and if $n$ is even, then*

$$a(n,m) = \sum_{k\geq 0}^{\frac{n}{2}} \frac{(-1)^k E_{n-2k}}{(2k)!(n-2k)!(2k+m+1)} \frac{1}{\binom{2k+m}{2k}}.$$

*Proof.* From Lemma 2.2 we obtain the following results. For the odd case $n = 2q+1$,

$$\begin{aligned} a(2q+1,m) &= \int_0^1 \sum_{k\geq 0} \frac{(-1)^k (1-t)^{2k+1} E_{2(q-k)}}{(2k+1)!(2q-2k)!} (1-t)^m dt \\ &= \sum_{k\geq 0} \frac{(-1)^k E_{2(q-k)}}{(2k+1)!(2q-2k)!} \int_0^1 (1-t)^{2k+1+m} dt \\ &= \sum_{k\geq 0} \frac{(-1)^k E_{2(q-k)}}{(2k+1)!(2q-2k)!(2k+2+m)}. \end{aligned}$$

For the even case $n = 2q$,

$$\begin{aligned} a(2q,m) &= \int_0^1 \sum_{k\geq 0} \frac{(-1)^k t^{2k} E_{2(q-k)}}{(2k)!(2q-2k)!} (1-t)^m dt \\ &= \sum_{k\geq 0} \frac{(-1)^k E_{2(q-k)}}{(2k)!(2q-2k)!} \int_0^1 t^{2k}(1-t)^m dt \\ &= \sum_{k\geq 0} \frac{(-1)^k E_{2(q-k)}}{(2k)!(2q-2k)!} B(2k+1, m+1) \\ &= \sum_{k\geq 0} \frac{(-1)^k E_{2q-2k}}{(2k)!(2q-2k)!(2k+m+1)} \frac{1}{\binom{2k+m}{2k}}, \end{aligned}$$

where $B(p,q)$ is the beta function defined by

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1} dt.$$

(See [5] about this.) □

Next, we consider the bivariate generating function on $a(n,m)$. Let

$$F(x,y) := \sum_{m\geq 0}\sum_{n\geq 0} a(n,m)x^n y^m.$$

Then, we get the next corollary.

**Corollary 3.3.**

$$F(x,y) = \int_0^1 \frac{\cos tx + \sin(1-t)x}{1-(1-t)y} \sec x dt.$$

*Proof.*

$$\begin{aligned}
F(x,y) &= \sum_{n\geq 0}\sum_{m\geq 0} a(n,m)x^n y^m \\
&= \sum_{n\geq 0}\sum_{m\geq 0} \int_0^1 f_n(t)(1-t)^m dt x^n y^m \\
&= \int_0^1 \sum_{m\geq 0} (\cos tx + \sin(1-t)x) \sec x \cdot (1-t)^m y^m dt \\
&\text{(by Lemma 2.1)} \\
&= \int_0^1 (\cos tx + \sin(1-t)x) \sec x \sum_{m\geq 0} (1-t)^m y^m dt \\
&= \int_0^1 \frac{\cos tx + \sin(1-t)x}{1-(1-t)y} \sec x dt. \quad \square
\end{aligned}$$

□

### 3.2. Double Star Graphs

Next, we consider the case when two graph are connected via a single edge (called a *bridge*). Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with $V_1 = \{x_1, \ldots, x_m, x\}$ and $V_2 = \{y_1, \ldots, y_n, y\}$. Then we define $G = G_1 \circ G_2$ by $G = (V, E)$ with $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup \{xy\}$. Then

$$vol(P(G_1)) = \int_0^1 \varphi_1(x)dx = \int_{Q_{m+1}} H_1(x_1, x_2, \ldots, x_m, x)dx_1dx_2 \cdots dx_m dx$$

$$vol(P(G_2)) = \int_0^1 \varphi_2(y)dy = \int_{Q_{n+1}} H_2(y_1, y_2, \ldots, y_n, y)dy_1dy_2 \cdots dy_n dy$$

$$\begin{aligned} vol(P(G)) &= \int_{Q_{m+n+2}} H(x_1, \ldots, x, y_1, \ldots, y)dx_1 \cdots dx dy_1 \cdots dy \\ &= \int\int_{[0,1]^2} \varphi_1(x)\varphi_2(y)K(x,y)dxdy \\ &= \int_0^1 [\int_0^{1-y} \varphi_1(x)dx]\varphi_2(y)dy \end{aligned}$$

