CONSTANT CURVATURES AND SURFACES OF REVOLUTION IN L³

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Abstract. In Minkowskian 3-spacetime $\mathbf{L}^3$ we find timelike or spacelike surface of revolution for the given Gauss curvature $K = -1, 0, 1$ and mean curvature $H = 0$. In fact, we set up the surface of revolution with the time axis for z-axis to be able to draw those surfaces on standard pictures in Minkowskian 3-spacetime $\mathbf{L}^3$.

1. Introduction

Herman Minkowski (1864-1909) developed the 4-dimensional differential geometry which consists of one time axis and three space axes behind Special Relativity. To draw the figures of the relativistic geometrical models in the space of three axes, we reduce Minkowskian 4-spacetime to 3-spacetime which consists of one time axis and two space axes. Hence we may observe the (2-dimensional) timelike or spacelike surfaces in 3-dimensional Minkowskian 3-spacetime $\mathbf{L}^3$(cf. [1], [3]). For the surfaces in Euclidean 3-space $\mathbf{E}^3$, there are several texts to study the Gaussian and mean curvatures(cf. [4], [5], [6]). We also observe the first and second fundamental forms and their coefficients E, F, G and L, M, N to define the Gaussian curvature K and mean curvature H of the 2-dimensional timelike and spacelike surfaces in $\mathbf{L}^3$. We now set up the surface of revolution M in the Minkowskian 3-space $\mathbf{L}^3$ whose revolution axis is the time axis z and the profile curve is $\alpha(u) = (b(u), 0, a(u))$ where $b(u) > 0$. Then the coordinate patch is given by $X(u, v) = (b(u) \cos v, b(u) \sin v, a(u))$ $0 < u < 2\pi$. To treat the Gaussian curvature simply, we use the unit speed profile curve and thus we have the Theorem 3.3 as follows.

For a unit speed timelike or spacelike profile curve of the surface of revolution M in $\mathbf{L}^3$, the Gaussian curvature is $K = \dfrac{b''(u)}{b(u)}$.

Note that for a unit speed profile curve of the surface of revolution M in $\mathbf{E}^3$, the Gaussian curvature is $K = -\dfrac{b''(u)}{b(u)}$. (cf. [5])

If $H = 0$ we may deal with the minimal or maximal surface of revolution in $\mathbf{L}^3$ which is the similar problem of the minimal surface of revolution in $\mathbf{E}^3$. To simplify the equation $H = 0$ we assume $a'(u) \neq 0$ and apply the inverse function theorem to obtain the Theorem 4.1 after reparametrization of the profile curve of the profile curve $\alpha(u) = (b(u), 0, a(u))$ by the form $\alpha(u) = (b(u), 0, u)$.

Theorem 4.1 *If M is a minimal timelike surface of revolution in $\mathbf{L}^3$, then M is a cone; $z = \sin^{-1}(\sqrt{x^2 + y^2})$. If M is a maximal spacelike surface of revolution in $\mathbf{L}^3$, then M is a pseudo-sphere; $z = \sinh^{-1}(\sqrt{x^2 + y^2})$.*

Finally, we find surfaces of revolution for the unit speed profile curve as given above in cases of Gaussian curvature $K = 1$, $K = 0$, and $K = -1$ as follows.

Theorem 5.1 *Let M be a surface of revolution with $K = 1$ in $\mathbf{L}^3$. If M is timelike, then M is a cone;*

$$z = \pm \frac{1}{c} \int \sqrt{x^2 + y^2 + 2} \, du.$$

If M is spacelike, then M is a paraboloid;

$$z = \pm \sqrt{x^2 + y^2 + 1}.$$

Theorem 5.2 *Let M be a surface of revolution with $K = 0$ in $\mathbf{L}^3$. If M is timelike, then M is a cone;*

$$z = A_1 \sqrt{x^2 + y^2} + B_1,$$

where $|A_1| > 1$ and B_1 are constants. If M is spacelike, then M is a cone;

$$z = A_2 \sqrt{x^2 + y^2} + B_2,$$

where $|A_2| < 1$ and B_2 are constants.

Theorem 5.3 *Let M be a surface of revolution with $K = -1$ in $\mathbf{L}^3$. If M is spacelike, then M is no longer a surface. If M is timelike, then M is a cone;*

$$z = \frac{1}{c} \int \sqrt{2c^2 - x^2 - y^2} du + d^2$$

where c, d are constants.

