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INT-SOFT SEMIGROUPS WITH TWO THRESHOLDS

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Abstract. In this paper, we study more general version of the paper [J. H. Lee, I. S. Kong, H. S. Kim and J. U. Jung, Generalized int-soft subsemigroups, Ann. Fuzzy Math. Inform. 8(6) (2014) 869–887]. We introduce the notion of int-soft semigroup with two thresholds ε and δ (briefly, (ε, δ)-int-soft semigroup) of a semigroup S, and investigate several related properties.

1. Introduction

Molodtsov [18] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [17] described the application of soft set theory to a decision making problem. Maji et al. [16] also studied several operations on the theory of soft sets. Feng [5] discussed soft rough sets applied to multicriteria group decision making. Many algebraic properties of soft sets are studied. (see [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 19, 21]). Song et al. [20] introduced the notion of an int-soft subsemigroup in a semigroup, and investigated their properties. Lee et al. [15] discussed further properties of int-soft subsemigroups, and then they considered generalizations of int-soft subsemigroups. They introduced the notion of θ-generalized int-soft subsemigroups in semigroups, and investigated several properties. They considered characterizations of a θ-generalized int-soft subsemigroup, and provided a condition for a special set to be a subsemigroup. They showed that the soft intersection of two θ-generalized int-soft subsemigroups over U is a θ-generalized int-soft subsemigroup over U, and

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discussed the soft pre-image and soft image of a θ-generalized int-soft subsemigroup under the homomorphism.

The aim of this paper is to study more general version of the paper [15]. We introduce the notion of int-soft semigroup with two thresholds ε and δ (briefly, (ε, δ)-int-soft semigroup) of a semigroup S, and provide many examples. We investigate the following items:

1. Relations between a δ-generalized int-soft semigroup over U and an (ε, δ)-int-soft semigroup over U.
2. Relations between an ε-generalized int-soft semigroup over U and an (ε, δ)-int-soft semigroup over U.
3. Relations between a δ-generalized int-soft semigroup over U and an $\varepsilon \cap \delta$-generalized int-soft semigroup over U.
4. Relations between an ε-generalized int-soft semigroup over U and an $\varepsilon \cap \delta$-generalized int-soft semigroup over U.
5. Relations between an $\varepsilon \cup \delta$-generalized int-soft semigroup over U and an ε-generalized int-soft semigroup over U.
6. Relations between an $\varepsilon \cup \delta$-generalized int-soft semigroup over U and δ-generalized int-soft semigroup over U.
7. Relations between an $\varepsilon \cup \delta$-generalized int-soft semigroup over U and an $\varepsilon \cap \delta$-generalized int-soft semigroup over U.
8. Relations between an $\varepsilon \cup \delta$-generalized int-soft semigroup over U and an (ε, δ)-int-soft semigroup over U.
9. Relations between an $\varepsilon \cap \delta$-generalized int-soft semigroup over U and an (ε, δ)-int-soft semigroup over U.
10. Relations between a δ-generalized int-soft semigroup over U and an ε-generalized int-soft semigroup over U.

2. Preliminaries

Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

A nonempty subset A of S is called a *subsemigroup* of S if $AA \subseteq A$, that is, $ab \in A$ for all $a, b \in A$.

Molodtsov [18] defined the soft set in the following way: Let U be an initial universe set and E be a set of parameters. Let $\mathscr{P}(U)$ denotes the power set of U and $A \subset E$.

A pair $(\tilde{f}, A)$ is called a *soft set* (see [18]) over U, where $\tilde{f}$ is a mapping given by

$$\tilde{f} : A \to \mathscr{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\tilde{f}(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set $(\tilde{f}, A)$. Clearly, a soft set is not a set. For illustration, Molodtsov considered several examples in [18].

The *soft union* of $(\tilde{f}, S)$ and $(\tilde{g}, S)$, denoted by $(\tilde{f}, S) \tilde{\cup} (\tilde{g}, S)$, is defined to be the soft set $\left(\tilde{f} \tilde{\cup} \tilde{g}, S\right)$ of S (over U) in which $\tilde{f} \tilde{\cup} \tilde{g}$ is defined by

$$\left(\tilde{f} \tilde{\cup} \tilde{g}\right)(x) = \tilde{f}(x) \cup \tilde{g}(x) \text{ for all } x \in S.$$

The *soft intersection* of $(\tilde{f}, S)$ and $(\tilde{g}, S)$, denoted by $(\tilde{f}, S) \tilde{\cap} (\tilde{g}, S)$, is defined to be the soft set $\left(\tilde{f} \tilde{\cap} \tilde{g}, S\right)$ of S (over U) in which $\tilde{f} \tilde{\cap} \tilde{g}$ is defined by

$$\left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x) = \tilde{f}(x) \cap \tilde{g}(x) \text{ for all } x \in S.$$

For a soft set $(\tilde{f}, A)$ over U and a subset γ of U, the *γ-inclusive set* of $(\tilde{f}, A)$, denoted by $(\tilde{f}, A)^{\supseteq}_{\gamma}$, is defined to be the set

$$(\tilde{f}, A)^{\supseteq}_{\gamma} := \left\{x \in A \mid \gamma \subseteq \tilde{f}(x)\right\}.$$

The *proper γ-inclusive set* of $(\tilde{f}, A)$, denoted by $(\tilde{f}, A)^{\supsetneq}_{\gamma}$, is defined to be the set

$$(\tilde{f}, A)^{\supsetneq}_{\gamma} := \left\{x \in A \mid \gamma \subsetneq \tilde{f}(x)\right\}.$$

3. Int-soft semigroups with two thresholds

In what follows, we take a semigroup S as a set of parameters, and let $\mathscr{P}^*(U) = \mathscr{P}(U) \setminus \{\emptyset\}$ unless otherwise specified.

Definition 3.1 ([20])**.** A soft set $(\tilde{f}, S)$ over U is called an *int-soft semigroup* over U if it satisfies:

$$(\forall x, y \in S)\left(\tilde{f}(xy) \supseteq \tilde{f}(x) \cap \tilde{f}(y)\right). \tag{3.1}$$

Definition 3.2 ([15]). A soft set $(\tilde{f}, S)$ over U is called a *θ-generalized int-soft semigroup* of S if there exists $\theta \in \mathscr{P}^*(U)$ such that

$$(\forall x, y \in S)\left(\tilde{f}(xy) \supseteq \theta \cap \tilde{f}(x) \cap \tilde{f}(y)\right). \tag{3.2}$$

Definition 3.3. A soft set $(\tilde{f}, S)$ over U is called an *int-soft semigroup with two thresholds ε and δ* (briefly, *(ε, δ)-int-soft semigroup*) of S if there exist $\varepsilon, \delta \in \mathscr{P}(U)$ such that

$$(\forall x, y \in S)\left(\tilde{f}(xy) \cup \varepsilon \supseteq \tilde{f}(x) \cap \tilde{f}(y) \cap \delta\right). \tag{3.3}$$

Example 3.4. Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Let $(\tilde{f}, S)$ be a soft set over $U = \mathbb{Z}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} 3\mathbb{Z} & \text{if } x = a, \\ 6\mathbb{N} & \text{if } x = b, \\ 12\mathbb{N} & \text{if } x = c. \end{cases}$$

It is routine to verify that $(\tilde{f}, S)$ is an (ε, δ)-int-soft semigroup of S with $\varepsilon = 24\mathbb{N}$ and $\delta = 12\mathbb{N}$.

Example 3.5. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	a
d	a	a	a	d

For the universe $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1, 2, 3, 6\} & \text{if } x = a, \\ U & \text{if } x = b, \\ \{6, 7, 8\} & \text{if } x \in \{c, d\}. \end{cases}$$

Then $(\tilde{f}, S)$ is an (ε, δ)-int-soft semigroup over U with $\varepsilon = \{1, 2, 3, 8\}$ and $\delta = \{3, 6, 8\}$ by the calculation in Table 1 (see Appendices).

