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ON RIEMANN DELTA-ALPHA FRACTIONAL INTEGRALS ON TIME SCALES

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Abstract. In this paper, we introduce and investigate the concept of Riemann Delta-alpha fractional integral on time scales. Many properties of this integral will be obtained.

1. Introduction

Fractional Calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. The subject is as old as the calculus of differentiation and goes back to times when Leibniz, Gauss, and Newton invented this kind of calculation. During three centuries, the theory of fractional calculus developed as a pure theoretical field, useful only for mathematicians. Nowadays, the fractional calculus attracts many scientists and engineers. There are several applications of this mathematical phenomenon in mechanics, physics, chemistry, control theory and so on [3,4,7-11].

Recently, the author in [6] define a new well-behaved simple fractional derivative called the conformable fractional derivative depending just on the basic limit definition of the derivative and $\alpha-$fractional integral. In this paper we define the Riemann Delta-alpha fractional integral on time scales, which gives a common generalization of the $\alpha-$fractional integral and the usual Riemann $\Delta-$integral [1,2,5].

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2. Preliminaries

A time scale $\mathbb{T}$ is a nonempty closed subset of real numbers $\mathbb{R}$ with the subspace topology inherited from the standard topology of $\mathbb{R}$. For $a, b \in \mathbb{T}$ we define the closed interval $[a, b]_{\mathbb{T}}$ by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma(t)$ by $\sigma(t) = \inf\{s > t : s \in \mathbb{T}\}$ where $\inf \emptyset = \sup\{\mathbb{T}\}$, while the backward jump operator $\rho(t)$ is defined by $\rho(t) = \sup\{s < t : s \in \mathbb{T}\}$ where $\sup \emptyset = \inf\{\mathbb{T}\}$.

If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$, we say that t is left-scattered. If $\sigma(t) = t$, we say that t is right-dense, while if $\rho(t) = t$, we say that t is left-dense. A point $t \in \mathbb{T}$ is dense if it is right and left dense; isolated if it is right and left scattered. The forward graininess function $\mu(t)$ and the backward graininess function $\eta(t)$ are defined by $\mu(t) = \sigma(t) - t$, $\eta(t) = t - \rho(t)$ for all $t \in \mathbb{T}$ respectively. If $\sup \mathbb{T}$ is finite and left-scattered, then we define $\mathbb{T}^k := \mathbb{T} \backslash \sup \mathbb{T}$, otherwise $\mathbb{T}^k := \mathbb{T}$; if $\inf \mathbb{T}$ is finite and right-scattered, then $\mathbb{T}_k := \mathbb{T} \backslash \inf \mathbb{T}$, otherwise $\mathbb{T}_k := \mathbb{T}$. We set $\mathbb{T}^k_k := \mathbb{T}^k \bigcap \mathbb{T}_k$.

A function $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ is called regulated provided its right-sided limit exists at all right-dense point of $[a, b)_{\mathbb{T}}$ and its left-sided limit exists at all left-dense point of $(a, b]_{\mathbb{T}}$

A partition of $[a, b]_{\mathbb{T}}$ is any finite ordered subset

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a, b]_{\mathbb{T}}, \quad where \quad a = t_0 < t_1 < \ldots < t_n = b.$$

Each partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]_{\mathbb{T}}$ decomposes it into subintervals $[t_{i-1}, t_i)_{\mathbb{T}}$, $i = 1, 2, \ldots, n$, such that for $i \neq j$ one has $[t_{i-1}, t_i)_{\mathbb{T}} \cap [t_{j-1}, t_j)_{\mathbb{T}} = \emptyset$.

By $\mathcal{P}([a, b]_{\mathbb{T}})$ we denote the set of all partitions of $[a, b]_{\mathbb{T}}$. Let $P_n, P_m \in \mathcal{P}([a, b]_{\mathbb{T}})$. If $P_n \subset P_m$ we call P_n a refinement of P_m. If P_n, P_m are independently chosen, then the partition $P_n \cup P_m$ is a common refinement of P_n and P_m.