2. Preliminaries

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. A bilinear form g on $\mathbb{R}^3$ is said to be the *Lorentz metric* defined by

$$g(x, y) = x_1 y_1 + x_2 y_2 - x_3 y_3.$$

From now on, $\mathbf{L}^3 = (\mathbb{R}^3, g)$ is said to be the *Minkowskian space.*

A regular curve $\alpha : I \to \mathbf{L}^3$ is said to be *timelike*(or *spacelike*) if its velocity vector $\alpha'(t)$ for all $t \in I$ is a timelike (or spacelike) vector respectively. Related to the metric g, the *cross product x and y* in $\mathbf{L}^3$ is given by

$$x \times y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & -\mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

In this paper, we may sometimes use just by $< , >$ for the inner product $g(,)$ in $\mathbf{L}^3$ without ambiguity.

Let D be an open subset of $\mathbb{R}^2$ and let $X : D \to \mathbb{R}^3$ be a differentiable function. Then $X : D \to \mathbf{E}^3$(or $\mathbf{L}^3$) is said to be *regular* if $X_u \times X_v \neq 0$ on D. Moreover, if X is a 1-1 regular map, then X is said to be a *coordinate patch.*

If a coordinate patch $X : D \to \mathbf{E}^3$(or $\mathbf{L}^3$) has a continuous inverse function $X^{-1} : X(D) \to D$, then X is said to be a *proper patch.* Further, we call M a *(regular) surface* in $\mathbf{E}^3$ if for each $p \in M$ there is a proper patch $X : D \to \mathbf{E}^3$ containing an open neighborhood of p. Also, a surface M in $\mathbf{L}^3$ is said to be *timelike*(or *spacelike*) if given any $p \in M$, there exists a coordinate patch $X : D \to M \subset \mathbf{L}^3$ with $p \in X(D)$ such that

$$U = \frac{X_u \times X_v}{|X_u \times X_v|}$$

is *spacelike*(or *timelike*) at p.

Suppose that M is a timelike or spacelike surface in $\mathbf{L}^3 = (\mathbb{R}^3, g)$. Since the the first fundamental form is defined by the inner product in

$\mathbf{L}^3$, it is defined by $I_p(v, w) = g(v, w)$ for $v, w \in T_pM$, $p \in M$ where g denotes the Lorentz metric on $\mathbb{R}^3$.

To study local properties of a surface in $\mathbf{L}^3$, the coefficients of the first fundamental form are given by $E = g(X_u, X_u)_p, F = g(X_u, X_v)_p, G = g(X_v, X_v)_p$, where g is the Lorentz metric on $\mathbb{R}^3$.

In $\mathbf{L}^3 = (\mathbf{R}^3, g)$, for the timelike(resp. spacelike) unit normal vector field $U = \dfrac{X_u \times_g X_v}{|X_u \times_g X_v|}$ defined on the spacelike(resp. timelike) surface M, we may define the timelike(resp. spacelike) *Gauss map* $U : M \to H^2$ where $H^2 = \{x \in \mathbf{L}^3 \mid |x|_g = 1\} = \{x \in \mathbf{L}^3 \mid g(x, x) = -1\} \cup \{x \in \mathbf{L}^3 \mid g(x, x) = 1\}$. Further, we have the differential map $dU_p : T_pM \to T_pM$ as in $\mathbf{E}^3$.

At a point $p \in M$ the *second fundamental form*

$$II_p : T_pM \to \mathbb{R}$$

is defined by $II_p(w) = - <dU(w), w>$ for $w \in T_pM$ for a surface M in $\mathbf{L}^3$ as in $\mathbf{E}^3$.

Given $w \in T_pM$, on a coordinate patch $X(u, v)$ containing a neighborhood of p, there is a curve $\alpha : (-\epsilon, \epsilon) \to M$ such that $\alpha(t) = X(u(t), v(t))$, $\alpha(0) = p$ and $\alpha'(0) = w$. Then, differentiating $< (U \circ \alpha)(t), \alpha'(t) >= 0$ we obtain

$$II_p(w) = - <dU(w), w>$$
$$= - <(U \circ \alpha)'(0), \alpha'(0) >=< (U \circ \alpha)(0), \alpha''(0) > .$$

Since $\alpha''(t) = \dfrac{d^2X}{dt^2} = X_{uu}(u'(t))^2 + 2X_{uv}u'(t)v'(t) + X_{vv}(v'(t))^2 + X_u u''(t) + X_v v''(t)$, the second fundamental form $II_{\alpha(t)}$ along $\alpha(t)$ is

$$\left\langle U \circ \alpha(t), X_{uu}\left(\frac{du}{dt}\right)^2 + 2X_{uv}\frac{du}{dt}\frac{dv}{dt} + X_{vv}\left(\frac{dv}{dt}\right)^2 \right\rangle$$

as $< U, X_u >=< U, X_v >= 0$. Therefore, the second fundamental form II on $X(u, v)$ which is independent of any curve is given by

$$II =< X_{uu}(du)^2 + 2X_{uv}dudv + X_{vv}(dv)^2, U >$$
$$= Ldu^2 + 2Mdudv + Ndv^2$$

where $L =< X_{uu}, U >$, $M =< X_{uv}, U >$, $N =< X_{vv}, U >$ are called the *coefficient of the 2nd fundamental form*. Gauss himself introduced the notations L, M, N for these quantities.

