Honam Mathematical J. **38** (2016), No. 1, pp. 127–139
http://dx.doi.org/10.5831/HMJ.2016.38.1.127

A NEW ITERATION SCHEME FOR A HYBRID PAIR OF NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we construct an iteration scheme involving a hybrid pair of nonexpansive mappings and utilize the same to prove some convergence theorems. In process, we remove a restricted condition (called end-point condition) in Sokhuma and Kaewkhao's results [Sokhuma and Kaewkhao, Fixed Point Theory Appl. 2010, Art. ID 618767, 9 pp.].

1. Introduction

Let X be a Banach space and K be a nonempty subset of X. Let $CB(K)$ be the family of nonempty closed bounded subsets of K while $C(K)$ be the family of nonempty compact convex subsets of K. A subset K of X is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x,k) = d(x,K) = \inf\{\|x-y\| : y \in K\}.$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote by $PB(K)$, the family of nonempty bounded proximinal subsets of K.

The Hausdorff metric H on $CB(K)$ is defined as

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\} \text{ for } A,\ B \in CB(K).$$

A mapping $f : K \to K$ is said to be nonexpansive if

$$\|fx - fy\| \leq \|x-y\|, \text{ for all } x,\ y \in K,$$

Received June 21, 2015. Accepted February 12, 2016.
2010 Mathematics Subject Classification. 47H10, 54H25.
Key words and phrases. Banach spaces, Fixed point, Nonexpansive mapping.
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while a multivalued mapping $T : K \to CB(K)$ is said to be nonexpansive if

$$H(Tx, Ty) \leq \|x - y\|, \text{ for all } x,\ y \in K.$$

We use the notation $F(T)$ for the set of fixed points of the mapping T while $F =: F(f) \cap F(T)$ denotes the set of common fixed points of f and T, i.e., a point x is said to be a common fixed point of t and T if $fx = x \in Tx$.

It is well known that sequence of Picard iteration (cf. [1]) defined as (for any $x_1 \in K$)

$$x_{n+1} = f^n x,\ n \in \mathbb{N} \tag{1.1}$$

need not be convergent in respect of a nonexpansive mapping. E.g., the sequence of iterates $x_{n+1} = fx_n$ for the mapping $f : [-1, 1] \to [-1, 1]$ defined by $fx = -x$ does not converge to 0 for any choice of non zero initial point which is indeed the fixed point of f. In an attempt to construct a convergent sequence of iterates in respect of a nonexpansive mapping, Mann [2] defined an iteration method as: (for any $x_1 \in K$)

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n f x_n,\ n \in \mathbb{N} \tag{1.2}$$

where $\alpha_n \in (0, 1)$.

In 1974, with a view to approximate fixed point of pseudo-contractive compact mappings in Hilbert spaces Ishikawa [3] introduced a new iteration procedure as follows: (for $x_1 \in K$)

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n f x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n f y_n,\ n \in \mathbb{N} \end{cases} \tag{1.3}$$

where $\alpha_n,\ \beta_n \in (0, 1)$.

For a comparison of the two iterative schemes in the one-dimensional case, we refer readers to Rhoades [4] wherein it is shown that under suitable conditions (see part-(a) of Theorem 3) rate of convergence of Ishikawa Iteration is better than that of Mann Iteration procedure. The study of fixed points for multivalued contractions as well as multivalued nonexpansive mappings was initiated by Nadler [5] and Markin [6] and by now there exists an extensive literature on multivalued fixed point theory which has applications in diverse area such as control theory, convex optimization, differential inclusion and economics (see [7] and references cited therein). Moreover, the existence of fixed points for multivalued nonexpansive mappings in uniformly convex Banach spaces was proved by Lim [8]. In recent years, different iterative processes have been used to approximate the fixed points of multivalued nonexpansive mappings. Among these iterative proceedures, iteration schemes due to Sastry and Babu [9], Panyanak [10], Song and Wang [11] and Shahzad and Zegeye [12] are notable generalizations of Mann and Ishikawa iteration process especially in the case of multivalued mappings. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings. For an almost upto date account of literature on iterative fixed points, we refer the readers to Berinde [13].