Let $f : [a, b]_{\mathbb{T}} \to \mathbb{R}$ be a real-valued bounded function on $[a, b]_{\mathbb{T}}$. We denote

$$M = \sup\{f(t)\sigma(t)^{\alpha-1} : t \in [a, b)_{\mathbb{T}}\}, \quad m = \inf\{f(t)\sigma(t)^{\alpha-1} : t \in [a, b)_{\mathbb{T}}\},$$

and for $1 \leq i \leq n$,

$$M_i = \sup\{f(t)\sigma(t)^{\alpha-1} : t \in [t_{i-1}, t_i)_{\mathbb{T}}\},$$

$$m_i = \inf\{f(t)\sigma(t)^{\alpha-1} : t \in [t_{i-1}, t_i)_{\mathbb{T}}\},$$

The upper Darboux $\Delta_\alpha-$sum of f with respect to the partition P, denoted by $U(f,P)$, is defined by

$$U(f,P)=\sum_{i=1}^{n} M_i(t_i-t_{i-1}),$$

while the lower Darboux $\Delta_\alpha-$sum of f with respect to the partition P, denoted by $L(f,P)$, is defined by

$$L(f,P)=\sum_{i=1}^{n} m_i(t_i-t_{i-1}).$$

Note that

$$U(f,P)\le\sum_{i=1}^{n} M(t_i-t_{i-1})=M(b-a)$$

and

$$L(f,P)\ge\sum_{i=1}^{n} m(t_i-t_{i-1})=m(b-a).$$

Thus, we have:

$$m(b-a)\le L(f,P)\le U(f,P)\le M(b-a).$$

3. The Riemann Delta-alpha fractional integral

Definition 3.1 *Let $I=[a,b]_{\mathbb{T}}$, where $a,b\in\mathbb{T}$. The upper Darboux $\Delta_\alpha-$fractional integral of f from a to b is defined by*

$$\overline{\int_a^b} f(t)\Delta_\alpha t=\inf_{P\in\mathcal{P}([a,b]_{\mathbb{T}})} U(f,P);$$

The lower Darboux $\Delta_\alpha-$fractional integral of f from a to b is defined by

$$\underline{\int_a^b} f(t)\Delta_\alpha t=\sup_{P\in\mathcal{P}([a,b]_{\mathbb{T}})} L(f,P).$$

If $\overline{\int_a^b} f(t)\Delta_\alpha t=\underline{\int_a^b} f(t)\Delta_\alpha t$, then we say that f is Darboux $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$, and the common value of the integrals, denoted by $\int_a^b f(t)\Delta_\alpha t$, is called the Darboux $\Delta_\alpha-$fractional integral.

Definition 3.2 *Let $I=[a,b]_{\mathbb{T}}$, where $a,b\in\mathbb{T}$. The upper Darboux $\Delta-$ integral of f from a to b is defined by*

$$\overline{\int_a^b} f(t)\Delta t = \inf_{P\in\mathcal{P}([a,b]_{\mathbb{T}})} U(f,P)$$

where $U(f,P)$ *denote the upper Darboux* $\Delta-$*sum of* f *with respect to the partition* P *and*

$$U(f,P)=\sum_{i=1}^{n} M_i(t_i-t_{i-1}), M_i=\sup\{f(t): t\in[t_{i-1},t_i)_{\mathbb{T}}\}.$$

The lower Darboux $\Delta-$ *integral of* f *from* a *to* b *is defined by*

$$\underline{\int_a^b} f(t)\Delta t = \sup_{P\in\mathcal{P}([a,b]_{\mathbb{T}})} L(f,P).$$

where $L(f,P)$ *denote the lower Darboux* $\Delta-$*sum of* f *with respect to the partition* P *and*

$$L(f,P)=\sum_{i=1}^{n} m_i(t_i-t_{i-1}), m_i=\inf\{f(t): t\in[t_{i-1},t_i)_{\mathbb{T}}\}.$$

If $\overline{\int_a^b} f(t)\Delta t = \underline{\int_a^b} f(t)\Delta t$, *then we say that* f *is Darboux* $\Delta-$*integrable on* $[a,b]_{\mathbb{T}}$, *and the common value of the integrals, denoted by* $\int_a^b f(t)\Delta t$, *is called the Darboux* $\Delta-$*integral.*

We can easily get the following theorem.

Theorem 3.3 *Let* $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ *is Darboux* $\Delta_\alpha-$*fractional integrable on* $[a,b]_{\mathbb{T}}$. *If* $\alpha=1$, *then* f *is Darboux* $\Delta-$*integrable on* $[a,b]_{\mathbb{T}}$.