Using the coefficients of the 1st and 2nd fundamental forms I, II, the *Gaussian curvature* K and the *mean curvature* H on a given surface are given by

$$K = \frac{LN - M^2}{EG - F^2}, \; H = \frac{EN + GL - 2FM}{2(EG - F^2)}.$$

For a point p on a surface in $\mathbf{E}^3$ we say

(1) p is an elliptic point if $K(p) > 0$,

(2) p is a hyperbolic point if $K(p) < 0$,

(3) p is a parabolic point if $K(p) = 0$ and if $L^2 + M^2 + N^2 \neq 0$,

and

(4) p is a planar point if $K(p) = 0$ and if $L = M = N = 0$.

Moreover, if $H = 0$ on a timelike (or spacelike) surface M in $\mathbf{L}^3$, then M is called the *minimal* (or *maximal*) surface.

3. Surfaces of revolution in $\mathbf{L}^3$

We construct the *timelike* (or *spacelike*) surfaces of revolution M by revolving a given *timelike* (or *spacelike*) curve α in xz-plane in $\mathbf{L}^3$. Corresponding to the Euclidean case, we take the profile curve $\alpha(u)$ of the form $\alpha(u) = (b(u), 0, a(u))$ with $a'(u) \neq 0$ and $b(u) > 0$. Then the parametrization of the surface M is obtained by

$$X(u, v) = (b(u) \cos v, b(u) \sin v, a(u)).$$

To compute the Gaussian curvature and the mean curvature, we should find the coefficients $\{E, F, G\}$ of the first fundamental form I and the coefficients $\{L, M, N\}$ of the second fundamental form II. Namely,

$X_u = (b'(u) \cos v, b'(u) \sin v, a'(u))$

$X_v = (-b(u) \sin v, b(u) \cos v, 0)$

$E = <X_u, X_u> = b'(u)^2 - a'(u)^2$

$F = <X_u, X_v> = 0$

$G = <X_v, X_v> = b(u)^2$

$X_u \times_g X_v = (-a'(u)b(u) \cos v, -a'(u)b(u) \sin v, -b(u)b'(u))$

$\qquad\qquad = b(u)(-a'(u) \cos v, -a'(u) \sin v, -b'(u))$

$|X_u \times_g X_v| = b(u)\sqrt{|a'(u)^2 - b'(u)^2|}$

$$U = \frac{X_u \times_g X_v}{|X_u \times_g X_v|} = \frac{(-a'(u)\cos v, -a'(u)\sin v, -b'(u))}{\sqrt{|a'(u)^2 - b'(u)^2|}}$$

$$X_{uu} = (b''(u)\cos v, b''(u)\sin v, a''(u))$$

$$X_{uv} = (-b'(u)\sin v, b'(u)\cos v, 0)$$

$$X_{vv} = (-b(u)\cos v, -b(u)\sin v, 0)$$

$$L = <X_{uu}, U> = \frac{-a'(u)b''(u) + a''(u)b'(u)}{\sqrt{|a'(u)^2 - b'(u)^2|}}$$

$$M = <X_{uv}, U> = 0$$

$$N = <X_{vv}, U> = \frac{a'(u)b(u)}{\sqrt{|a'(u)^2 - b'(u)^2|}}.$$

Remark 3.1. *Since*

$$<U, U> = \frac{a'(u)^2 - b'(u)^2}{|a'(u)^2 - b'(u)^2|} = -\frac{<\alpha'(u), \alpha'(u)>}{|a'(u)^2 - b'(u)^2|},$$

α *is timelike (or spacelike) if and only if U is spacelike (or timelike). Equivalently, M is a timelike (or spacelike) surface respectively in $\mathbf{L}^3$.*

The mean curvature of the surface of revolution M in $\mathbf{L}^3$ is given by

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

$$= \frac{(b'(u)^2 - a'(u)^2)a'(u)b(u)}{2b(u)^2\sqrt{|a'(u)^2 - b'(u)^2|}(b'(u)^2 - a'(u)^2)}$$

$$+ \frac{b(u)^2(-a'(u)b''(u) + a''(u)b'(u))}{2b(u)^2\sqrt{|a'(u)^2 - b'(u)^2|}(b'(u)^2 - a'(u)^2)}$$

$$= \frac{1}{2}\left(\frac{a'(u)}{b(u)\sqrt{|(a'(u))^2 - (b'(u))^2|}} + \frac{-a'(u)b''(u) + a''(u)b'(u)}{\sqrt{|(a'(u))^2 - (b'(u))^2|}(b'(u)^2 - a'(u)^2)}\right).$$