Theorem 3.6. *For any $\varepsilon, \delta \in \mathscr{P}^*(U)$. the following assertions are valid.*

(1) *Every δ-generalized int-soft semigroup over U is both an (ε, δ)-int-soft semigroup over U and an $\varepsilon \cap \delta$-generalized int-soft semigroup over U .*
(2) *Every ε-generalized int-soft semigroup over U is an $\varepsilon \cap \delta$-generalized int-soft semigroup over U.*
(3) *Every $\varepsilon \cup \delta$-generalized int-soft semigroup over U is both an ε-generalized int-soft semigroup and a δ-generalized int-soft semigroup over U.*
(4) *Every $\varepsilon \cup \delta$-generalized int-soft semigroup over U is an $\varepsilon \cap \delta$-generalized int-soft semigroup over U.*
(5) *Every $\varepsilon \cup \delta$-generalized int-soft semigroup over U is an (ε, δ)-int-soft semigroup over U.*

Proof. Straightforward. □

Theorem 3.7. *For any $\varepsilon, \delta \in \mathscr{P}^*(U)$ with $\varepsilon \cap \delta \neq \emptyset$, $\varepsilon \nsubseteq \delta$ and $\delta \nsubseteq \varepsilon$, if $(\tilde{f}, S)$ is both an (ε, δ)-int-soft semigroup of S and an $\varepsilon \cap \delta$-generalized int-soft semigroup of S, then it is a δ-generalized int-soft semigroup of S.*

Proof. Let $(\tilde{f}, S)$ be both an (ε, δ)-int-soft semigroup of S and an $\varepsilon \cap \delta$-generalized int-soft semigroup of S. For any $x, y \in S$, we have

$$\tilde{f}(xy) \supseteq (\tilde{f}(xy) \cup \varepsilon) \cap \varepsilon^c \supseteq (\tilde{f}(x) \cap \tilde{f}(y) \cap \delta) \cap \varepsilon^c,$$

and so

$$\begin{aligned}
\tilde{f}(xy) &\supseteq \Big(\tilde{f}(x) \cap \tilde{f}(y) \cap \delta \cap \varepsilon^c\Big) \cup \Big(\tilde{f}(x) \cap \tilde{f}(y) \cap \delta \cap \varepsilon\Big) \\
&= \Big(\tilde{f}(x) \cap \tilde{f}(y) \cap \delta\Big) \cap (\varepsilon \cup \varepsilon^c) \\
&= \tilde{f}(x) \cap \tilde{f}(y) \cap \delta.
\end{aligned}$$

Hence $(\tilde{f}, S)$ is a δ-generalized int-soft semigroup of S. □

Corollary 3.8. *For any $\varepsilon, \delta \in \mathscr{P}^*(U)$ with $\varepsilon \cap \delta \neq \emptyset$, $\varepsilon \nsubseteq \delta$ and $\delta \nsubseteq \varepsilon$, if $(\tilde{f}, S)$ is both an (ε, δ)-int-soft semigroup of S and an ε-generalized int-soft semigroup of S, then it is an $\varepsilon \cup \delta$-generalized int-soft semigroup and so a δ-generalized int-soft semigroup of S.*

Proof. Since every ε-generalized int-soft semigroup is an $\varepsilon \cap \delta$-generalized int-soft semigroup, it is obvious from Theorem 3.7. □

Question. For any $\varepsilon, \delta \in \mathscr{P}^*(U)$ with $\varepsilon \cap \delta \neq \emptyset$, $\varepsilon \nsubseteq \delta$ and $\varepsilon \nsupseteq \delta$, we have the following questions.

(1) If $(\tilde{f}, S)$ is an (ε, δ)-int-soft semigroup of S, then is it an ε-generalized int-soft semigroup of S?
(2) If $(\tilde{f}, S)$ is an (ε, δ)-int-soft semigroup of S, then is it a δ-generalized int-soft semigroup of S?
(3) If $(\tilde{f}, S)$ is an (ε, δ)-int-soft semigroup of S, then is it an $\varepsilon \cap \delta$-generalized int-soft semigroup of S?
(4) If $(\tilde{f}, S)$ is an (ε, δ)-int-soft semigroup of S, then is it an $\varepsilon \cup \delta$-generalized int-soft semigroup of S?
(5) If $(\tilde{f}, S)$ is both an (ε, δ)-int-soft semigroup of S and an $\varepsilon \cap \delta$-generalized int-soft semigroup of S, then is it an ε-generalized int-soft semigroup of S?
(6) If $(\tilde{f}, S)$ is a δ-generalized int-soft semigroup of S, then is it an ε-generalized int-soft semigroup of S?
(7) If $(\tilde{f}, S)$ is a δ-generalized int-soft semigroup of S, then is it an $\varepsilon \cup \delta$-generalized int-soft semigroup of S?
(8) If $(\tilde{f}, S)$ is an ε-generalized int-soft semigroup of S, then is it an (ε, δ)-int-soft semigroup of S?
(9) If $(\tilde{f}, S)$ is an ε-generalized int-soft semigroup of S, then is it a δ-generalized int-soft semigroup of S?
(10) If $(\tilde{f}, S)$ is an ε-generalized int-soft semigroup of S, then is it an $\varepsilon \cup \delta$-generalized int-soft semigroup of S?
(11) If $(\tilde{f}, S)$ is an $\varepsilon \cap \delta$-generalized int-soft semigroup of S, then is it an (ε, δ)-int-soft semigroup of S?
(12) If $(\tilde{f}, S)$ is an $\varepsilon \cap \delta$-generalized int-soft semigroup of S, then is it a δ-generalized int-soft semigroup of S?
(13) If $(\tilde{f}, S)$ is an $\varepsilon \cap \delta$-generalized int-soft semigroup of S, then is it an ε-generalized int-soft semigroup of S?
(14) If $(\tilde{f}, S)$ is an $\varepsilon \cap \delta$-generalized int-soft semigroup of S, then is it an $\varepsilon \cup \delta$-generalized int-soft semigroup of S?

The answer to the question above is negative as seen in the following examples.

Example 3.9. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	b	b	a	b
b	b	b	b	b
c	a	b	c	b
d	b	b	d	b

For the universe $U = \{1, 2, 3, 4, 5\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1,2,3\} & \text{if } x = a, \\ \{4,5\} & \text{if } x = b, \\ \{2,3,4\} & \text{if } x \in \{c, d\}. \end{cases}$$

Let $\varepsilon = \{1, 2, 3\}$ and $\delta = \{1, 4\}$. Then $\varepsilon \cap \delta = \{1\}$, $\varepsilon \cup \delta = \{1, 2, 3, 4\}$ and $(\tilde{f}, S)$ is an (ε, δ)-int-soft subsemigroup over U by the calculation in Table 2 (see Appendices).

(1) $(\tilde{f}, S)$ is not an ε-generalized int-soft semigroup over U since

$$\tilde{f}(aa) = \tilde{f}(b) = \{4, 5\} \not\supseteq \{1, 2, 3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \varepsilon.$$

(2) $(\tilde{f}, S)$ is not a δ-generalized int-soft semigroup over U since

$$\tilde{f}(aa) = \tilde{f}(b) = \{4, 5\} \not\supseteq \{1\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \delta.$$

Also, the (ε, δ)-int-soft semigroup $(\tilde{f}, S)$ over U in Example 3.5 is not a δ-generalized int-soft semigroup over U since

$$\tilde{f}(bc) = \tilde{f}(a) = \{1, 2, 3, 6\} \not\supseteq \{6, 8\} = \tilde{f}(b) \cap \tilde{f}(c) \cap \delta.$$

(3) $(\tilde{f}, S)$ is not an $\varepsilon \cap \delta$–generalized int-soft semigroup over U since

$$\tilde{f}(aa) = \tilde{f}(b) = \{4, 5\} \not\supseteq \{1\} = \tilde{f}(a) \cap \tilde{f}(a) \cap (\varepsilon \cap \delta).$$

(4) $(\tilde{f}, S)$ is not an $\varepsilon \cup \delta$–generalized int-soft semigroup over U since

$$\tilde{f}(aa) = \tilde{f}(b) = \{4, 5\} \not\supseteq \{1, 2, 3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap (\varepsilon \cup \delta).$$

Example 3.10. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	b	a	a	b
b	a	b	b	a
c	a	b	c	a
d	b	a	d	b

For the universe $U = \{1,2,3,4,5,6,7,8,9,10\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{2,3,5\} & \text{if } x = a, \\ \{2,3,4\} & \text{if } x = b, \\ \{1,4\} & \text{if } x = c, \\ \{3,4\} & \text{if } x = d. \end{cases}$$

Let $\varepsilon = \{3,5\}$ and $\delta = \{1,3\}$. Then $\varepsilon \cap \delta = \{3\}$, $\varepsilon \cup \delta = \{1,3,5\}$ and $(\tilde{f}, S)$ is both an (ε, δ)-int-soft subsemigroup over U and an $\varepsilon \cap \delta$-generalized int-soft semigroup over U by the calculation in Table 3 and Table 4, respectively (see Appendices).

Then $(\tilde{f}, S)$ is not an ε-generalized int-soft semigroup over U since

$$\tilde{f}(aa) = \tilde{f}(b) = \{2,3,4\} \not\supseteq \{3,5\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \varepsilon,$$

and so it is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U.

Example 3.11. (1) Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	b	b
b	b	b	b
c	b	b	c

Let $(\tilde{f}, S)$ be a soft set over $U = \{1,2,3\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1,2,3\} & \text{if } x = a, \\ \{1,2\} & \text{if } x = b, \\ \{1\} & \text{if } x = c. \end{cases}$$

If $\varepsilon = \{2,3\}$ and $\delta = \{1,2\}$, then $(\tilde{f}, S)$ is both a δ-generalized int-soft semigroup and an $\varepsilon \cap \delta$-generalized int-soft semigroup over U. But it is neither an ε-generalized int-soft semigroup nor an $\varepsilon \cup \delta$-generalized int-soft semigroup over U since

$$\tilde{f}(aa) = \tilde{f}(b) = \{1,2\} \not\supseteq \{2,3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \varepsilon$$

and

$$\tilde{f}(aa) = \tilde{f}(b) = \{1,2\} \not\supseteq \{1,2,3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap (\varepsilon \cup \delta)$$

respectively.

