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ON THE TATE-SHAFAREVICH GROUPS OVER DEGREE 3 NON-GALOIS EXTENSIONS

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Abstract. Let A be an abelian variety defined over a number field K and let L be a degree 3 non-Galois extension of K. Let $Ш(A/K)$ and $Ш(A/L)$ denote, respectively, the Tate-Shafarevich groups of A over K and over L. Assuming that $Ш(A/L)$ is finite, we compute $[Ш(A/K)][Ш(A_\varphi/K)]/[Ш(A/L)]$, where $[X]$ is the order of a finite abelian group X.

1. Introduction

Let K be a number field and let L be a degree 3 non-Galois extension of K. Let L_0 be the number field containing L which is Galois over K such that the Galois gruop $\mathrm{Gal}(L_0/K)$ is isomorphic to a symmetric group S_3. Write $\overline{K}$, G_K, M_K, K_v for the algebraic closure of K, $\mathrm{Gal}(\overline{K}/K)$, a complete set of places on K, the completion of K at the place $v \in M_K$, respectively. Fix $\sigma \in G_K - G_L$ and fix $\tau \in G_L - G_{L_0}$.

Let A be an abelian variety defined over K. We define the restriction of scalars $Res_{L/K}(A)$ of A from L to K as Weil did in [11, p.5]. There is an abelian variety $Res_{L/K}(A)$ defined over K with an isomorphism $\phi\colon Res_{L/K}(A) \to A$ defined over L such that

$$(\phi, \sigma(\phi), \sigma^2(\phi))\colon Res_{L/K}(A) \to A \times A \times A$$

is an isomorphism. For the properties of the restriction of scalars, see [11, p.5].

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Let Φ be a 3-dimensional integral representation of S_3 defined by

$$\Phi(\sigma) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Phi(\tau) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and let φ be a 2-dimensional integral representation of S_3 defined by

$$\varphi(\sigma) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \varphi(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Assume that 1 is the identity automorphism on A. Note that $Res_{L/K}(A)$ is a twist of $A^3 := A \times A \times A$ defined over K such that

$$\tau \begin{pmatrix} \phi \\ \sigma(\phi) \\ \sigma^2(\phi) \end{pmatrix} = \Phi(\tau) \begin{pmatrix} \phi \\ \sigma(\phi) \\ \sigma^2(\phi) \end{pmatrix}.$$

There is a twist A_φ of $A^2 := A \times A$ defined over K with an isomorphism $\widetilde{\varphi}\colon A_\varphi \to A^2$ defined over L_0 satisfying $\sigma(\widetilde{\varphi}) = \varphi(\sigma) \circ \widetilde{\varphi}$ and $\tau(\widetilde{\varphi}) = \varphi(\tau) \circ \widetilde{\varphi}$.

Let Ш(A/K) and Ш(A/L) denote the Tate-Shafarevich groups of A over K and over L, respectively. We assume throughout that these groups are finite. We write $[X]$ for the order of a finite abelian group X.

Note that $Res_{L/K}(A)$ is isomorphic to $A \times A_\varphi$ over L_0 and is isogeneous to $A \times A_\varphi$ over K. But the Tate-Shafarevich group is not an isogeny invariant and in general,

$$[\text{Ш}(A/L)] = [\text{Ш}(Res_{L/K}(A)/K)] \neq [\text{Ш}(A/K)][\text{Ш}(A_\varphi/K)].$$

The difference was computed for quadratic extensions in [12, Main Theorem] and for cyclic extensions in [13, Main Theorem]. In this paper we derive a formula relating $[\text{Ш}(A/L)]$, $[\text{Ш}(A/K)]$ and $[\text{Ш}(A_\varphi/K)]$ for the non-Galois extension L/K.