The proofs of the following two Theorems are standard and similar to [11,Theorem 5.5 and 5.6].

Theorem 3.4 *Let* $L(f,P)=U(f,P)$ *for some* $P\in\mathcal{P}([a,b]_{\mathbb{T}})$, *then the function* f *is Darboux* $\Delta_\alpha-$*fractional integrable on* $[a,b]_{\mathbb{T}}$ *and*

$$\int_a^b f(t)\Delta_\alpha t = L(f,P)=U(f,P).$$

Theorem 3.5 *(Cauchy criterion) Let* $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ *be a bounded function on* $[a,b]_{\mathbb{T}}$. *Then the function* f *is Darboux* $\Delta_\alpha-$*fractional integrable on* $[a,b]_{\mathbb{T}}$ *if and only if for every* $\epsilon>0$ *there exists a partition* $P\in\mathcal{P}([a,b]_{\mathbb{T}})$ *such that* $U(f,P)-L(f,P)<\epsilon$.

The following lemma can be found in [11].

Lemma 3.6 *Let* $I=[a,b]_{\mathbb{T}}$ *be a closed (bounded) interval in* $\mathbb{T}$. *For every* $\delta>0$ *there is a partition* $P_\delta=\{t_0,t_1,\ldots,t_n\}\in\mathcal{P}([a,b]_{\mathbb{T}})$ *such* *that for each i one has:*

$$t_i - t_{i-1} \leq \delta \quad or \quad t_i - t_{i-1} > \delta \wedge \rho(t_i) = t_{i-1}.$$

For each $\delta > 0$, we denote by $P_\delta \in \mathcal{P}_\delta([a,b]_{\mathbb{T}})$ the set of all $P \in \mathcal{P}([a,b]_{\mathbb{T}})$ that possess the property indicated in Lemma 3.6.

The next theorem gives another Cauchy criterion for integrability.

Theorem 3.7 *A bounded function f on $[a,b]_{\mathbb{T}}$ is Darboux $\Delta_\alpha-$fractional integrable if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that $P \in \mathcal{P}_\delta([a,b]_{\mathbb{T}})$ implies*

$$U(f,P) - L(f,P) < \epsilon.$$

Proof. *If for each $\epsilon > 0$ there exists $\delta > 0$ such that $P_\delta \in \mathcal{P}([a,b]_{\mathbb{T}})$ implies*

$$U(f,P_\delta) - L(f,P_\delta) < \epsilon,$$

then we have that f is Darboux $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$ by Theorem 3.4.

Conversely, suppose that f is Darboux $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$. Let $\epsilon > 0$ and $P_0 \in \mathcal{P}([a,b]_{\mathbb{T}})$ given by

$$a = t_0 < t_1 < \ldots < t_n = b$$

such that

$$U(f,P_0) - L(f,P_0) < \epsilon.$$

Let $\delta = \frac{\epsilon}{8nM}$, where $M = \sup\{|f(t)\sigma(t)^{\alpha-1}| : t \in [a,b]_{\mathbb{T}}\}$.

Now we consider any $P \in \mathcal{P}_\delta([a,b]_{\mathbb{T}})$ given by

$$a = t'_0 < t'_1 < \ldots < t'_m = b.$$

Let $P' = P \cup P_0$. If P' has one more element, say t, than P, then we will have $t \in (t'_{i-1}, t'_i)$ for some $i \in \{1,2,\ldots,m\}$, where $t'_i - t'_{i-1} < \delta$.

Then we have

$$\begin{aligned} L(f,P') - L(f,P) &= m_i^{(1)}(t - t'_{i-1}) + m_i^{(2)}(t'_i - t) - m_i(t'_i - t'_{i-1}) \\ &\leq M(t - t'_{i-1} + t'_i - t + t'_i - t'_{i-1}) \\ &\leq 2M\delta. \end{aligned}$$

Since P' has at most $n-1$ elements that are not in P, an induction argument shows that

$$L(f,P') - L(f,P) \leq 2(n-1)M\delta = 2(n-1)M\frac{\epsilon}{8nM} < \frac{\epsilon}{4}.$$

We also have $L(f,P_0) - L(f,P) < \frac{\epsilon}{4}$ and $U(f,P) - U(f,P_0) < \frac{\epsilon}{4}$.