(2) Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	a	a
b	a	b	b
c	a	b	b

Let $(\tilde{f}, S)$ be a soft set over $U = \{1, 2, 3, 4\}$ defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1\} & \text{if } x = a, \\ \{1,2\} & \text{if } x = b, \\ U & \text{if } x = c. \end{cases}$$

If $\varepsilon = \{1, 2\}$ and $\delta = \{2, 3\}$, then $(\tilde{f}, S)$ is an ε-generalized int-soft semigroup and hence an $\varepsilon \cap \delta$-generalized int-soft semigroup over U. Since

$$\tilde{f}(cc) \cup \varepsilon = \tilde{f}(b) \cup \varepsilon = \{1, 2\} \not\supseteq \{2, 3\} = \tilde{f}(c) \cap \tilde{f}(c) \cap \delta,$$

it is not an (ε, δ)-int-soft semigroup over U. Note that

$$\tilde{f}(cc) = \tilde{f}(b) = \{1, 2\} \not\supseteq \{2, 3\} = \tilde{f}(c) \cap \tilde{f}(c) \cap \delta,$$

and so it is not a δ-generalized int-soft semigroup over U. Since

$$\tilde{f}(cc) = \tilde{f}(b) = \{1, 2\} \not\supseteq \{1, 2, 3\} = \tilde{f}(c) \cap \tilde{f}(c) \cap (\varepsilon \cup \delta),$$

it is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U.

The following example shows that there are $\varepsilon, \delta \in \mathscr{P}^*(U)$ and a soft set $(\tilde{f}, S)$ over U such that

(1) $\varepsilon \cap \delta \neq \emptyset$, $\varepsilon \not\subseteq \delta$ and $\varepsilon \not\supseteq \delta$,

(2) $(\tilde{f}, S)$ is both an (ε, δ)-int-soft semigroup over U and a δ-generalized int-soft semigroup over U.

Example 3.12. Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	a	a
b	a	b	b
c	a	b	c

For the universe $U = \{1, 2, 3, 4, 5\}$, let $(\tilde{f}, S)$ be a soft set over U given by $\tilde{f}(a) = \{1, 2\}$, $\tilde{f}(b) = \{1, 2, 3\}$ and $\tilde{f}(c) = \{3, 4\}$. Using the Table 5 (see Appendices), we know that $(\tilde{f}, S)$ is both an (ε, δ)-int-soft semigroup over U and a δ-generalized int-soft semigroup over U with $\varepsilon = \{1, 2, 3\}$ and $\delta = \{3, 4, 5\}$.

Theorem 3.13. *For any $\varepsilon, \delta \in \mathscr{P}^*(U)$, if ε and δ are disjoint, then every (ε, δ)-int-soft semigroup over U is a δ-generalized int-soft semigroup over U.*

Proof. Let $(\tilde{f}, S)$ be an (ε, δ)-int-soft semigroup over U. Then

$$\tilde{f}(xy) \cup \varepsilon \supseteq \tilde{f}(x) \cap \tilde{f}(y) \cap \delta$$

for all $x, y \in S$. If $a \in \tilde{f}(x) \cap \tilde{f}(y) \cap \delta$, then $a \in \tilde{f}(xy) \cup \varepsilon$ and $a \in \delta$. Since ε and δ are disjoint, it follows that $a \notin \varepsilon$ and that $a \in \tilde{f}(xy)$. Hence $\tilde{f}(xy) \supseteq \tilde{f}(x) \cap \tilde{f}(y) \cap \delta$ for all $x, y \in S$. Therefore $(\tilde{f}, S)$ is a δ-generalized int-soft semigroup over U. □

Remark 3.14. For any $\varepsilon, \delta \in \mathscr{P}^*(U)$ which are disjoint, we can verify the following contents by showing examples.

(1) Any (ε, δ)-int-soft semigroup over U is neither an ε-generalized int-soft semigroup over U nor an $\varepsilon \cup \delta$-generalized int-soft semigroup over U.
(2) Any δ-generalized int-soft semigroup over U is neither an ε-generalized int-soft semigroup over U nor an $\varepsilon \cup \delta$-generalized int-soft semigroup over U.
(3) Any ε-generalized int-soft semigroup over U is neither a δ-generalized int-soft semigroup over U nor an (ε, δ)-int-soft semigroup over U.
(4) Any ε-generalized int-soft semigroup over U is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U.

Example 3.15. (1) Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	b	b
b	b	b	b
c	b	b	b

For the universe $U = \{1, 2, 3, 4, 5\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1, 4, 5\} & \text{if } x = a, \\ \{2, 4, 5\} & \text{if } x = b, \\ \{3, 4, 5\} & \text{if } x = c. \end{cases}$$

For $\varepsilon = \{2, 3\}$ and $\delta = \{4, 5\}$, $(\tilde{f}, S)$ is both an (ε, δ)-int-soft semigroup and a δ-generalized int-soft semigroup over U by the calculation in Table 6 (see Appendices).

Since $\tilde{f}(cc) = \tilde{f}(b) = \{2,4,5\} \not\supseteq \{3\} = \tilde{f}(c) \cap \tilde{f}(c) \cap \varepsilon$, $(\tilde{f}, S)$ is not an ε-generalized int-soft semigroup over U. It is also not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U since $\tilde{f}(cc) = \tilde{f}(b) = \{2,4,5\} \not\supseteq \{3,4,5\} = \tilde{f}(c) \cap \tilde{f}(c) \cap (\varepsilon \cup \delta)$.

(2) Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	b	b
b	b	b	b
c	c	c	c

For the universe $U = \{1,2,3,4,5\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{3\} & \text{if } x = a, \\ \{1,2\} & \text{if } x = b, \\ \{3,4\} & \text{if } x = c. \end{cases}$$

For $\varepsilon = \{1\}$ and $\delta = \{2,3\}$, $(\tilde{f}, S)$ is an ε-generalized int-soft semigroup over U by the calculation in Table 7 (see Appendices).

But, $(\tilde{f}, S)$ is not an (ε, δ)-int-soft semigroup over U since

$$\tilde{f}(aa) \cup \varepsilon = \tilde{f}(b) \cup \varepsilon = \{1,2\} \not\supseteq \{3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \delta.$$

Since $\tilde{f}(aa) = \tilde{f}(b) = \{1,2\} \not\supseteq \{3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \delta$, it is not a δ-generalized int-soft semigroup over U. Also, it is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U since $\tilde{f}(aa) = \tilde{f}(b) = \{1,2\} \not\supseteq \{3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap (\varepsilon \cup \delta)$.

Theorem 3.16. *For any $\varepsilon, \delta \in \mathscr{P}^*(U)$ with $\varepsilon \subsetneq \delta$, the following assertions are valid.*

(1) *Every δ-generalized int-soft semigroup over U is an ε-generalized int-soft semigroup over U.*
(2) *Every δ-generalized int-soft semigroup over U is an $\varepsilon \cup \delta$-generalized int-soft semigroup over U.*
(3) *Every $\varepsilon \cap \delta$-generalized int-soft semigroup over U is an ε-generalized int-soft semigroup over U.*

Proof. The proof is straightforward. □

Remark 3.17. For any $\varepsilon, \delta \in \mathscr{P}^*(U)$ with $\varepsilon \subsetneq \delta$, we can verify the following contents by showing examples.

(1) Any (ε, δ)-int-soft semigroup over U is neither a δ-generalized int-soft semigroup over U nor an ε-generalized int-soft semigroup over U.

(2) Any ε-generalized int-soft semigroup over U is neither a δ-generalized int-soft semigroup over U nor an (ε,δ)-int-soft semigroup over U.

Example 3.18. (1) Let $S=\{a,b,c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	b	a
b	b	b	b
c	b	b	c

For the universe $U=\{1,2,3\}$, let $(\tilde{f},S)$ be a soft set over U defined as follows:

$$\tilde{f}:S\to\mathscr{P}(U),\ x\mapsto\begin{cases}\{1,2,3\} & \text{if } x=a,\\ \{1,2\} & \text{if } x=b,\\ \{1\} & \text{if } x=c.\end{cases}$$

For $\varepsilon=\{2,3\}$ and $\delta=\{1,2,3\}$, $(\tilde{f},S)$ is an (ε,δ)-int-soft semigroup over U. Since

$$\tilde{f}(aa)=\{1,2\}\not\supseteq\{1,2,3\}=\tilde{f}(a)\cap\tilde{f}(a)\cap\delta$$

and

$$\tilde{f}(aa)=\{1,2\}\not\supseteq\{2,3\}=\tilde{f}(a)\cap\tilde{f}(a)\cap\varepsilon,$$

$(\tilde{f},S)$ is neither a δ-generalized int-soft semigroup over U nor an ε-generalized int-soft semigroup over U.