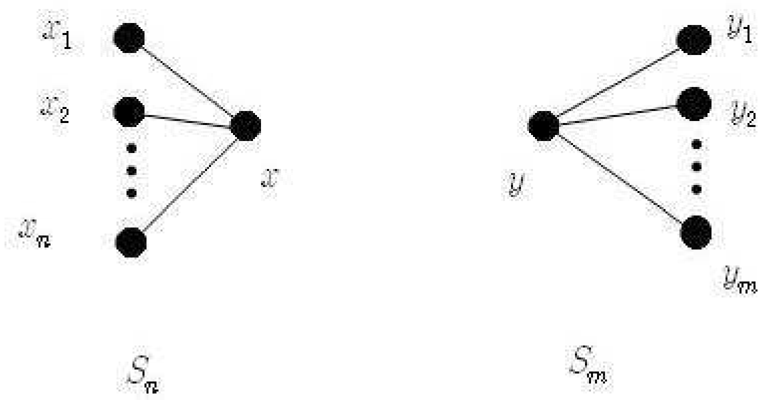


FIGURE 2

A double star graph is the graph $DS(n,m)$ where two star graphs ($S_n$ and $S_m$) are joined by a single edge which connects both center vertices. In other words, $DS(n,m) := S_n \circ S_m$. Remind that $vol(P(S_n)) = \int_0^1 (1-x)^n dx$ and $vol(P(S_m)) = \int_0^1 (1-y)^m dy$.

**Theorem 3.4.** *The volume of the graph polytope for the double star graph $DS(n,m)$ is*

$$vol(P(DS(n,m))) = \frac{1}{(m+1)(n+1)}(1 - \frac{1}{\binom{m+n+2}{m+1}}).$$

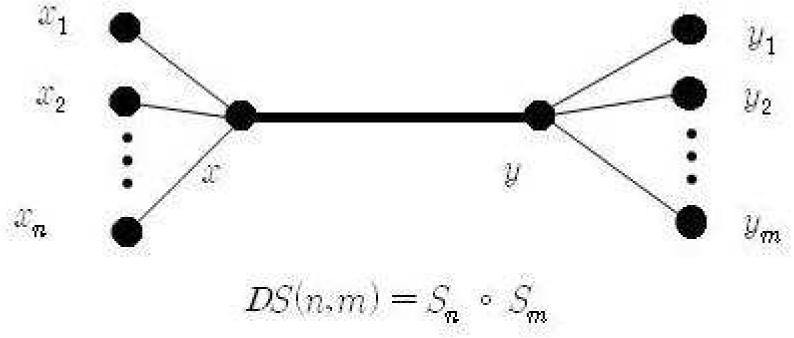


FIGURE 3

*Proof.*

$$\begin{aligned}
vol(P(DS(n,m))) &= \int_0^1 [\int_0^{1-y} (1-x)^n dx](1-y)^m dy \\
&= \int_0^1 [-\frac{1}{n+1}(1-x)^{n+1}]_0^{1-y}(1-y)^m dy \\
&= \int_0^1 (-\frac{1}{n+1}y^{n+1} + \frac{1}{n+1})(1-y)^m dy \\
&= \frac{1}{n+1}\int_0^1 (1-y^{n+1})(1-y)^m dy \\
&= \frac{1}{n+1}\{\int_0^1 (1-y)^m dy - \int_0^1 y^{n+1}(1-y)^m dy\} \\
&= \frac{1}{(m+1)(n+1)} - \frac{1}{n+1}B(n+2,m+1) \\
&= \frac{1}{(m+1)(n+1)}(1 - \frac{1}{\binom{m+n+2}{m+1}})
\end{aligned}$$

□

Remark that $vol(P(DS(n,m))) = vol(P(DS(m,n)))$.