Recently, Sokhuma and Kaewkhao [14] introduced the following modified Ishikawa iteration scheme for a pair of single valued and multivalued mapping.
Let K be a nonempty closed and bounded convex subset of Banach space X and let $f : K \to K$ be a single valued nonexpansive mapping and let $T : K \to CB(K)$ be a multivalued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{cases} y_n = \beta_n z_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n f y_n + (1-\alpha_n)x_n, \end{cases} \tag{1.4}$$

where $x_0 \in K$, $z_n \in Tx_n$ and $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$.

This scheme has been studied by several authors [14, 15, 16, 17, 18] with respect to various classes of mappings in different classes of spaces. All the authors proved their results with end point condition $Tw = w$ for all $w \in F(T)$, where T is multivalued mapping. With a motivation to remove this strong condition, in this paper we introduce a new iteration scheme for a pair of hybrid mapping and prove some convergence theorems. In this way, we are not only able to remove restricted condition but also generalized the class of functions. In process, several relevant results, especially those contained in [14, 15, 16, 17, 18] are generalized and improved.

2. Preliminaries

With a view to make, our presentation self contained, we collect some relevant basic definitions, results and iterative methods which will be used frequently in the text later.

In 2005, Sastry and Babu [9] defined Ishikawa iteration scheme for multivalued mappings. Let $T : K \to PB(K)$ be a multivalued mapping

and $p \in F(T)$. Then the sequence of Ishikawa iteration is defined as follows:
Choose $x_0 \in K$,

$$y_n = \beta_n z_n + (1-\beta_n)x_n,\ \beta_n \in [0,1],\ n \geq 0,$$

where $z_n \in Tx_n$ such that $\|z_n - p\| = d(p, Tx_n)$ and

$$x_{n+1} = \alpha_n z'_n + (1-\alpha_n)x_n,\ \alpha_n \in [0,1],\ n \geq 0,$$

where $z'_n \in Ty_n$ such that $\|z'_n - p\| = d(p, Ty_n)$.

Sastry and Babu [9] proved that Ishikawa iteration scheme for a multivalued nonexpansive mapping T with a fixed point p converges to a fixed point q of T under certain conditions. In 2007, Panyanak [10] extended the results of Sastry and Babu to uniformly convex Banach space for multivalued nonexpansive mappings. Panyanak also modified the iteration scheme of Sastry and Babu and imposed the question of convergence of this scheme. He introduced the following modified Ishikawa iteration method:
For $x_0 \in K$, write

$$y_n = \beta_n z_n + (1-\beta_n)x_n,\ \beta_n \in [a,b],\ 0 < a < b < 1,\ n \geq 0,$$

where $z_n \in Tx_n$ is such that $\|z_n - u_n\| = d(u_n, Tx_n)$, and $u_n \in F(T)$ such that $\|x_n - u_n\| = d(x_n, F(T))$, and

$$x_{n+1} = \alpha_n z'_n + (1-\alpha_n)x_n\ \alpha_n \in [a,b],$$

where $z'_n \in Ty_n$ such that $\|z'_n - v_n\| = d(v_n, Ty_n)$, and $v_n \in F(T)$ such that $\|y_n - v_n\| = d(y_n, F(T))$.

In 2009, Song and Wang [11] pointed out a gap in the result of Panyanak [10]. In attempt to remove this gap, they gave a partial answer to the question raised by Panyanak by using the following iteration scheme. Let $\alpha_n,\ \beta_n \in [0,1]$ and $\gamma_n \in (0,\infty)$ such that $\lim\limits_{n\to\infty} \gamma_n = 0$ with $x_0 \in K$, write

$$\begin{array}{rcl} y_n & = & \beta_n z_n + (1-\beta_n)x_n, \\ x_{n+1} & = & \alpha_n z'_n + (1-\alpha_n)x_n, \end{array}$$

where $\|z_n - z'_n\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - z'_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$ for $z_n \in Tx_n$ and $z'_n \in Ty_n$.