(2) Let $S=\{a,b,c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	b	c
b	b	b	c
c	c	c	c

For the universe $U=\{1,2,3\}$, let $(\tilde{f},S)$ be a soft set over U defined as follows:

$$\tilde{f}:S\to\mathscr{P}(U),\ x\mapsto\begin{cases}\{1,2,3\} & \text{if } x=a,\\ \{1,2\} & \text{if } x=b,\\ \{2\} & \text{if } x=c.\end{cases}$$

For $\varepsilon=\{1,2\}$ and $\delta=\{1,2,3\}$, $(\tilde{f},S)$ is an ε-generalized int-soft semigroup over U. Since

$$\tilde{f}(aa)\cup\varepsilon=\tilde{f}(b)\cup\varepsilon=\{1,2\}\not\supseteq\{1,2,3\}=\tilde{f}(a)\cap\tilde{f}(a)\cap\delta$$

and

$$\tilde{f}(aa)=\tilde{f}(b)=\{1,2\}\not\supseteq\{1,2,3\}=\tilde{f}(a)\cap\tilde{f}(a)\cap\delta,$$

$(\tilde{f}, S)$ is neither an (ε, δ)-int-soft semigroup over U nor a δ-generalized int-soft semigroup over U.

Theorem 3.19. *For any $\varepsilon, \delta \in \mathscr{P}^*(U)$ with $\delta \subsetneq \varepsilon$, the following assertions are valid.*

(1) *Every ε-generalized int-soft semigroup over U is a δ-generalized int-soft semigroup, an (ε, δ)-int-soft semigroup and an $\varepsilon \cup \delta$-generalized int-soft semigroup over U.*
(2) *Every $\varepsilon \cap \delta$-generalized int-soft semigroup over U is a δ-generalized int-soft semigroup and an (ε, δ)-int-soft semigroup over U.*

Proof. Straightforward. □

Remark 3.20. For any $\varepsilon, \delta \in \mathscr{P}^*(U)$ with $\delta \subsetneq \varepsilon$, we can verify the following contents by showing examples.

(1) Any (ε, δ)-int-soft semigroup over U is neither an ε-generalized int-soft semigroup over U nor a δ-generalized int-soft semigroup over U.
(2) Any δ-generalized int-soft semigroup over U is not an ε-generalized int-soft semigroup over U.

Example 3.21. (1) Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	a	c
b	a	b	c
c	c	c	c

For the universe $U = \{1, 2, 3\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1,2,3\} & \text{if } x = a, \\ \{1,2\} & \text{if } x = b, \\ \{2\} & \text{if } x = c. \end{cases}$$

For $\varepsilon = \{1, 2, 3\}$ and $\delta = \{2, 3\}$, $(\tilde{f}, S)$ is an (ε, δ)-int-soft semigroup over U. Since

$$\tilde{f}(aa) = \tilde{f}(b) = \{1, 2\} \not\supseteq \{2, 3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \delta$$

and

$$\tilde{f}(aa) = \tilde{f}(b) = \{1, 2\} \not\supseteq \{1, 2, 3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \varepsilon,$$

$(\tilde{f}, S)$ is neither a δ-generalized int-soft semigroup over U nor an ε-generalized int-soft semigroup over U.

(2) Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	a	b
b	a	b	a
c	b	a	b

For the universe $U = \{1, 2, 3, 4, 5\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{3, 5\} & \text{if } x = a, \\ \{1, 2\} & \text{if } x = b, \\ \{2\} & \text{if } x = c. \end{cases}$$

For $\varepsilon = \{1, 2, 3\}$ and $\delta = \{1\}$, $(\tilde{f}, S)$ is a δ-generalized int-soft semigroup over U. But it is not an ε-generalized int-soft semigroup over U since

$$\tilde{f}(aa) = \tilde{f}(b) = \{1, 2\} \not\supseteq \{3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \varepsilon.$$

Let $\varepsilon, \delta \in \mathscr{P}^*(U)$ be such that $\delta \nsubseteq \varepsilon$. For a soft set $(\tilde{f}, S)$ over U and fixed $w \in S$, we consider a set, so called an $(\varepsilon, \delta)^w$-set with respect to w, or simply (ε, δ)-set

$$S^w_{(\varepsilon,\delta)} := \left\{ x \in S \mid \tilde{f}(x) \cup \varepsilon \supseteq \tilde{f}(w) \cap \delta \right\}.$$

In the following example, we know that the $(\varepsilon, \delta)^w$-set $S^w_{(\varepsilon,\delta)}$ is a subsemigroup of S for some $w = a \in S$, but not a subsemigroup of S for some $w = b \in S$.

Example 3.22. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	a	a	c	a
d	a	a	d	d

For the universe $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1, 2, 3, 4, 5\} & \text{if } x = a, \\ \{2, 4, 6, 8, 10\} & \text{if } x = b, \\ \{2, 8, 10\} & \text{if } x = c, \\ \{5, 7, 8, 10\} & \text{if } x = d. \end{cases}$$

Given $\varepsilon := \{3,4,8\}$ and $\delta := \{2,3,4,8,10\}$, $(\tilde{f}, S)$ is not an (ε,δ)-int-soft semigroup over U since

$$\tilde{f}(db) \cup \varepsilon = \{1,2,3,4,5\} \cup \{3,4,8\} \not\supseteq \{8,10\} = \tilde{f}(d) \cap \tilde{f}(b) \cap \delta.$$

We know that the set $S^a_{(\varepsilon,\delta)} = \{a,b,c\}$ is a subsemigroup of S. But $S^b_{(\varepsilon,\delta)} = S^c_{(\varepsilon,\delta)} = \{b,c\}$ and $S^d_{(\varepsilon,\delta)} = \{b,c,d\}$ are not subsemigroups of S.

If the $(\varepsilon,\delta)^w$-set $S^w_{(\varepsilon,\delta)}$ is a subsemigroup of S for all $w \in S$, then we say that it is an (ε,δ)-*subsemigroup* of S.

Theorem 3.23. *If $(\tilde{f}, S)$ is an (ε,δ)-int-soft semigroup over U, then the nonempty $(\varepsilon,\delta)^a$-set is a subsemigroup of S for all $a \in S$.*

Proof. Let $a \in S$ and assume that $S^a_{(\varepsilon,\delta)} \neq \emptyset$. Let $x, y \in S^a_{(\varepsilon,\delta)}$. Then $\tilde{f}(x) \cup \varepsilon \supseteq \tilde{f}(a) \cap \delta$ and $\tilde{f}(y) \cup \varepsilon \supseteq \tilde{f}(a) \cap \delta$. It follows from (3.3) that

$$\begin{aligned}
\tilde{f}(xy) \cup \varepsilon &\supseteq \left(\tilde{f}(x) \cap \tilde{f}(y) \cap \delta\right) \cup \varepsilon \\
&= \left(\tilde{f}(x) \cup \varepsilon\right) \cap \left(\tilde{f}(y) \cup \varepsilon\right) \cap (\delta \cup \varepsilon) \\
&\supseteq \left(\tilde{f}(a) \cap \delta\right) \cap (\delta \cup \varepsilon) \\
&\supseteq \left(\tilde{f}(a) \cap \delta\right) \cap \delta \\
&= \tilde{f}(a) \cap \delta.
\end{aligned}$$

Hence $xy \in S^a_{(\varepsilon,\delta)}$ and $S^a_{(\varepsilon,\delta)}$ is a subsemigroup of S. □

The following example shows that there exist $a \in S$, $\varepsilon, \delta \in \mathscr{P}^*(U)$ and a soft set $(\tilde{f}, S)$ over U such that

(i) The $(\varepsilon,\delta)^a$-set $S^a_{(\varepsilon,\delta)}$ is a subsemigroup of S.
(ii) $(\tilde{f}, S)$ is not an (ε,δ)-int-soft semigroup over U.

Example 3.24. Let $S = \{a,b,c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	b	b
b	b	b	b
c	b	b	a

For the universe $U = \{1,2,3,4,5,6,7,8,9,10\}$, let $(\tilde{f}, S)$ be a soft set over U given by

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1,2,3,4\} & \text{if } x = a, \\ \{4,5,6,7\} & \text{if } x = b, \\ \{9,10\} & \text{if } x = c,. \end{cases}$$

Given $\varepsilon := \{5,6\}$ and $\delta := \{4,9\}$, $(\tilde{f}, S)$ is not an (ε,δ)-int-soft semigroup over U since

$$\tilde{f}(cc) \cup \varepsilon = \{1,2,3,4,5,6\} \not\supseteq \{9\} = \tilde{f}(c) \cap \delta.$$

The set $S^a_{(\varepsilon,\delta)} = S^b_{(\varepsilon,\delta)} = \{a,b\}$ is a subsemigroup of S. Note that $S^c_{(\varepsilon,\delta)} = \{c\}$ is not a subsemigroup of S.