### 3.3. Star-Path-Star Graphs

A star-path-star graph is the graph $SPS(l,n,m)$ where two star graphs ($S_l$ and $S_m$) are joined by a path ($L_n$) which connects both center vertices. (See Figures 4 and 5.)

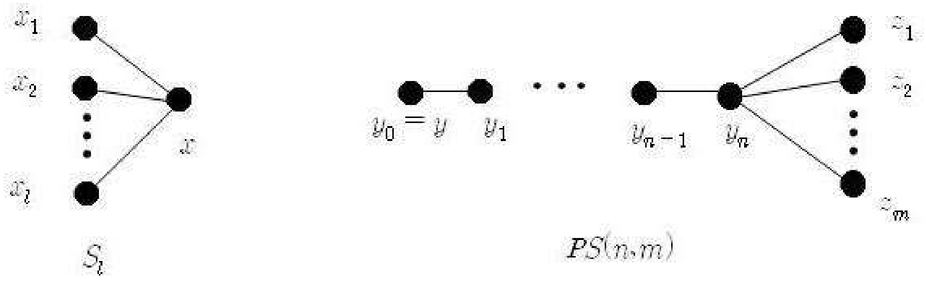


FIGURE 4

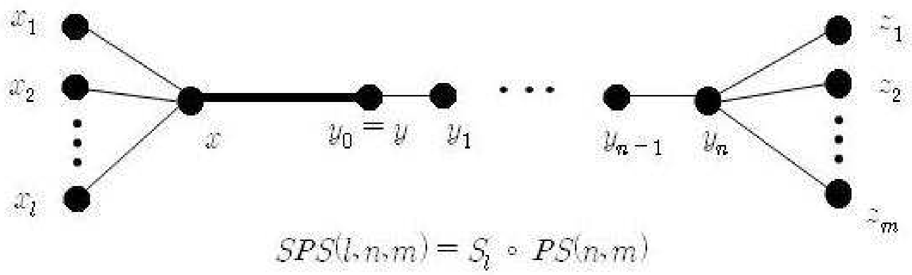


FIGURE 5

**Theorem 3.5.** *The volume of the graph polytope for a star-path-star graph $SPS(l,n,m)$ is the following. If $n$ is odd,*

$$vol(P(SPS(\ell,n,m))) = \frac{1}{\ell+1}\Big[\sum_{k\geq 0}^{\frac{n-1}{2}} \frac{(-1)^k E_{n-2k-1}}{(2k+1)!(n-2k-1)!(2k+m+2)} - \sum_{k\geq 0} \frac{(-1)^k E_{n-2k-1}(\ell+1)!(2k+m+1)!}{(2k+1)!(n-2k-1)!(\ell+2k+m+3)!}\Big],$$

*and if $n$ is even,*

$$vol(P(SPS(\ell,n,m))) = \frac{1}{\ell+1}\Big[\sum_{k\geq 0}^{\frac{n}{2}} \frac{(-1)^k E_{n-2k}}{(2k)!(n-2k)!(2k+m+1)} \cdot \frac{1}{\binom{2k+m}{2k}} - \sum_{k\geq 0} \frac{(-1)^k E_{n-2k}(\ell+2k+1)!m!}{(2k)!(n-2k)!(\ell+2k+m+2)!}\Big].$$

*Proof.* The proof is immediate if we use the formulas $vol(P(S_l)) = \int_0^1 (1-x)^l dx$ and $vol(P(PS(n,m)) = \int_0^1 f_n(y)(1-y)^l dy$. □

## 4. Posets and Linear Extensions([7])

A *poset*, or partially ordered set is an ordered pair $P = (X, \leq)$ where $X$ is a nonempty set and $\leq$ is a relation on $X$ satisfying reflexivity, antisymmetry and transitivity. A *chain* of a poset $P = (X, \leq)$ is a nonempty subset of $X$ containing pairwise comparable elements. The *height* of a poset is the maximum size of chains in the poset. Every bipartite graph can be regarded as a poset of height 2. A *linear extension* of a poset is a linear order (or total order) that is comparable with the partial order in the poset.