Simultaneously, Shahzad and Zegeye [12] extended the corresponding results of Sastry and Babu [9], Panyanak [10] and Song and Wang [11] to quasi-nonexpansive multivalued mappings and also relaxed the end point condition and compactness of the domain by using the following modified iteration scheme and gave an affirmative answer to the Panyanak question in a more general setting wherein

$$y_n = \beta_n z_n + (1-\beta_n)x_n,\ \beta_n \in [0,1],\ n \geq 0,$$
$$x_{n+1} = \alpha_n z_n' + (1-\alpha_n)x_n,\ \alpha_n \in [0,1],\ n \geq 0,$$

where $z_n \in Tx_n$ and $z_n' \in Ty_n$.

Now, we list the following important property of a uniformly convex Banach space essentially due to Schu [19] and an important lemma due to Khaewkhao and Sokhuma [14].

Lemma 2.10. ([19]) Let X be a uniformly convex Banach space, let $\{u_n\}$ be a sequence of real numbers such that $0 < b \leq u_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$, and $\lim_{n\to\infty} \|u_n x_n + (1-u_n)y_n\| = a$ for some $a > 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.11. ([14]) Let X be a Banach space, and let K be a nonempty closed convex subset of X. Then,

$$d(y,Ty) \leq \|y-x\| + d(x,Tx) + H(Tx,Ty),$$

where $x, y \in K$ and T is a multivalued nonexpansive mapping from K into $CB(K)$.
The following very useful theorem is due to Song and Cho [20] .

Lemma 2.12. Let $T : K \to P(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : \|x-y\| = d(x,Tx)\}$. Then the following are equivalent.
(i) $x \in F(T)$,
(ii) $P_T(x) = \{x\}$,
(iii) $x \in F(P_T)$. Moreover, $F(T) = F(P_T)$.

3. Main Results

In this paper, we introduce the following iteration scheme; Let K be a nonempty closed, bounded and convex subset of Banach space X, let $f : K \to K$ be a single valued nonexpansive mapping and let

$T : K \to CB(K)$ be a multivalued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n f y_n + (1-\beta_n)x_n, \end{cases} \tag{1}$$

where $x_0 \in K$, $z_n \in P_T x_n$ and $0 < a \leq \alpha_n,\ \beta_n \leq b < 1$.
Now, we start with the following lemma:

Lemma 3.1. Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X while $T : K \to P(K)$ be a multivalued mapping with $F(f,T) \neq \emptyset$ such that P_T is a nonexpansive mapping. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (1) then, $\lim\limits_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(f,T)$.

Proof. Let $w \in F(f,T)$ and $\{x_n\}$ be the sequence described by (1). Then, in view Lemma 2.3 we have

$$w \in P_T(w) = \{w\}.$$

Now, consider

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1-\beta_n)x_n + \beta_n f y_n - w\| \\ &\leq (1-\beta_n)\|x_n - w\| + \beta_n\|fy_n - fw\| \\ &\leq (1-\beta_n)\|x_n - w\| + \beta_n\|y_n - w\|. \end{aligned} \tag{3.1}$$

But

$$\begin{aligned} \|y_n - w\| &= \|(1-\alpha_n)x_n + \alpha_n z_n - w\| \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|z_n - w\| \\ &= (1-\alpha_n)\|x_n - w\| + \alpha_n d(z_n, P_T w) \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n H(P_T x_n, P_T w) \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|x_n - w\| \\ &= \|x_n - w\|. \end{aligned} \tag{3.2}$$

In view of Equations (3.1) and (3.2), we have

$$\|x_{n+1} - w\| \leq \|x_n - w\| \tag{3.3}$$

which shows that $\{\|x_n - p\|\}$ is a decreasing sequence of non-negative reals. Thus in all, sequence $\{\|x_n - p\|\}$ is bounded below and decreasing, therefore remains convergent.