Theorem 3.25. *The soft intersection of two (ε,δ)-int-soft semigroups over U is an (ε,δ)-int-soft semigroup over U.*

Proof. Let $(\tilde{f}, S)$ and $(\tilde{g}, S)$ be (ε,δ)-int-soft semigroups over U. For any $x, y \in S$, we have

$$\begin{aligned} \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(xy) \cup \varepsilon &= \left(\tilde{f}(xy) \cap \tilde{g}(xy)\right) \cup \varepsilon \\ &= \left(\tilde{f}(xy) \cup \varepsilon\right) \cap (\tilde{g}(xy) \cup \varepsilon) \\ &\supseteq \left(\tilde{f}(x) \cap \tilde{f}(y) \cap \delta\right) \cap (\tilde{g}(x) \cap \tilde{g}(y) \cap \delta) \\ &= \left(\tilde{f}(x) \cap \tilde{g}(x) \cap \delta\right) \cap \left(\tilde{f}(y) \cap \tilde{g}(y) \cap \delta\right) \\ &= \left(\tilde{f}(x) \cap \tilde{g}(x)\right) \cap \left(\tilde{f}(y) \cap \tilde{g}(y)\right) \cap \delta \\ &= \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(x) \cap \left(\tilde{f} \tilde{\cap} \tilde{g}\right)(y) \cap \delta. \end{aligned}$$

Hence $(\tilde{f}, S) \tilde{\cap} (\tilde{g}, S)$ is an (ε,δ)-int-soft semigroup over U. □

The converse of Theorem 3.25 is not true in general as seen in the following example.

Example 3.26. Let $S = \{a,b,c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	a	a
b	a	b	b
c	a	b	b

For the universe $U = \{1, 2, 3, 4, 5, \}$, let $(\tilde{f}, S)$ and $(\tilde{g}, S)$ be soft sets over U given by

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{2,3,4\} & \text{if } x = a, \\ U & \text{if } x = b, \\ \{1\} & \text{if } x = c, \end{cases}$$

and

$$\tilde{g} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{2,3,5\} & \text{if } x = a, \\ \{1,2,3,5\} & \text{if } x = b, \\ \{1,4\} & \text{if } x = c, \end{cases}$$

respectively. The soft intersection $(\tilde{f}, S) \tilde{\cap} (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is given as follows:

$$\tilde{f} \tilde{\cap} \tilde{g} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{2,3\} & \text{if } x = a, \\ \{1,2,3,5\} & \text{if } x = b, \\ \{1\} & \text{if } x = c. \end{cases}$$

If we take $\varepsilon := \{1\}$ and $\delta := \{1, 4, 5\}$, then $(\tilde{g}, S)$ is not an (ε, δ)-int-soft semigroup over U since $\tilde{g}(cc) \cup \varepsilon = \tilde{g}(b) \cup \varepsilon = \{1, 2, 3, 5\} \not\supseteq \{1, 4\} = \tilde{g}(c) \cap \tilde{g}(c) \cap \delta$. Tables 8 and 9 (see Appendices) show that $(\tilde{f}, S)$ and $(\tilde{f}, S) \tilde{\cap} (\tilde{g}, S)$ are (ε, δ)-int-soft semigroups over U.

The following example shows that the soft union of two (ε, δ)-int-soft semigroups over U is not an (ε, δ)-int-soft semigroup over U in general.

Example 3.27. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

For the universe $U = \{1, 2, 3, 4, 5\}$, let $(\tilde{f}, S)$ and $(\tilde{g}, S)$ be soft sets over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{2,3\} & \text{if } x = a, \\ \{3,4\} & \text{if } x = b, \\ U & \text{if } x = c, \\ \{2,4,5\} & \text{if } x = d, \end{cases}$$

and

$$\tilde{g}: S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{2,3\} & \text{if } x = a, \\ U & \text{if } x = b, \\ \{2,4\} & \text{if } x = c, \\ \{2,3,5\} & \text{if } x = d, \end{cases}$$

respectively. Then $(\tilde{f}, S)$ and $(\tilde{g}, S)$ are (ε, δ)-int-soft semigroups over U with $\varepsilon = \{4\}$ and $\delta = \{1,2,3,4\}$. But

$$\begin{aligned} \left(\tilde{f} \,\tilde{\cup}\, \tilde{g}\right)(bc) \cup \varepsilon = \left(\tilde{f} \,\tilde{\cup}\, \tilde{g}\right)(a) \cup \varepsilon &= \{2,3,4\} \not\supseteq \{1,2,3,4\} \\ &= \left(\tilde{f} \,\tilde{\cup}\, \tilde{g}\right)(b) \cap \left(\tilde{f} \,\tilde{\cup}\, \tilde{g}\right)(c) \cap \delta. \end{aligned}$$

Hence the soft union $(\tilde{f}, S) \,\tilde{\cup}\, (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is not an (ε, δ)-int-soft semigroup over U.

In the following example, we know that there exist soft sets $(\tilde{f}, S)$ and $(\tilde{g}, S)$ over U such that the soft union $(\tilde{f}, S) \,\tilde{\cup}\, (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is an (ε, δ)-int-soft semigroup over U, but $(\tilde{f}, S)$ is not an (ε, δ)-int-soft semigroup over U or $(\tilde{g}, S)$ is not an (ε, δ)-int-soft semigroup over U.

Example 3.28. Let $S = \{a, b, c\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c
a	b	a	a
b	a	b	b
c	a	b	c

Let $(\tilde{f}, S)$ and $(\tilde{g}, S)$ be soft sets over $U = \{1,2,3,4,5,6,7,8,9\}$ defined as follows:

$$\tilde{f}: S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{2,3,4\} & \text{if } x = a, \\ \{4,5\} & \text{if } x = b, \\ \{2,4,6\} & \text{if } x = c, \end{cases}$$

and

$$\tilde{g}: S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1,2,3,6\} & \text{if } x = a, \\ \{6,7\} & \text{if } x = b, \\ \{2,4,5,8\} & \text{if } x = c, \end{cases}$$

respectively. Then the soft union $(\tilde{f}, S) \tilde{\cup} (\tilde{g}, S)$ of $(\tilde{f}, S)$ and $(\tilde{g}, S)$ is given as follows:

$$\tilde{f} \tilde{\cup} \tilde{g} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1,2,3,4,6\} & \text{if } x = a, \\ \{4,5,6,7\} & \text{if } x = b, \\ \{2,4,5,6,8\} & \text{if } x = c, \end{cases}$$

which is an (ε, δ)-int-soft semigroup over U with $\varepsilon = \{1, 2\}$ and $\delta = \{2, 3\}$. But $(\tilde{f}, S)$ and $(\tilde{g}, S)$ are not (ε, δ)-int-soft semigroups over U since

$$\tilde{f}(aa) \cup \varepsilon = \tilde{f}(b) \cup \varepsilon = \{1, 2, 4, 5\} \not\supseteq \{2, 3\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \delta$$

and

$$\tilde{g}(aa) \cup \varepsilon = \tilde{g}(b) \cup \varepsilon = \{1, 2, 6, 7\} \not\supseteq \{2, 3\} = \tilde{g}(a) \cap \tilde{g}(a) \cap \delta.$$

Theorem 3.29. *If $(\tilde{f}, S)$ is an (ε, δ)-int-soft semigroup over U, then the γ-inclusive set*

$$(\tilde{f}, S)^{\supseteq}_{\gamma} := \{x \in S \mid \tilde{f}(x) \supseteq \gamma\}$$

is a subsemigroup of S for all $\gamma \in \mathscr{P}^*(U)$ with $\gamma \subseteq \delta \setminus \varepsilon$.

Proof. Let $\gamma \in \mathscr{P}^*(U)$ be such that $\gamma \subseteq \delta \setminus \varepsilon$. Assume that $(\tilde{f}, S)$ is an (ε, δ)-int-soft semigroup over U. Let $x, y \in (\tilde{f}, S)^{\supseteq}_{\gamma}$. Then $\tilde{f}(x) \supseteq \gamma$ and $\tilde{f}(y) \supseteq \gamma$. It follows from (3.3) that $\tilde{f}(xy) \cup \varepsilon \supseteq \tilde{f}(x) \cap \tilde{f}(y) \cap \delta \supseteq \gamma \cap \delta \supseteq \gamma$. Hence $xy \in (\tilde{f}, S)^{\supseteq}_{\gamma}$, and so $(\tilde{f}, S)^{\supseteq}_{\gamma}$ is a subsemigroup of S. □

Theorem 3.30. *If $(\tilde{f}, S)$ is an (ε, δ)-int-soft semigroup over U with $\varepsilon \subseteq \delta$, then the set*

$$S_a := \{x \in S \mid (\tilde{f}(x) \cap \delta) \cup \varepsilon \supseteq (\tilde{f}(a) \cap \delta) \cup \varepsilon\}$$

is a subsemigroup of S for all $a \in S$.