We construct a short exact sequence of abelian varieties defined over K

(1) $$0 \longrightarrow A \xrightarrow{f_1} Res_{L/K}(A) \xrightarrow{f_2} A_\varphi \longrightarrow 0$$

and its dual exact sequence

$$0 \longrightarrow A_\varphi^\vee \xrightarrow{f_2^\vee} Res_{L/K}(A^\vee) \xrightarrow{f_1^\vee} A^\vee \longrightarrow 0,$$

where $X^\vee$ is the dual abelian variety of an abelian variety X.

Denote by $\mathrm{H}^1(f_1)\colon \mathrm{H}^1(K, A) \to \mathrm{H}^1(K, Res_{L/K}(A))$ the induced morphism from $f_1\colon A \to Res_{L/K}(A)$ and denote by $\mathrm{H}^1(f_{1,v})\colon \mathrm{H}^1(K_v, A) \to$

$\mathrm{H}^1(K_v, Res_{L/K}(A))$ the induced morphism from the map $f_{1,v}\colon A(\overline{K}_v) \to Res_{L/K}(\overline{K}_v)$.

For a morphism $f\colon A_1 \to A_2$ defined over K, write $f(K)\colon A_1(K) \to A_2(K)$.

Main Theorem. *Assume that the Tate-Shafarevich groups are finite. Then*

$$\frac{[Ш(A/K)][Ш(A_\varphi/K)]}{[Ш(A/L)]} = \frac{[\mathrm{Ker}(\mathrm{H}^1(f_1))][\mathrm{Ker}(\mathrm{H}^1(f_2^\vee))]}{\prod_{v\in M_K}[\mathrm{Ker}(\mathrm{H}^1(f_{1,v}))]} = \frac{[\mathrm{Coker}(f_2(K))][\mathrm{Coker}(f_1^\vee(K))]}{\prod_{v\in M_K}[\mathrm{Coker}(f_{2,v}(K_v))]}.$$

Proof. The first equality of the theorem is obvious from Theorem 6 and Corollay 8 . From the short exact sequence (1) we have the exact sequence

$$Res_{L/K}(A)(K) \xrightarrow{f_2(K)} A_\varphi(K) \to \mathrm{H}^1(K,A) \xrightarrow{\mathrm{H}^1(f_1)} \mathrm{H}^1(K, Res_{L/K}(A))$$

and thus

$$\mathrm{Ker}(\mathrm{H}^1(f_1)) \cong \mathrm{Coker}(f_2(K)\colon Res_{L/K}(A)(K) \to A_\varphi(K)).$$

Similarly, $\mathrm{Ker}(\mathrm{H}^1(f_{1,v})) \cong \mathrm{Coker}(f_{2,v}(K)\colon Res_{L/K}(A)(K_v) \to A_\varphi(K_v))$ and $\mathrm{Ker}(\mathrm{H}^1(f_2^\vee)) \cong \mathrm{Coker}(f_1^\vee(K)\colon Res_{L/K}(A^\vee)(K) \to A^\vee(K))$. □

2. Tate-Shafarevich groups over exact sequences

Define 1 to be the identity automorphism on A. Define 3×1 matrix $M_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ and 2×3 matrix $M_2 = \begin{pmatrix}1 & 0 & -1\\ 1 & -1 & 0\end{pmatrix}$. Then we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{M_1} A^3 \xrightarrow{M_2} A^2 \longrightarrow 0.$$

With the representations Φ and φ of S_3, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{M_1} & A^3 & \xrightarrow{M_2} & A^2 & \longrightarrow & 0\\ & & \| & & \downarrow{\scriptstyle \Phi(\bullet)} & & \downarrow{\scriptstyle \varphi(\bullet)} & & \\ 0 & \longrightarrow & A & \xrightarrow{M_1} & A^3 & \xrightarrow{M_2} & A^2 & \longrightarrow & 0, \end{array}$$

which induces a short exact sequence of abelian varieties over K

$$0 \longrightarrow A \xrightarrow{f_1} Res_{L/K}(A) \xrightarrow{f_2} A_\varphi \longrightarrow 0 \tag{2}$$

such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f_1} & Res_{L/K}(A) & \xrightarrow{f_2} & A_\varphi & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\widehat{\Phi}} & & \downarrow{\scriptstyle\widetilde{\varphi}} & & \\ 0 & \longrightarrow & A & \xrightarrow{M_1} & A^3 & \xrightarrow{M_2} & A^2 & \longrightarrow & 0, \end{array}$$

where $\widehat{\Phi} = (\phi, \sigma(\phi), \sigma^2(\phi))$.