**Example 4.1.** *A zigzag poset (or called a fence) is an well-known example of the posets, where the order relations form a path with alternating orientations:*

$$x_1 < x_2 > x_3 < x_4 > x_5 < \cdots > (<) x_n.$$

*A linear extension of a zigzag poset is called an alternating permutation. The number of different linear extensions in this poset is well-known and is the Euler zigzag number or up/down number(sequence A000111 in OEIS* [4]*). The sequence is as follows:*

$$1, 1, 1, 2, 5, 16, 61, 272, 1385, \cdots.$$

*For the case $n = 4$,*

$$x_1x_3x_2x_4, x_1x_3x_4x_2, x_3x_1x_2x_4, x_3x_1x_4x_2, x_3x_4x_1x_2$$

*are all of the linear extensions.*

## 5. Relationship between Volumes of the Graph Polytopes and Number of Linear Extensions

By Theorem 2.3, for a given bipartite graph we can obtain a new formula to get the number of linear extensions of the poset corresponding to the given bipartite graph. In order to prove this we need some preliminary knowledge.

Let $G = ([m] \sqcup ([n]\backslash[m]), E)$ be a bipartite graph, $Q = [0,1]^n$. Then we can regard it as a poset like in the following. Consider $n-$vector

$$(x_1, x_2, \ldots, x_m, y_{m+1}, \ldots, y_n) \in Q.$$

We consider the inequalities $x_i + y_j \leq 1$ whenever $ij \in E$. Now, let $x_i := 1 - y_i$ for all $i \in V_2$. Then $x_i \leq x_j$ must holds whenever $ij \in E$. Bipartite poset $(V(G), \leq)$ associated with the bipartite graph $G = ([m] \sqcup ([n]\backslash[m]), E)$ is the ground set $[n]$ with the partial order:

$$i \leq j \text{ whenever } i \in V_1, j \in V_2 \text{ and } ij \in E.$$

Note that bipartite posets defined as above are of height 2.
The volume of the graph polytope

$$P(G) = \{(x_1, x_2, \ldots, x_m, y_{m+1}, \ldots, y_n) \in [0,1]^n \mid x_i + y_j \leq 1 \forall ij \in E\}$$

can be described as

$$\begin{aligned} vol(P(G)) &= vol(\{(x_1, x_2, \ldots, x_m, y_{m+1}, \ldots, y_n) \in Q \mid x_i + y_j \leq 1 \forall ij \in E\}) \\ &= vol(\{x \in Q \mid x_i \leq x_j \forall ij \in E\}) \\ &= vol(\{x \in Q \mid x_i \leq x_j \text{ whenever } j \text{ covers } i \text{ in the poset } V(G)\}) \\ &\quad \text{where } x = (x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n). \end{aligned}$$

Let $e(V(G))$ be the set of all linear extensions in the bipartite poset $V(G)$. Then we have

$$\{x \in Q \mid x_i \leq x_j \text{ whenever } j \text{ covers } i \text{ in the poset } V(G)\} = \bigcup_{\sigma \in e(V(G))} P(\sigma),$$

where

$$P(\sigma) = \{x \in Q \mid 0 \leq x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n} \leq 1\}_{\sigma=(i_1, i_2, \ldots, i_n)}.$$

Note that $vol(P(\sigma)) = \frac{1}{n!}$ and $vol(P(\sigma) \cap P(\tau) = 0$ if $\sigma, \tau \in e(V(G))$ with $\sigma \neq \tau$. Let $e(G)$ be the number of linear extensions in the bipartite poset $V(G)$. Then we have the conclusion stated in the next theorem.