Proof. Note that $a \in S_a$ for all $a \in S$. Let $x, y \in S_a$. Then

$$(\tilde{f}(x) \cap \delta) \cup \varepsilon \supseteq (\tilde{f}(a) \cap \delta) \cup \varepsilon \text{ and } (\tilde{f}(y) \cap \delta) \cup \varepsilon \supseteq (\tilde{f}(a) \cap \delta) \cup \varepsilon.$$

Since $\varepsilon \subseteq \delta$, it follows from (3.3) that

$$\begin{aligned}(\tilde{f}(a) \cap \delta) \cup \varepsilon &\subseteq \Big((\tilde{f}(x) \cap \delta) \cup \varepsilon\Big) \cap \Big((\tilde{f}(y) \cap \delta) \cup \varepsilon\Big) \\ &= \Big((\tilde{f}(x) \cap \tilde{f}(y)) \cap \delta\Big) \cup \varepsilon \\ &= \Big(((\tilde{f}(x) \cap \tilde{f}(y)) \cap \delta) \cap \delta\Big) \cup \varepsilon \\ &\subseteq \Big((\tilde{f}(xy) \cup \varepsilon) \cap \delta\Big) \cup \varepsilon \\ &= \Big((\tilde{f}(xy) \cap \delta) \cup (\varepsilon \cap \delta)\Big) \cup \varepsilon \\ &= \Big(\tilde{f}(xy) \cap \delta\Big) \cup \varepsilon.\end{aligned}$$

Thus $xy \in S_a$, and S_a is a subsemigroup of S for all $a \in S$. □

The following example shows that there exist $a \in S$ and a soft set $(\tilde{f}, S)$ over U such that

(i) The set S_a is a subsemigroup of S.
(ii) $(\tilde{f}, S)$ is not an (ε, δ)-int-soft semigroup over U with $\varepsilon \subseteq \delta$.

Example 3.31. (1) Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	c	c	c	c
d	a	b	a	d

For the universe $U = \{1, 2, 3, 4, 5\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{1, 4, 5\} & \text{if } x = a, \\ \{1, 2, 5\} & \text{if } x = b, \\ \{1, 2, 3\} & \text{if } x \in \{c, d\}. \end{cases}$$

If we take $\varepsilon = \{1\}$ and $\delta = \{1, 2\}$, $S_a = S$ is a subsemigroup of S, but $S_b = S_c = S_d = \{b, c, d\}$ is not a subsemigroup of S. Also, $(\tilde{f}, S)$ is not an (ε, δ)-int-soft semigroup over U since $\tilde{f}(bc) \cup \varepsilon = \tilde{f}(a) \cup \varepsilon = \{1, 4, 5\} \not\supseteq \{1, 2\} = \tilde{f}(b) \cap \tilde{f}(c) \cap \delta$.

(2) Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	b	a	a	b
b	a	b	b	a
c	a	b	c	d
d	b	a	d	b

For the universe $U = \{1, 2, 3, 4, 5\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} \{2,3,4\} & \text{if } x = a, \\ \{1,2\} & \text{if } x = d, \\ \{2,3,5\} & \text{if } x \in \{b, c\}. \end{cases}$$

Then $(\tilde{f}, S)$ is not an (ε, δ)-int-soft semigroup over U with $\varepsilon = \{1\}$ and $\delta = \{1, 2, 3, 4\}$ since $\tilde{f}(aa) \cup \varepsilon = \tilde{f}(b) \cup \varepsilon = \{1, 2, 3, 5\} \not\supseteq \{2, 3, 4\} = \tilde{f}(a) \cap \tilde{f}(a) \cap \delta$. Note that $S_k = \{a, b, c\}$ for $k = b, c$ and $S_d = S$ are subsemigroups of S. But $S_a = \{a\}$ is not a subsemigroup of S.

Note that if ε and δ are disjoint, then every (ε, δ)-int-soft semigroup over U is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U and every ε-generalized int-soft semigroup over U is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U (See Example 3.15). But we have the following theorem.

Theorem 3.32. *Let $\varepsilon, \delta \in \mathscr{P}^*(U)$ be disjoint. If a soft set $(\tilde{f}, S)$ over U is both (ε, δ)-int-soft semigroup and ε-generalized int-soft semigroup over U, then it is an $\varepsilon \cup \delta$-generalized int-soft semigroup over U.*

Proof. For any $x, y \in S$, we have

$$\begin{aligned} \tilde{f}(xy) &\supseteq (\tilde{f}(xy) \cup \varepsilon) \cap \varepsilon^c \\ &\supseteq (\tilde{f}(x) \cap \tilde{f}(y) \cap \delta) \cap \varepsilon^c \end{aligned} \tag{3.4}$$

by (3.3). Combining (3.2) and (3.4), we get

$$\begin{aligned} \tilde{f}(xy) &\supseteq (\tilde{f}(x) \cap \tilde{f}(y) \cap \varepsilon) \cup (\tilde{f}(x) \cap \tilde{f}(y) \cap \delta \cap \varepsilon^c) \\ &= \tilde{f}(x) \cap \tilde{f}(y) \cap (\varepsilon \cup \delta). \end{aligned}$$

Therefore $(\tilde{f}, S)$ is an $\varepsilon \cup \delta$-generalized int-soft semigroup over U. □

Note that if ε and δ are disjoint, then every ε-generalized int-soft semigroup over U is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over

U and every δ-generalized int-soft semigroup over U is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U (See Example 3.15). But we have the following theorem.

Theorem 3.33. *Let $\varepsilon, \delta \in \mathscr{P}^*(U)$ be disjoint. If a soft set $(\tilde{f}, S)$ over U is both ε-generalized int-soft semigroup and δ-generalized int-soft semigroup over U, then it is an $\varepsilon \cup \delta$-generalized int-soft semigroup over U.*

Proof. Straightforward. □

For any $\varepsilon, \delta \in \mathscr{P}^*(U)$ with $\varepsilon \cap \delta \neq \emptyset$, $\varepsilon \nsubseteq \delta$ and $\delta \nsubseteq \varepsilon$, we have

(1) Any (ε, δ)-int-soft semigroup over U is neither a δ-generalized int-soft semigroup nor an ε-generalized int-soft semigroup over U (see Example 3.9).
(2) Any ε-generalized semigroup over U is not a δ-generalized int-soft semigroup over U (see Example 3.11).
(3) Any (ε, δ)-int-soft semigroup over U is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U (see Example 3.9).
(4) Any ε-generalized semigroup over U is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U (see Example 3.11).
(5) Any $\varepsilon \cap \delta$-generalized semigroup over U is not a δ-generalized int-soft semigroup over U (see Example 3.11).
(6) Any (ε, δ)-generalized int-soft semigroup over U is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U (see Example 3.9).
(7) Any δ-generalized semigroup over U is not an ε-generalized semigroup over U, and so it is not an $\varepsilon \cup \delta$-generalized int-soft semigroup over U (see Example 3.11).

Let $\varepsilon, \delta \in \mathscr{P}^*(U)$ be such that $\varepsilon \cap \delta \neq \emptyset$, $\varepsilon \nsubseteq \delta$ and $\delta \nsubseteq \varepsilon$. Then any (ε, δ)-int-soft semigroup over U is neither an ε-generalized int-soft semigroup nor an $\varepsilon \cup \delta$-generalized int-soft semigroup over U (see Example 3.9). Also, any $\varepsilon \cap \delta$-generalized int-soft semigroup over U is neither an ε-generalized int-soft semigroup nor an $\varepsilon \cup \delta$-generalized int-soft semigroup over U (see Example 3.10).

Now we have the following question: If a soft set $(\tilde{f}, S)$ over U is both an (ε, δ)-int-soft semigroup and an $\varepsilon \cap \delta$-generalized int-soft semigroup over U, then is it both an ε-generalized int-soft semigroup and an $\varepsilon \cup \delta$-generalized int-soft semigroup over U?

The following example shows that the answer to this question is negative.