Now we have the dual exact sequence of the exact sequence (2)

$$0 \longrightarrow A^\vee_\varphi \xrightarrow{f_2^\vee} Res_{L/K}(A^\vee) \xrightarrow{f_1^\vee} A^\vee \longrightarrow 0, \tag{3}$$

which induces a long exact sequence

$$\begin{aligned} 0 &\longrightarrow A^\vee_\varphi(K) \longrightarrow Res_{L/K}(A^\vee)(K) \xrightarrow{f_1^\vee(K)} A^\vee(K) \\ &\longrightarrow \mathrm{H}^1(K, A^\vee_\varphi) \xrightarrow{\mathrm{H}^1(f_2^\vee)} \mathrm{H}^1(K, Res_{L/K}(A^\vee)) \longrightarrow \mathrm{H}^1(K, A^\vee). \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Ker}(\mathrm{H}^1(f_2^\vee)) &\cong \mathrm{Coker}(f_1^\vee(K) \colon Res_{L/K}(A^\vee)(K) \to A^\vee(K)) \\ &= \frac{A^\vee(K)}{f_1^\vee\left(Res_{L/K}(A^\vee)(K)\right)}. \end{aligned}$$

Now the exact sequence

$$0 \longrightarrow \mathrm{Ker}(\mathrm{H}^1(f_2^\vee)) \longrightarrow \mathrm{H}^1(K, A^\vee_\varphi) \xrightarrow{\mathrm{H}^1(f_2^\vee)} \mathrm{H}^1(K, Res_{L/K}(A^\vee)).$$

induces a natural commutative diagram:

(4)

$$\begin{array}{ccccc} \mathrm{Ker}(\mathrm{H}^1(f_2^\vee)) & \hookrightarrow & \mathrm{H}^1(K, A^\vee_\varphi) & \xrightarrow{\mathrm{H}^1(f_2^\vee)} & f_2^\vee(\mathrm{H}^1(K, A^\vee_\varphi)) \longrightarrow 0 \\ \downarrow{\scriptstyle\mathcal{F}^\vee} & & \downarrow{\scriptstyle\mathcal{G}^\vee} & & \downarrow{\scriptstyle\mathcal{H}^\vee} \\ \bigoplus_{v\in M_K} \mathrm{Ker}(\mathrm{H}^1(f_{2,v}^\vee)) & \hookrightarrow & \bigoplus_{v\in M_K} \mathrm{H}^1(K_v, A^\vee_\varphi) & \xrightarrow{\oplus \mathrm{H}^1(f_{2,v}^\vee)} & \bigoplus_{v\in M_K} \mathrm{H}^1(K_v, Res_{L/K}(A^\vee)), \end{array}$$

where $f_{1,v} \colon A(\overline{K}_v) \to Res_{L/K}(\overline{K}_v)$ is derived from f_1.

We denote by $\mathcal{I}$ the map $\mathrm{Coker}(\mathcal{F}^\vee) \to \mathrm{Coker}(\mathcal{G}^\vee)$ induced from above diagram (4). Therefore,

$$\begin{aligned} \mathrm{Ker}(\mathcal{H}^\vee) &= Ш(Res_{L/K}(A^\vee)/K) \cap \mathrm{H}^1(f_2^\vee)(\mathrm{H}^1(K, A^\vee_\varphi)) \\ &= Ш(Res_{L/K}(A^\vee)/K) \cap \mathrm{Ker}(\mathrm{H}^1(f_1^\vee)). \end{aligned}$$

Note that $\mathrm{Ker}(\mathcal{G}^\vee) = Ш(A_\varphi^\vee/K)$. Then the *Kernel-Cokernel* sequence of diagram (4) is