Lemma 3.2. Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X while $T : K \to P(K)$ be a multivalued mapping with $F(f,T) \neq \emptyset$ such that P_T is a nonexpansive mapping. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (1) then, $\lim_{n\to\infty} \|fy_n - x_n\| = 0$.

Proof. In view of Lemma 3.1, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(f,T)$. Write $\lim_{n\to\infty} \|x_n - w\| = c$.

Now, consider

$$\begin{aligned}
\|fy_n - w\| &\leq \|y_n - w\| \\
&\leq \|(1-\alpha_n)x_n + \alpha_n z_n - w\| \\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|z_n - w\| \\
&= (1-\alpha_n)\|x_n - w\| + \alpha_n d(z_n, P_T w) \\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n H(P_T x_n, P_T w) \\
&\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|x_n - w\| \\
&= \|x_n - w\|. \qquad (3.4)
\end{aligned}$$

On taking lim sup of both the sides, we obtain

$$\limsup_{n\to\infty} \|fy_n - w\| \leq c. \qquad (3.5)$$

Also,

$$\begin{aligned}
c &= \lim_{n\to\infty} \|x_{n+1} - w\| \\
&= \lim_{n\to\infty} \|(1-\beta_n)x_n + \beta_n fy_n - w\| \\
&= \lim_{n\to\infty} \|(1-\beta_n)(x_n - w) + \beta_n(fy_n - w)\| \qquad (3.6)
\end{aligned}$$

In view of Equations (3.4), (3.5) and (3.6) and Lemma 2.12, we get

$$\lim_{n\to\infty} \|(fy_n - w) - (x_n - w)\| = \lim_{n\to\infty} \|fy_n - x_n\| = 0.$$

Lemma 3.3. Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X while $T : K \to P(K)$ be a multivalued mapping with $F(f,T) \neq \emptyset$ such that P_T is a nonexpansive mapping. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (1) then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Proof. Let $w \in F(f,T)$ and $\{x_n\}$ be the sequence described by (1). Then, in view of Lemma 2.3 we have

$$w \in P_T(w) = \{w\}.$$

Now, consider

$$\begin{aligned}\|x_{n+1}-w\| &= \|(1-\beta_n)x_n+\beta_n f y_n - w\| \\ &\le (1-\beta_n)\|x_n-w\|+\beta_n\|fy_n-fw\| \\ &\le (1-\beta_n)\|x_n-w\|+\beta_n\|y_n-w\|. \end{aligned}\tag{3.7}$$

so that

$$\|x_{n+1}-w\|-\|x_n-w\| \le \beta_n(\|y_n-w\|-\|x_n-w\|),$$

$$or\ \frac{\|x_{n+1}-w\|-\|x_n-w\|}{\beta_n} \le \|y_n-w\|-\|x_n-w\|.$$

Since $0<a\le\beta_n\le b<1$, we have

$$\liminf_{n\to\infty}\Big\{\Big(\frac{\|x_{n+1}-w\|-\|x_n-w\|}{\beta_n}\Big)+\|x_n-w\|\Big\} \le \liminf_{n\to\infty}\|y_n-w\|.$$

It follows that

$$c \le \liminf_{n\to\infty}\|y_n-w\|.$$

Owing to Equation (3.2) $\limsup\limits_{n\to\infty}\|y_n-w\|\le c$ so that

$$\begin{aligned} c &= \lim_{n\to\infty}\|y_n-w\| \\ &= \lim_{n\to\infty}\|(1-\alpha_n)x_n+\alpha_n z_n-w\| \\ &= \lim_{n\to\infty}\|(1-\alpha_n)(x_n-w)+\alpha_n(z_n-w)\|. \end{aligned}\tag{3.8}$$