Therefore

$$U(f,P) - L(f,P) < U(f,P_0) - L(f,P_0) + \frac{\epsilon}{4} + \frac{\epsilon}{4} < \epsilon.$$

We now give Riemanns definition of integrability.

Definition 3.8 *Let $f:\mathbb{T}\to\mathbb{R}$ be a bounded function on $[a,b]_{\mathbb{T}}$, and let $P\in\mathcal{P}_\delta([a,b]_{\mathbb{T}})$ be given by*

$$a=t_0<t_1<\ldots<t_n=b.$$

For $1\le i\le n$, choose arbitrary points $\xi_i\in[t_{i-1},t_i)_{\mathbb{T}}$ and form the sum

$$S=\sum_{i=1}^{n} f(\xi_i)\sigma(\xi_i)^{\alpha-1}(t_i-t_{i-1}).$$

We call S a Riemann $\Delta_\alpha-$sum of f corresponding to $P\in\mathcal{P}_\delta([a,b]_{\mathbb{T}})$. We say that f is Riemann $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$ if there exists a number I with the following property: for each $\epsilon>0$ there exists $\delta>0$ such that

$$|S-I|<\epsilon$$

for every Riemann $\Delta_\alpha-$sum of f corresponding to $P\in\mathcal{P}_\delta([a,b]_{\mathbb{T}})$ and independent of the choice of $\xi_i\in[t_{i-1},t_i)_{\mathbb{T}}$. The number I is called the Riemann $\Delta_\alpha-$fractional integral of f on $[a,b]_{\mathbb{T}}$. The Riemann $\Delta_\alpha-$fractional integral reduces to the Riemann $\Delta-$fractional integral for $\alpha=1$.

From Definitions 3.1 and 3.8, we can get the following theorem.

Theorem 3.9 *A bounded functions $f:\mathbb{T}\to\mathbb{R}$ on $[a,b]_{\mathbb{T}}$ is Riemann $\Delta_\alpha-$fractional integrable if and only if it is Darboux $\Delta_\alpha-$fractional integrable, in which case the values of the integrals are equal.*

The Riemann $\Delta_\alpha-$fractional integral has the following properties. Here we will not dwell with the proofs.

Theorem 3.10 *Let functions $f,\ g:\mathbb{T}\to\mathbb{R}$ be $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$, $a<b<c$ and λ_1,λ_2 be arbitrary real numbers. Then,*

(1) $\lambda_1 f\pm\ \lambda_2 g$ is Riemann $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$ and

$$\int_a^b(\lambda_1 f(t)\pm\ \lambda_2 g(t))\Delta_\alpha t=\lambda_1\int_a^b f(t)\Delta_\alpha t\pm\lambda_2\int_a^b g(t)\Delta_\alpha t.$$

(2) $\int_a^c f(t)\Delta_\alpha t+\int_c^b f(t)\Delta_\alpha t=\int_a^b f(t)\Delta_\alpha t$.

(3) if $f\le g$ for $t\in[a,b]_{\mathbb{T}}$, then $\int_a^b f(t)\Delta_\alpha t\le\int_a^b g(t)\Delta_\alpha t$.

(4) $|f|$ is $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$ and $|\int_a^b f(t)\Delta_\alpha t|\le\int_a^b|f(t)|\Delta_\alpha t$.

(5) fg is Riemann $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$.

The following theorem may be proved in much the same way as [11, Theorem 5.18, 5.19, 5.20, 5.21.].

Theorem 3.11 *Let $I=[a,b]_{\mathbb{T}}$, where $a,b\in\mathbb{T}$.*

(¡) Every monotone function f is Riemann $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$.

(¡¡) Every continuous function f is Riemann $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$.

(¡¡¡) Every bounded function f with only finitely many discontinuity points is Riemann $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$.

(¡¡¡¡) Every regulated function f is Riemann $\Delta_\alpha-$fractional integrable on $[a,b]_{\mathbb{T}}$.