(5)
$$0 \to \mathrm{Ker}(\mathcal{F}^\vee) \longrightarrow Ш(A_\varphi^\vee/K) \longrightarrow Ш(Res_{L/K}(A^\vee)/K) \cap \mathrm{Ker}(\mathrm{H}^1(f_1^\vee))$$
$$\longrightarrow \mathrm{Coker}(\mathcal{F}^\vee) \longrightarrow \mathcal{I}(\mathrm{Coker}(\mathcal{F}^\vee)) \longrightarrow 0.$$

For a topological abelian group M, let $\widehat{M}$ be the completion of M with respect to the topology defined by the subgroups of finite index. Write M^* for the group of continuous characters of finite order of M, i.e. $M^* = \mathrm{Hom}_{\mathrm{cts}}(M, \mathbf{Q}/\mathbf{Z})$.

Theorem 1 (Global Duality Theorem). *Assume that $Ш(A/K)$ is finite. Then there is an exact sequence:*

$$0 \to Ш(A/K) \to \mathrm{H}^1(K, A) \to \bigoplus_{v \in M_K} \mathrm{H}^1(K_v, A) \to \widehat{A^\vee(K)}^{\,*} \to 0.$$

Proof. See [1, Corollary 1], [3, Theorem 1.1] or [5, I.6.14(b)]. □

Theorem 2 (Local Duality Theorem). *For a place $v \in M_K$ there exists a bilinear, non-degenerate pairing*

$$\langle\ ,\ \rangle : \mathrm{H}^0(K_v, A^\vee) \times \mathrm{H}^1(K_v, A) \longrightarrow \mathbf{Q}/\mathbf{Z}.$$

Proof. See [9, p.156-04], [10, p.289] and [5, I.3.4 and I.3.7]. □

Here $\mathrm{H}^0(K_v, A^\vee) = A^\vee(K_v)$ unless v is archimedian, in which case it equals the quotient of $A^\vee(K_v)$ by its identity component (see [10, p.289]).

Lemma 3. *The dual of the exact sequence*

$$0 \longrightarrow \mathrm{Ker}(\mathrm{H}^1(f_{2,v}^\vee)) \longrightarrow \mathrm{H}^1(K_v, A_\varphi^\vee) \xrightarrow{\mathrm{H}^1(f_{2,v}^\vee)} \mathrm{H}^1(K_v, Res_{L/K}(A^\vee))$$

is the exact sequence

$$0 \longleftarrow \mathrm{Ker}(\mathrm{H}^1(f_{1,v})) \longleftarrow \mathrm{H}^0(K_v, A_\varphi) \xleftarrow{\mathrm{H}^0(f_{2,v})} \mathrm{H}^0(K_v, Res_{L/K}(A)).$$

Proof. The exact sequence

$$0 \longrightarrow A \xrightarrow{f_1} Res_{L/K}(A) \xrightarrow{f_2} A_\varphi \longrightarrow 0$$

induces the exact sequence

$$\begin{array}{ccc} \mathrm{H}^1(K_v, Res_{L/K}(A)) \xleftarrow{\mathrm{H}^1(f_{1,v})} & \mathrm{H}^1(K_v, A) & \\ & \uparrow & \\ & \mathrm{H}^0(K_v, A_\varphi) & \longleftarrow \mathrm{H}^0(K_v, Res_{L/K}(A)). \end{array}$$

Then the lemma follows from the local duality theorem. □

Lemma 4. *Suppose that M is a finite abelian group and that M' is an abelian group. Let $f\colon M \to M'$ be a group homomorphism and let $\mathrm{Hom}(f,\cdot)\colon \mathrm{Hom}(M',\mathbf{Q}/\mathbf{Z}) \to \mathrm{Hom}(M,\mathbf{Q}/\mathbf{Z})$ be the dual of f. Then $[\text{image of } \mathrm{Hom}(f,\cdot)] = [\text{image of } f]$.*