Example 3.34. Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

$\cdot$	a	b	c	d
a	b	b	b	b
b	b	b	b	b
c	b	b	c	b
d	b	b	c	c

For the universe $U = \{1, 2, 3, 4, 5\}$, let $(\tilde{f}, S)$ be a soft set over U defined as follows:

$$\tilde{f} : S \to \mathscr{P}(U),\ x \mapsto \begin{cases} U & \text{if } x \in \{a, b\} \\ \{2, 3, 4, 5\} & \text{if } x = c, \\ \{1, 5\} & \text{if } x = d. \end{cases}$$

If we take $\varepsilon = \{1, 3\}$ and $\delta = \{3, 5\}$, then $(\tilde{f}, S)$ is both an (ε, δ)-int-soft semigroup and an $\varepsilon \cap \delta$-generalized int-soft semigroup over U. But it is neither an ε-generalized int-soft semigroup nor an $\varepsilon \cup \delta$-generalized int-soft semigroup over U since

$$\tilde{f}(dd) = \tilde{f}(c) = \{2, 3, 4, 5\} \not\supseteq \{1\} = \tilde{f}(d) \cap \tilde{f}(d) \cap \varepsilon$$

and

$$\tilde{f}(dd) = \tilde{f}(c) = \{2, 3, 4, 5\} \not\supseteq \{1, 5\} = \tilde{f}(d) \cap \tilde{f}(d) \cap (\varepsilon \cup \delta).$$

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Appendices

TABLE 1. Relations between $\tilde{f}(xy) \cup \varepsilon$ and $\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$

x	y	xy	$\tilde{f}(xy)$	$\tilde{f}(x)$	$\tilde{f}(y)$	$\tilde{f}(xy) \cup \varepsilon$	$\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$
a	a	a	$\{1,2,3,6\}$	$\{1,2,3,6\}$	$\{1,2,3,6\}$	$\{1,2,3,6,8\}$	$\{3,6\}$
a	b	a	$\{1,2,3,6\}$	$\{1,2,3,6\}$	U	$\{1,2,3,6,8\}$	$\{3,6\}$
a	c	a	$\{1,2,3,6\}$	$\{1,2,3,6\}$	$\{6,7,8\}$	$\{1,2,3,6,8\}$	$\{6\}$
a	d	a	$\{1,2,3,6\}$	$\{1,2,3,6\}$	$\{6,7,8\}$	$\{1,2,3,6,8\}$	$\{6\}$
b	a	a	$\{1,2,3,6\}$	U	$\{1,2,3,6\}$	$\{1,2,3,6,8\}$	$\{3,6\}$
b	b	b	U	U	U	U	$\{3,6,8\}$
b	c	a	$\{1,2,3,6\}$	U	$\{6,7,8\}$	$\{1,2,3,6,8\}$	$\{6,8\}$
b	d	a	$\{1,2,3,6\}$	U	$\{6,7,8\}$	$\{1,2,3,6,8\}$	$\{6,8\}$
c	a	a	$\{1,2,3,6\}$	$\{6,7,8\}$	$\{1,2,3,6\}$	$\{1,2,3,6,8\}$	$\{6\}$
c	b	a	$\{1,2,3,6\}$	$\{6,7,8\}$	U	$\{1,2,3,6,8\}$	$\{6,8\}$
c	c	c	$\{6,7,8\}$	$\{6,7,8\}$	$\{6,7,8\}$	$\{1,2,3,6,7,8\}$	$\{6,8\}$
c	d	a	$\{1,2,3,6\}$	$\{6,7,8\}$	$\{6,7,8\}$	$\{1,2,3,6,8\}$	$\{6,8\}$
d	a	a	$\{1,2,3,6\}$	$\{6,7,8\}$	$\{1,2,3,6\}$	$\{1,2,3,6,8\}$	$\{6\}$
d	b	a	$\{1,2,3,6\}$	$\{6,7,8\}$	U	$\{1,2,3,6,8\}$	$\{6,8\}$
d	c	a	$\{1,2,3,6\}$	$\{6,7,8\}$	$\{6,7,8\}$	$\{1,2,3,6,8\}$	$\{6,8\}$
d	d	d	$\{6,7,8\}$	$\{6,7,8\}$	$\{6,7,8\}$	$\{1,2,3,6,7,8\}$	$\{6,8\}$

TABLE 2. Relations between $\tilde{f}(xy) \cup \varepsilon$ and $\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$

x	y	xy	$\tilde{f}(xy)$	$\tilde{f}(x)$	$\tilde{f}(y)$	$\tilde{f}(xy) \cup \varepsilon$	$\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$
a	a	b	$\{4,5\}$	$\{1,2,3\}$	$\{1,2,3\}$	U	$\{1\}$
a	b	b	$\{4,5\}$	$\{1,2,3\}$	$\{4,5\}$	U	$\emptyset$
a	c	a	$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3\}$	$\emptyset$
a	d	b	$\{4,5\}$	$\{1,2,3\}$	$\{2,3,4\}$	U	$\emptyset$
b	a	b	$\{4,5\}$	$\{4,5\}$	$\{1,2,3\}$	U	$\emptyset$
b	b	b	$\{4,5\}$	$\{4,5\}$	$\{4,5\}$	U	$\{4\}$
b	c	b	$\{4,5\}$	$\{4,5\}$	$\{2,3,4\}$	U	$\{4\}$
b	d	b	$\{4,5\}$	$\{4,5\}$	$\{2,3,4\}$	U	$\{4\}$
c	a	a	$\{1,2,3\}$	$\{2,3,4\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\emptyset$
c	b	b	$\{4,5\}$	$\{2,3,4\}$	$\{4,5\}$	U	$\{4\}$
c	c	c	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$	$\{4\}$
c	d	b	$\{4,5\}$	$\{2,3,4\}$	$\{2,3,4\}$	U	$\{4\}$
d	a	b	$\{4,5\}$	$\{2,3,4\}$	$\{1,2,3\}$	U	$\emptyset$
d	b	b	$\{4,5\}$	$\{2,3,4\}$	$\{4,5\}$	U	$\{4\}$
d	c	d	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$	$\{4\}$
d	d	b	$\{4,5\}$	$\{2,3,4\}$	$\{2,3,4\}$	U	$\{4\}$

TABLE 3. Relations between $\tilde{f}(xy) \cup \varepsilon$ and $\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$

x	y	xy	$\tilde{f}(xy)$	$\tilde{f}(x)$	$\tilde{f}(y)$	$\tilde{f}(xy) \cup \varepsilon$	$\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$
a	a	b	$\{2,3,4\}$	$\{2,3,5\}$	$\{2,3,5\}$	$\{2,3,4,5\}$	$\{3\}$
a	b	a	$\{2,3,5\}$	$\{2,3,5\}$	$\{2,3,4\}$	$\{2,3,5\}$	$\{3\}$
a	c	a	$\{2,3,5\}$	$\{2,3,5\}$	$\{1,4\}$	$\{2,3,5\}$	$\emptyset$
a	d	b	$\{2,3,4\}$	$\{2,3,5\}$	$\{3,4\}$	$\{2,3,4,5\}$	$\{3\}$
b	a	a	$\{2,3,5\}$	$\{2,3,4\}$	$\{2,3,5\}$	$\{2,3,5\}$	$\{3\}$
b	b	b	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4,5\}$	$\{3\}$
b	c	b	$\{2,3,4\}$	$\{2,3,4\}$	$\{1,4\}$	$\{2,3,4,5\}$	$\emptyset$
b	d	a	$\{2,3,5\}$	$\{2,3,4\}$	$\{3,4\}$	$\{2,3,5\}$	$\{3\}$
c	a	a	$\{2,3,5\}$	$\{1,4\}$	$\{2,3,5\}$	$\{2,3,5\}$	$\emptyset$
c	b	b	$\{2,3,4\}$	$\{1,4\}$	$\{2,3,4\}$	$\{2,3,4,5\}$	$\emptyset$
c	c	c	$\{1,4\}$	$\{1,4\}$	$\{1,4\}$	$\{1,3,4,5\}$	$\{1\}$
c	d	a	$\{2,3,5\}$	$\{1,4\}$	$\{3,4\}$	$\{2,3,5\}$	$\emptyset$
d	a	b	$\{2,3,4\}$	$\{3,4\}$	$\{2,3,5\}$	$\{2,3,4,5\}$	$\{3\}$
d	b	a	$\{2,3,5\}$	$\{3,4\}$	$\{2,3,4\}$	$\{2,3,5\}$	$\{3\}$
d	c	d	$\{3,4\}$	$\{3,4\}$	$\{1,4\}$	$\{3,4,5\}$	$\emptyset$
d	d	b	$\{2,3,4\}$	$\{3,4\}$	$\{3,4\}$	$\{2,3,4,5\}$	$\{3\}$

TABLE 4. Relations between $\tilde{f}(xy) \cup \varepsilon$ and $(\varepsilon \cap \delta) \cap \tilde{f}(x) \cap \tilde{f}(y)$

x	y	xy	$\tilde{f}(xy)$	$\tilde{f}(x)$	$\tilde{f}(y)$	$\tilde{f}(xy) \cup \varepsilon$	$(\varepsilon \cap \delta) \cap \tilde{f}(x) \cap \tilde{f}(y)$
a	a	b	$\{2,3,4\}$	$\{2,3,5\}$	$\{2,3,5\}$	$\{2,3,4,5\}$	$\{3\}$
a	b	a	$\{2,3,5\}$	$\{2,3,5\}$	$\{2,3,4\}$	$\{2,3,5\}$	$\{3\}$
a	c	a	$\{2,3,5\}$	$\{2,3,5\}$	$\{1,4\}$	$\{2,3,5\}$	$\emptyset$
a	d	b	$\{2,3,4\}$	$\{2,3,5\}$	$\{3,4\}$	$\{2,3,4,5\}$	$\{3\}$
b	a	a	$\{2,3,5\}$	$\{2,3,4\}$	$\{2,3,5\}$	$\{2,3,5\}$	$\{3\}$
b	b	b	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\{2,3,4,5\}$	$\{3\}$
b	c	b	$\{2,3,4\}$	$\{2,3,4\}$	$\{1,4\}$	$\{2,3,4,5\}$	$\emptyset$
b	d	a	$\{2,3,5\}$	$\{2,3,4\}$	$\{3,4\}$	$\{2,3,5\}$	$\{3\}$
c	a	a	$\{2,3,5\}$	$\{1,4\}$	$\{2,3,5\}$	$\{2,3,5\}$	$\emptyset$
c	b	b	$\{2,3,4\}$	$\{1,4\}$	$\{2,3,4\}$	$\{2,3,4,5\}$	$\emptyset$
c	c	c	$\{1,4\}$	$\{1,4\}$	$\{1,4\}$	$\{1,3,4,5\}$	$\emptyset$
c	d	a	$\{2,3,5\}$	$\{1,4\}$	$\{3,4\}$	$\{2,3,5\}$	$\emptyset$
d	a	b	$\{2,3,4\}$	$\{3,4\}$	$\{2,3,5\}$	$\{2,3,4,5\}$	$\{3\}$
d	b	a	$\{2,3,5\}$	$\{3,4\}$	$\{2,3,4\}$	$\{2,3,5\}$	$\{3\}$
d	c	d	$\{3,4\}$	$\{3,4\}$	$\{1,4\}$	$\{3,4,5\}$	$\emptyset$
d	d	b	$\{2,3,4\}$	$\{3,4\}$	$\{3,4\}$	$\{2,3,4,5\}$	$\{3\}$