**Proposition 5.1.** *Let $G = (V = V_1 \sqcup V_2, E)$ be a bipartite graph, $P(G)$ its associated graph polytope, $(V(G), \leq)$ its bipartite poset, and $e(G)$ the number of linear extensions for the bipartite poset $(V(G), \leq)$. Then the formula*

$$e(G) = n! \cdot vol(P(G))$$

*holds.*

**Theorem 5.2.** *Let $G = (V = V_1 \sqcup V_2, E)$ be a bipartite graph. Then the recursion formula on the number of linear extensions holds:*

$$e(G) = \frac{1}{2} \sum_{i \in V} e(G - i),$$

*where $G - i$ is the graph with the vertex set $V \backslash \{i\}$ and, accordingly, with the inherited edge set in the original edge set $E$.*

*Proof.* From Theorem 2.3 and Proposition 5.1 our conclusion follows immediately. □

Recently we knew that the formula obtained in Theorem 5.2 is exactly the one that Stachowiak [3] had found in 1988. However, we get the formula directly from the recursive volume formula(RVF) given in Theorem 2.3.

**Example 5.3.** *Let $G = (V, E)$ where $V = \{1, 2\} \cup \{3, 4\}, E = \{13, 23, 24\}$. From the bipartite graph $G$ we obtain the bipartite poset $(V(G), \leq)$ whose order relation is as follows: $1 \leq 3 \geq 2 \leq 4$. There are 5 linear extensions $1234, 1243, 2134, 2143, 2413$ as we mentioned in the previous example.*

$$P(G) = \{(x_1, x_2, y_3, y_4) \in [0, 1]^4 | x_1 + y_3 \leq 1, x_2 + y_3 \leq 1, x_2 + y_4 \leq 1\}.$$

*We let $x_3 := 1 - y_3, x_4 := 1 - y_4$. Then we get a new polytope*

$$P'(G) = \{(x_1, x_2, x_3, x_4) \in [0, 1]^4 | x_1 \leq x_3, x_2 \leq x_3, x_2 \leq x_4\}.$$

*Note that $vol(P(G)) = vol(P'(G))$. We can decompose $P'(G))$ into 5 simplices (each of them has volume $1/4!$), like*

$$
\begin{array}{ccc}
1234 & \leftrightarrow & \{0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1\} \\
1243 & \leftrightarrow & \{0 \leq x_1 \leq x_2 \leq x_4 \leq x_3 \leq 1\} \\
2134 & \leftrightarrow & \{0 \leq x_2 \leq x_1 \leq x_3 \leq x_4 \leq 1\} \\
2143 & \leftrightarrow & \{0 \leq x_2 \leq x_1 \leq x_4 \leq x_3 \leq 1\} \\
2413 & \leftrightarrow & \{0 \leq x_2 \leq x_4 \leq x_1 \leq x_3 \leq 1\}.
\end{array}
$$

| $n\backslash m$ | m=0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| n=0 | 1 | 1 | 2 | 6 | 24 | 120 | 720 | 5040 |
| 1 | 1 | 2 | 6 | 24 | 120 | 720 | 5040 | 40320 |
| 2 | 2 | 5 | 18 | 84 | 480 | 3240 | 25200 | 221760 |
| 3 | 5 | 16 | 70 | 384 | 2520 | 19200 | 166320 | 1612800 |
| 4 | 16 | 61 | 310 | 1938 | 14280 | 120720 | 1149120 | 12146400 |
| 5 | 61 | 272 | 1582 | 11136 | 91224 | 848640 | 8814960 | 100961280 |
| 6 | 272 | 1385 | 9058 | 70824 | 638064 | 6474480 | 72848160 | 898470720 |
| 7 | 1385 | 7936 | 57678 | 496128 | 4876344 | 53606400 | 649574640 | 8583966720 |

TABLE 1. Table of $e(PS(n,m))(0 \leq m, n \leq 7)$

Theorem 3.2 and Theorem 5.2 give next results concerning the number of linear extensions.