Proof. It is obvious. □

Lemma 5. *Define $\mathcal{F}\colon \mathrm{Ker}(\mathrm{H}^1(f_1)) \to \prod_{v\in M_K} \mathrm{Ker}(\mathrm{H}^1(f_{1,v}))$. Then*

$$\left[\mathcal{I}(\mathrm{Coker}(\mathcal{F}^\vee))\right] = \left[\mathrm{Ker}(\mathrm{H}^1(f_1))/\,\mathrm{Ker}(\mathcal{F})\right].$$

Proof. From diagram (4) there is the following commutative diagram:

$$\begin{array}{ccc} \bigoplus_{v\in M_K} \mathrm{Ker}(\mathrm{H}^1(f_{2,v}^\vee)) & \longrightarrow & \bigoplus_{v\in M_K} \mathrm{H}^1(K_v, A_\varphi^\vee) \\ {\scriptstyle\text{surjective}}\downarrow & & \downarrow \\ \mathrm{Coker}(\mathcal{F}^\vee) & \xrightarrow{\mathcal{I}} & \mathrm{Coker}(\mathcal{G}^\vee). \end{array} \tag{6}$$

From Lemma 3 and [5, I.6.14(b)], the dual of a composition map in the above diagram

$$\bigoplus_{v\in M_K} \mathrm{Ker}(\mathrm{H}^1(f_{2,v}^\vee)) \longrightarrow \bigoplus_{v\in M_K} \mathrm{H}^1(K_v, A_\varphi^\vee) \longrightarrow \mathrm{Coker}(\mathcal{G}^\vee) \tag{7}$$

is the composition map

$$\prod_{v\in M_K} \mathrm{Ker}(\mathrm{H}^1(f_{1,v})) \longleftarrow \prod_{v\in M_K} \mathrm{H}^0(K_v, A_\varphi) \longleftarrow \widehat{A_\varphi(K)}. \tag{8}$$

Diagram (6) implies that $[\mathcal{I}(\mathrm{Coker}(\mathcal{F}^\vee))] = [\text{image of the map (7)}]$ and Lemma 4 implies [image of the map (7)] = [image of the map (8)]. From the following natural commutative diagram:

$$\begin{array}{ccc} \prod_{v\in M_K} \mathrm{Ker}(\mathrm{H}^1(f_{1,v})) & \longleftarrow & \prod_{v\in M_K} \mathrm{H}^0(K_v, A_\varphi) \\ {\scriptstyle\mathcal{F}}\uparrow & & \uparrow \\ \mathrm{Ker}(\mathrm{H}^1(f_1)) & \underset{\text{surjective}}{\longleftarrow} & \widehat{A_\varphi(K)}, \end{array}$$

[image of the map (8)] = [image of $\mathcal{F}$] = $[\mathrm{Ker}(\mathrm{H}^1(f_1))/\,\mathrm{Ker}(\mathcal{F})]$. Then the lemma follows. □

Theorem 6. *Assume that Ш(A/L) is finite. Then*

$$\frac{[\text{Ш}(A_\varphi^\vee/K)][\mathrm{Ker}(\mathcal{F})]}{[\text{Ш}(Res_{L/K}(A^\vee)/K) \cap \mathrm{Ker}(\mathrm{H}^1(f_1^\vee))]} = \frac{[\mathrm{Ker}(\mathrm{H}^1(f_1))][\mathrm{Ker}(\mathrm{H}^1(f_2^\vee))]}{\prod_{v\in M_K}[\mathrm{Ker}(\mathrm{H}^1(f_{1,v}))]}.$$

Proof. From the map $\mathcal{F}^\vee$ in diagram (4), we have

$$\frac{[\mathrm{Coker}(\mathcal{F}^\vee)]}{[\mathrm{Ker}(\mathcal{F}^\vee)]} = \frac{[\bigoplus_v \mathrm{Ker}(\mathrm{H}^1(f_{1,v}))]}{[\mathrm{H}^1(G, A(L))]}.$$

Then from the sequence (5) and Lemma 5, the theorem is immediate. □

3. Cassels pairing

When Ш(A/K) is finite, there is a canonical pairing

$$Ш(A/K) \times Ш(A^\vee/K) \longrightarrow \mathbf{Q}/\mathbf{Z},$$

which is non-degenerate. This pairing will be called Cassels pairing. For details, see [4], [10, p.292], [5, pp.96–99] and [6, 12.2].