As, $\|z_n-w\| = d(z_n,P_Tw)\le H(P_Tx_n,P_Tw)\le\|x_n-w\|$, we have

$$\limsup_{n\to\infty}\|z_n-w\| \le \limsup_{n\to\infty}\|x_n-w\| = c. \tag{3.9}$$

Owing to Lemma 2.12, Equations (3.8) and (3.9) we obtain $\lim\limits_{n\to\infty}\|x_n - z_n\| = 0$.

Lemma 3.4. Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X while $T: K\to P(K)$ be a multivalued mapping with $F(f,T)\neq\emptyset$ such that P_T is a nonexpansive mapping. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (1) then, $\lim\limits_{n\to\infty}\|fx_n-x_n\|=0$.

Proof. As,

$$\begin{aligned}\|fx_n-x_n\| &= \|fx_n-fy_n+fy_n-x_n\| \\ &\le \|fx_n-fy_n\|+\|fy_n-x_n\| \\ &\le \|x_n-y_n\|+\|fy_n-x_n\| \\ &\le \|x_n-(1-\alpha_n)x_n+\alpha_n z_n\|+\|fy_n-x_n\| \\ &= \alpha_n\|x_n-z_n\|+\|fy_n-x_n\| \end{aligned}$$

therefore,

$$\lim_{n\to\infty}\|fx_n - x_n\| \le \lim_{n\to\infty}\alpha_n\|x_n - z_n\| + \lim_{n\to\infty}\|fy_n - x_n\|.$$

On using Lemma 3.2 and Lemma 3.3, we get

$$\lim_{n\to\infty}\|fx_n - x_n\| = 0.$$

Theorem 3.5. Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X while $T : K \to P(K)$ be a multivalued mapping with $F(f,T) \neq \emptyset$ such that P_T is a nonexpansive mapping. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (1) then, $\{x_{n_i}\} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ implies $y \in F(f,T)$.

Proof. Assume that $\lim_{i\to\infty}\|x_{n_i} - y\| = 0$. By Lemma 3.4, we obtain $0 = \lim_{i\to\infty}\|fx_{n_i} - x_{n_i}\| = \lim_{i\to\infty}\|(I-f)(x_{n_i})\|$. Since $I - f$ is demiclosed at 0, we have $(I-f)(y) = 0$. Thus $y = fy$, i.e., $y \in F(f)$. By Lemma 2.2, we have

$$\begin{aligned} d(y, P_T y) &\le \|y - x_{n_i}\| + d(x_{n_i}, P_T x_{n_i}) + H(P_T x_{n_i}, P_T y) \\ &\le \|y - x_{n_i}\| + \|x_{n_i} - z_{n_i}\| + \|x_{n_i} - y\| \ \to 0 \end{aligned}$$

as $i \to \infty$. It follows that $y \in F(P_T) = F(T)$. Thus $y \in F(f,T)$.

Theorem 3.6. Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X while $T : K \to P(K)$ be a multivalued mapping with $F(f,T) \neq \emptyset$ such that P_T is a nonexpansive mapping. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (1) then, $\{x_n\}$ converges strongly to a common fixed point of f and T.

Proof. Since $\{x_n\}$ is contained in a compact subset K which is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in K$, that is, $\lim_{i\to\infty}\|x_{n_i} - y\| = 0$. Now, in view of Theorem 3.5, $y \in F(f,T)$ while owing owing to Lemma 3.1 $\lim_{n\to\infty}\|x_n - y\|$ exists. Thus, in all $\lim_{n\to\infty}\|x_n - y\| = \lim_{i\to\infty}\|x_{n_i} - y\| = 0$ so that $\{x_n\}$ converges strongly to $y \in F(f,T)$.