**Corollary 5.4.** *The number $e(PS(n,m))$ of linear extensions in the path-star graph of order $n, m$ is given by the following formula:*

$$e(PS(n,m)) = \begin{cases} (n+m+1)! \sum_{k\geq 0} \frac{(-1)^k E_{n-2k-1}}{(2k+1)!(n-2k-1)!(2k+m+2)}, & n \text{ is odd}, \\ (n+m+1)! \sum_{k\geq 0} \frac{(-1)^k E_{n-2k}}{(2k)!(n-2k)!(2k+m+1)} \cdot \frac{1}{\binom{2k+m}{2k}}, & n \text{ is even}. \end{cases}$$

**Remark 5.5.** *Table 1 shows values of $e(PS(n,m))$ for small values of $n$ and $m$. There is an information (see OEIS[4]) about the sequences given by the first three rows and columns as below:*

- $m = 0 \leftrightarrow E_{n+1}$ *(A000111)*
- $m = 1 \leftrightarrow E_{n+2}$ *(A000111)*
- $m = 2 \leftrightarrow [2(x-1)\tan(\frac{x}{2} + \frac{\pi}{4}) - x^2 + 2]/x^3$ *(A131281)*
- $n = 0 \leftrightarrow n!$ *(A000142)*
- $n = 1 \leftrightarrow (m+1)!$ *(A000142)*
- $n = 2 \leftrightarrow \frac{(m+1)!(m+4)}{2}$ *(A038720)*

*There is no information in the other sequences in Table 1.*

## 6. Concluding Remarks

One of the most curious problems is the characterization of the volume size of the graph polytopes for the trees of $n$-vertices. So far, all the volumes of the graph polytopes in the trees are distinct for different trees up to $n = 11$(we checked it!). Is there any general method to order the volumes in the tree case linearly? All the path-star type graphs are specific kinds of trees. If we generalize our formula in such a way that at each vertex of the path we attach suitable size of star graph, we guess that we can describe the volume formulas for some other types of the trees. We also can raise the problem for the cycles rather than paths as well as the complete graphs, which seems to be quite nontrivial. It would be interesting to analyze the poset structures which consist of the faces of the graph polytopes for the path-star type graphs.

**Acknowledgements**

The authors would like to express deep gratitude to the anonymous referees for a very careful reading of the original manuscript and useful suggestions.

## References

[1] M. Bóna, H.-K. Ju and R. Yoshida, *On the enumeration of a certain weighted graphs*, Discrete Applied Math., 155(2007), 1481-1496.
[2] H.-K. Ju, S. Kim and S.-J. Seo, *On the Volume of Graph Polytopes*, Honam Math. J. **37**(2015), No.3, 361-376.
[3] D. Lee and H.-K. Ju, *Different volume computation methods of graph polytopes*. Preprint(arXiv:1507.07623v1 [math.**CO**]), 2015, submitted.
[4] *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://oeis.org.
[5] W. Rudin, Principles of Mathematical Analysis (3rd ed.), McGraw-Hill, 1976.
[6] G. Stachowiak, *The number of linear extensions of bipartite graphs*, Order, **5**(1988), 257-259.
[7] R. Stanley, Enumerative Combinatorics vol.1 (2nd ed.), Cambridge Univ. Press, Cambridge, 2012.

Hyeong-Kwan Ju
Department of Mathematics, Chonnam National University,
Gwangju 500-757, Republic of Korea.
E-mail: hkju@jnu.ac.kr

Soo-Jeong Seo
Department of Mathematics, Chonnam National University,
Gwangju 500-757, Republic of Korea.
E-mail: yahwoo98@gmail.com