Let $\langle -, - \rangle_K : Ш(A/K) \times Ш(A^\vee/K) \to \mathbf{Q}/\mathbf{Z}$ be the Cassels pairing for A defined over K, and let $\langle -, - \rangle_L : Ш(A/L) \times Ш(A^\vee/L) \to \mathbf{Q}/\mathbf{Z}$ be the Cassels pairing for A defined over L.

Write res_A for the restriction map $\mathrm{H}^1(K, A) \to \mathrm{H}^1(L, A)$ and write $\mathrm{cores}_{A^\vee}$ for the corestriction map $\mathrm{H}^1(L, A^\vee) \to \mathrm{H}^1(K, A^\vee)$ (for the definition see [7] or [8, p.259]).

Theorem 7. *For $a \in Ш(A/K)$ and $b^\vee \in Ш(A^\vee/L)$*

$$\langle a, \mathrm{cores}(b^\vee) \rangle_K = \langle \mathrm{res}(a), b^\vee \rangle_L .$$

Proof. See [12, p.216]. □

Corollary 8. *We get*

$$\frac{[\mathrm{Ker}(\mathcal{F})]}{[Ш(Res_{L/K}(A^\vee)/K) \cap \mathrm{Ker}(\mathrm{H}^1(f_1^\vee))]} = \frac{[Ш(A/K)]}{[Ш(A/L)]}.$$

Proof. From Shapiro's lemma (see [2, (6.2) Proposition]) we have an isomorphism $\mathrm{H}^1(K, Res_{L/K}(A^\vee)) \cong \mathrm{H}^1(L, A^\vee)$. From the previous theorem and the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^1(K, Res_{L/K}(A^\vee)) & \xrightarrow{\cong} & \mathrm{H}^1(L, A^\vee) \\ & \searrow^{f_1^\vee} & \downarrow \mathrm{cores}_{A^\vee} \\ & & \mathrm{H}^1(K, A^\vee), \end{array}$$

we have the isomorphism:

$$\begin{aligned} Ш(Res_{L/K}(A^\vee)/K) \cap \mathrm{Ker}(\mathrm{H}^1(f_1^\vee)) &\cong Ш(A^\vee/L) \cap \mathrm{Ker}(\mathrm{cores}_{A^\vee}) \\ &\cong \mathrm{Hom}(Ш(A/L)/\,\mathrm{res}_A(Ш(A/K)), \mathbf{Q}/\mathbf{Z}). \end{aligned}$$

From the isomorphism $Ш(A/K)/\operatorname{Ker}(\mathrm{res}_A) \cong \mathrm{res}_A(Ш(A/K))$,

$$\frac{[\operatorname{Ker}(\mathrm{res}_A \colon Ш(A/K) \to Ш(A/L))]}{[Ш(Res_{L/K}(A^\vee)/K) \cap \operatorname{Ker}(\mathrm{H}^1(f_1^\vee))]} = \frac{[Ш(A/K)]}{[Ш(A)/L)]}.$$

Then Shapiro's lemma implies

$$\begin{aligned}\operatorname{Ker}(\mathcal{F}\colon\ \operatorname{Ker}(\mathrm{H}^1(f_1)) \to \prod_{v\in M_K} &\operatorname{Ker}(\mathrm{H}^1(f_{1,v}))\\ &= \operatorname{Ker}(\mathrm{H}^1(f_1)\colon Ш(A/K) \to Ш(Res_{L/K}(A)/K))\\ &\cong \operatorname{Ker}(\mathrm{res}_A \colon Ш(A/K) \to Ш(A/L))\end{aligned}$$

and the corollary follows. □

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