Khan and Fukhar-ud-din [21] introduced the so-called condition (A') for two mappings and gave an improved version in [22] of condition (I) of Senter and Dotson [23]. A hybrid version of condition (A') involving a pair of single valued and multivalued mapping which is weaker than compactness of the domain, is given as follows:

A pair of single valued mapping $f : K \to K$ and a multivalued mapping $T : K \to CB(K)$ is said to satisfy condition (A') if there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(r) > 0$ for all $r \in (0, \infty)$ such that either $d(x, fx) \geq g(d(x, F(f, T)))$ or $d(x, Tx) \geq g(d(x, F(f, T)))$ for all $x \in K$.

Theorem 3.7. Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X while $T : K \to P(K)$ be a multivalued mapping with $F(f, T) \neq \emptyset$ such that P_T is a nonexpansive mapping. If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (1) and pair (f, T) satisfies condition A', then $\{x_n\}$ converges strongly to a common fixed point of f and T.
Proof. Firstly, we show that $F(f, T)$ is closed. Let $\{x_n\}$ be a sequence in $F(f, T)$ converging to some point $z \in K$. Since

$$\begin{aligned} \|x_n - fz\| &= \|fx_n - fz\| \\ &\leq \|x_n - z\|, \end{aligned}$$

so that

$$\limsup_n \|x_n - fz\| \leq \limsup \|x_n - z\| = 0.$$

Owing to uniqueness of limit, we have $fz = z$. Also,

$$d(x_n, P_T z) \leq H(P_T x_n, P_T z) \leq \|x_n - z\| \to 0 \ as \ n \to \infty.$$

This implies that $\{x_n\}$ converges to some point of $P_T z$ and hence $z \in F(P_T) = F(T)$.
By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(f, T)$ and let us take to be c. If $c = 0$, then there is nothing to prove. If $c > 0$, then in view of Equation (3.3) for all $p \in F(f, T)$, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|,$$

so that

$$\inf_{p\in F(f,T)} \|x_{n+1} - p\| \leq \inf_{p\in F(f,T)} \|x_n - p\|,$$

which amounts to say that

$$d(x_{n+1}, F(f, T)) \leq d(x_n, F(f, T))$$

and hence $\lim_{n\to\infty} d(x_n, F(f, T))$ exists. Owing to Condition (A') there exists a nondecreasing function g such that

$$\lim_{n\to\infty} g(d(x_n, F(f, T))) \leq \lim_{n\to\infty} \|x_n - fx_n\| = 0$$

or,

$$\lim_{n\to\infty} g(d(x_n, F(f,T))) \leq \lim_{n\to\infty} d(x_n, P_T x_n) \leq \lim_{n\to\infty} \|x_n - z_n\| = 0$$

so that in both the cases $\lim_{n\to\infty} g(d(x_n, F(f,T))) = 0$. Since, g is a non-decreasing function and $g(0) = 0$, therefore $\lim_{n\to\infty} d(x_n, F(f,T)) = 0$. This implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\|x_{n_k} - p_k\| \leq \frac{1}{2^k} \text{ for all k} \geq 1$$

wherein $\{p_k\}$ is in $F(f,T)$. By Lemma 3.1, we have

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| \leq \frac{1}{2^k},$$

so that

$$\begin{aligned} \|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}, \end{aligned}$$

which implies that $\{p_k\}$ is a Cauchy sequence. Since $F(f,T)$ is closed, therefore $\{p_k\}$ is a convergent sequence. Write $\lim_{k\to\infty} p_k = p$. Now, in order to show that $\{x_n\}$ converges to p lets proceed as follows:

$$\|x_{n_k} - p\| \leq \|x_{n_k} - p_k\| + \|p_k - p\| \to 0 \ as\ k \to \infty,$$

so that that $\lim_{k\to\infty} \|x_{n_k} - p\| = 0$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, therefore $x_n \to p$.

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