Theorem 3.12 *Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}$. Then, f is Riemann $\Delta_\alpha-$fractional integrable on $[t,\sigma(t)]_{\mathbb{T}}$ and*

$$\int_t^{\sigma(t)} f(s)\Delta_\alpha s = \mu(t)f(t)\sigma(t)^{\alpha-1}.$$

Proof. *If $t=\sigma(t)$, then the equality is obvious. If $t<\sigma(t)$, then $\mathcal{P}([t,\sigma(t)]_{\mathbb{T}})$ contains only one element given by*

$$t = s_0 < s_1 = \sigma(t).$$

Since $[s_0,s_1)_{\mathbb{T}}=\{t\}$, we have

$$U(f,P)=L(f,P)=f(t)\sigma(t)^{\alpha-1}(\sigma(t)-t)=\mu(t)f(t)\sigma(t)^{\alpha-1}.$$

By Theorems 3.5 and 3.9, f is Riemann $\Delta_\alpha-$fractional integrable on $[t,\sigma(t)]_{\mathbb{T}}$ and

$$\int_t^{\sigma(t)} f(s)\Delta_\alpha s = \mu(t)f(t)\sigma(t)^{\alpha-1}.$$

Theorem 3.13 *Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}$. Then, f is Riemann $\Delta_\alpha-$fractional integrable on $[\rho(t),t]_{\mathbb{T}}$ and*

$$\int_{\rho(t)}^{t} f(s)\Delta_\alpha s = \eta(t)f^\rho(t)\sigma(\rho(t))^{\alpha-1}.$$

Proof. *If $t=\rho(t)$, then the equality is obvious. If $t>\rho(t)$, then $[\rho(t),t]_{\mathbb{T}}$ contains only one element given by*

$$\rho(t) = s_0 < s_1 = t.$$

Since $[s_0,s_1)_{\mathbb{T}}=\{\rho(t)\}$, we have

$$U(f,P)=L(f,P)=f^\rho(t)\sigma(\rho(t))^{\alpha-1}(t-\rho(t))=\eta(t)f^\rho(t)\sigma(\rho(t))^{\alpha-1}.$$

By Theorems 3.5 and 3.9, f is Riemann $\Delta_\alpha-$fractional integrable on $[\rho(t),t]_{\mathbb{T}}$ and

$$\int_{\rho(t)}^{t} f(s)\Delta_\alpha s = \eta(t)f^\rho(t)\sigma(\rho(t))^{\alpha-1}.$$

By the definition of the Riemann $\Delta_\alpha-$fractional integral and Theorem 3.13, we have the following corollary:

Corollary 3.14 *Let $a, b \in \mathbb{T}$ and $a < b$. Then we have the following:*

(i). *If $\mathbb{T} = \mathbb{R}$, then a bounded function f is Riemann $\Delta_\alpha-$fractional integrable on $[a, b]_{\mathbb{T}}$ if and only if $f(t)t^{\alpha-1}$ is Riemann integrable on $[a, b]_{\mathbb{T}}$ in the classical sense, and in this case*

$$\int_a^b f(t)\Delta_\alpha t = \int_a^b f(t)t^{\alpha-1}dt.$$

(ii). *If $\mathbb{T} = \mathbb{Z}$, then each function $f : \mathbb{Z} \to \mathbb{R}$ is Riemann $\Delta_\alpha-$fractional integrable on $[a, b]_{\mathbb{T}}$. Moreover*

$$\int_a^b f(t)\Delta_\alpha t = \sum_{t=a}^{b-1} f(t)(t+1)^{\alpha-1}.$$

(iii). *If $\mathbb{T} = h\mathbb{Z}$, then each function $f : h\mathbb{Z} \to \mathbb{R}$ is Riemann $\Delta_\alpha-$fractional integrable on $[a, b]_{\mathbb{T}}$. Moreover*

$$\int_a^b f(t)\Delta_\alpha t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)(kh+h)^{\alpha-1}h.$$

Example. *Let $f : [1, 2] \cup \{3, 4\} \to \mathbb{R}$ be defined by $f(t) = t$, $\alpha = \frac{1}{2}$. Then,*

$$\begin{aligned}\int_1^4 f(t)\Delta_{\frac{1}{2}} t &= \int_1^2 f(t)t^{-\frac{1}{2}}dt + \int_3^4 f(t)\Delta_{\frac{1}{2}} t\\ &= \int_1^2 t^{\frac{1}{2}}dt + f(3)4^{-\frac{1}{2}}\\ &= \frac{4\sqrt{2}}{3} + \frac{5}{6}.\end{aligned}$$

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