TABLE 5. Relations between $\tilde{f}(xy) \cup \varepsilon$ and $\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$

x	y	xy	$\tilde{f}(xy)$	$\tilde{f}(x)$	$\tilde{f}(y)$	$\tilde{f}(xy) \cup \varepsilon$	$\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$
a	a	b	$\{1,2,3\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$	$\emptyset$
a	b	a	$\{1,2\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\emptyset$
a	c	a	$\{1,2\}$	$\{1,2\}$	$\{3,4\}$	$\{1,2,3\}$	$\emptyset$
b	a	a	$\{1,2\}$	$\{1,2,3\}$	$\{1,2\}$	$\{1,2,3\}$	$\emptyset$
b	b	b	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{3\}$
b	c	b	$\{1,2,3\}$	$\{1,2,3\}$	$\{3,4\}$	$\{1,2,3\}$	$\{3\}$
c	a	a	$\{1,2\}$	$\{3,4\}$	$\{1,2\}$	$\{1,2,3\}$	$\emptyset$
c	b	b	$\{1,2,3\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,3\}$	$\{3\}$
c	c	c	$\{3,4\}$	$\{3,4\}$	$\{3,4\}$	$\{1,2,3,4\}$	$\{3,4\}$

TABLE 6. Relations between $\tilde{f}(xy) \cup \varepsilon$ and $\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$

x	y	xy	$\tilde{f}(xy)$	$\tilde{f}(x)$	$\tilde{f}(y)$	$\tilde{f}(xy) \cup \varepsilon$	$\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$
a	a	b	$\{2,4,5\}$	$\{1,4,5\}$	$\{1,4,5\}$	$\{2,3,4,5\}$	$\{4,5\}$
a	b	b	$\{2,4,5\}$	$\{1,4,5\}$	$\{2,4,5\}$	$\{2,3,4,5\}$	$\{4,5\}$
a	c	b	$\{2,4,5\}$	$\{1,4,5\}$	$\{3,4,5\}$	$\{2,3,4,5\}$	$\{4,5\}$
b	a	b	$\{2,4,5\}$	$\{2,4,5\}$	$\{1,4,5\}$	$\{2,3,4,5\}$	$\{4,5\}$
b	b	b	$\{2,4,5\}$	$\{2,4,5\}$	$\{2,4,5\}$	$\{2,3,4,5\}$	$\{4,5\}$
b	c	b	$\{2,4,5\}$	$\{2,4,5\}$	$\{3,4,5\}$	$\{2,3,4,5\}$	$\{4,5\}$
c	a	b	$\{2,4,5\}$	$\{3,4,5\}$	$\{1,4,5\}$	$\{2,3,4,5\}$	$\{4,5\}$
c	b	b	$\{2,4,5\}$	$\{3,4,5\}$	$\{2,4,5\}$	$\{2,3,4,5\}$	$\{4,5\}$
c	c	b	$\{2,4,5\}$	$\{3,4,5\}$	$\{3,4,5\}$	$\{2,3,4,5\}$	$\{4,5\}$

Table 7. Relations between $\tilde{f}(xy)$ and $\tilde{f}(x) \cap \tilde{f}(y) \cap \varepsilon$

x	y	xy	$\tilde{f}(xy)$	$\tilde{f}(x)$	$\tilde{f}(y)$	$\tilde{f}(x) \cap \tilde{f}(y) \cap \varepsilon$
a	a	b	$\{1,2\}$	$\{3\}$	$\{3\}$	$\emptyset$
a	b	b	$\{1,2\}$	$\{3\}$	$\{1,2\}$	$\emptyset$
a	c	b	$\{1,2\}$	$\{3\}$	$\{3,4\}$	$\emptyset$
b	a	b	$\{1,2\}$	$\{1,2\}$	$\{3\}$	$\emptyset$
b	b	b	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1\}$
b	c	b	$\{1,2\}$	$\{1,2\}$	$\{3,4\}$	$\emptyset$
c	a	c	$\{3,4\}$	$\{3,4\}$	$\{3\}$	$\emptyset$
c	b	c	$\{3,4\}$	$\{3,4\}$	$\{1,2\}$	$\emptyset$
c	c	c	$\{3,4\}$	$\{3,4\}$	$\{3,4\}$	$\emptyset$

Table 8. Relations between $\tilde{f}(xy) \cup \varepsilon$ and $\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$

x	y	xy	$\tilde{f}(xy)$	$\tilde{f}(x)$	$\tilde{f}(y)$	$\tilde{f}(xy) \cup \varepsilon$	$\tilde{f}(x) \cap \tilde{f}(y) \cap \delta$
a	a	b	U	$\{2,3,4\}$	$\{2,3,4\}$	U	$\{4\}$
a	b	a	$\{2,3,4\}$	$\{2,3,4\}$	U	$\{1,2,3,4\}$	$\{4\}$
a	c	a	$\{2,3,4\}$	$\{2,3,4\}$	$\{1\}$	$\{1,2,3,4\}$	$\emptyset$
b	a	a	$\{2,3,4\}$	U	$\{2,3,4\}$	$\{1,2,3,4\}$	$\{4\}$
b	b	b	U	U	U	U	$\{1,4,5\}$
b	c	b	U	U	$\{1\}$	U	$\{1\}$
c	a	a	$\{2,3,4\}$	$\{1\}$	$\{2,3,4\}$	$\{1,2,3,4\}$	$\emptyset$
c	b	b	U	$\{1\}$	U	U	$\{1\}$
c	c	b	U	$\{1\}$	$\{1\}$	U	$\{1\}$

TABLE 9. Relations between $(\tilde{f} \tilde{\cap} \tilde{g})(xy) \cup \varepsilon$ and $(\tilde{f} \tilde{\cap} \tilde{g})(x) \cap (\tilde{f} \tilde{\cap} \tilde{g})(y) \cap \delta$

x	y	xy	$(\tilde{f} \tilde{\cap} \tilde{g})(xy)$	$(\tilde{f} \tilde{\cap} \tilde{g})(x)$	$(\tilde{f} \tilde{\cap} \tilde{g})(y)$	$(\tilde{f} \tilde{\cap} \tilde{g})(xy) \cup \varepsilon$	$(\tilde{f} \tilde{\cap} \tilde{g})(x) \cap (\tilde{f} \tilde{\cap} \tilde{g})(y) \cap \delta$
a	a	b	$\{1,2,3,5\}$	$\{2,3\}$	$\{2,3\}$	$\{1,2,3,5\}$	$\emptyset$
a	b	a	$\{2,3\}$	$\{2,3\}$	$\{1,2,3,5\}$	$\{1,2,3\}$	$\emptyset$
a	c	a	$\{2,3\}$	$\{2,3\}$	$\{1\}$	$\{1,2,3\}$	$\emptyset$
b	a	a	$\{2,3\}$	$\{1,2,3,5\}$	$\{2,3\}$	$\{1,2,3\}$	$\emptyset$
b	b	b	$\{1,2,3,5\}$	$\{1,2,3,5\}$	$\{1,2,3,5\}$	$\{1,2,3,5\}$	$\{1,5\}$
b	c	b	$\{1,2,3,5\}$	$\{1,2,3,5\}$	$\{1\}$	$\{1,2,3,5\}$	$\{1\}$
c	a	a	$\{2,3\}$	$\{1\}$	$\{2,3\}$	$\{1,2,3\}$	$\emptyset$
c	b	b	$\{1,2,3,5\}$	$\{1\}$	$\{1,2,3,5\}$	$\{1,2,3,5\}$	$\{1\}$
c	c	b	$\{1,2,3,5\}$	$\{1\}$	$\{1\}$	$\{1,2,3,5\}$	$\{1\}$