Since $a'(u) \neq 0$ by hypothesis, using the inverse function theorem we may rearrange $a(u)$ to be $a(u) = u$. Hence $a'(u) = 1$, and $a''(u) = 0$. Thus,

$$H = \frac{1}{2}\left(\frac{1}{b(u)\sqrt{|(1 - (b'(u))^2|}} + \frac{-b''(u)}{\sqrt{|1 - (b'(u))^2|}(b'(u)^2 - 1)}\right)$$

$$(3.1) \qquad = \frac{b'(u)^2 - 1 - b(u)b''(u)}{2b(u)\sqrt{|1 - (b'(u))^2|}(b'(u)^2 - 1)}.$$

Theorem 3.2. *For the timelike or spacelike profile curve of type* $\alpha(u) = (b(u), 0, u)$, *the surface of revolution* M *in* $\mathbf{L}^3$, *the mean curvature is*

$$H = \frac{b'(u)^2 - 1 - b(u)b''(u)}{2b(u)\sqrt{|1 - (b'(u))^2|}(b'(u)^2 - 1)}.$$

Now, the Gaussian curvature of the surface of revolution M is given by

$$\begin{aligned}
K &= \frac{LN - M^2}{EG - F^2} \\
&= \frac{-a'(u)a'(u)b(u)b''(u) + a'(u)a''(u)b(u)b'(u)}{b(u)^2|b'(u)^2 - a'(u)^2|(b'(u)^2 - a'(u)^2)} \\
&= \frac{-a'(u)a'(u)b''(u) + a'(u)a''(u)b'(u)}{b(u)|b'(u)^2 - a'(u)^2|(b'(u)^2 - a'(u)^2)}
\end{aligned}$$

in $\mathbf{L}^3$.

To simplify the Gaussian curvatures we assume that $\alpha(t)$ is unit speed. Thus,

(3.2) $$(b'(u))^2 - (a'(u))^2 = \pm 1.$$

Differentiating (3.2),

$$2b'(u)b''(u) - 2a'(u)a''(u) = 0$$

(3.3) $$b'(u)b''(u) = a'(u)a''(u).$$

Hence

$$\begin{aligned}
K &= \frac{-a'(u)a'(u)b''(u) + a'(u)a''(u)b'(u)}{b(u)(b'(u)^2 - a'(u)^2)} \\
&= \frac{-a'(u)a'(u)b''(u) + b'(u)b''(u)b'(u)}{b(u)(b'(u)^2 - a'(u)^2)} \\
&= \frac{b''(u)(-a'(u)^2 + b'(u)^2)}{b(u)(b'(u)^2 - a'(u)^2)} \\
&= \frac{b''(u)}{b(u)}.
\end{aligned}$$

Then we have

Theorem 3.3. *For a unit speed timelike or spacelike profile curve of the surface of revolution* M *in* $\mathbf{L}^3$, *the Gaussian curvature is* $K = \dfrac{b''(u)}{b(u)}$.

4. Minimal or Maximal Surfaces of revolution in $\mathbf{L}^3$

Suppose M be a timelike or spacelike surface of revolution in $\mathbf{L}^3$. Since M is minimal or maximal, $H = 0$. And hence we have a differential equation derived from (3.1),

(4.1) $$b'(u)^2 - b(u)b''(u) - 1 = 0.$$

To solve this equation, set $w = b'$. Then $b'' = \dfrac{d}{du}\left(\dfrac{db}{du}\right) = \dfrac{dw}{du}$, By the chain rule, $b'' = \dfrac{dw}{db}\dfrac{db}{du} = w\dfrac{dw}{db}$. Hence, the equation (4.1) implies

(4.2) $$w^2 - bw\frac{dw}{db} - 1 = 0.$$

By the separation of variables,

$$\int \frac{w}{w^2 - 1}dw = \int \frac{1}{b}db,$$

$\dfrac{1}{2}\ln|w^2 - 1| = \ln|bC_1|$, where C_1 is constant. Thus,

(4.3) $$|w^2 - 1| = (bC_1)^2.$$

Since the profile curve is $\alpha(u) = (b(u), 0, u)$, $\alpha'(u) = (b'(u), 0, 1)$, and hence $<\alpha'(u), \alpha'(u)> = b'(u)^2 - 1 = w^2 - 1$.

Theorem 4.1. *If M is a minimal timelike surface of revolution in $\mathbf{L}^3$, then M is a cone; $z = \sin^{-1}(\sqrt{x^2 + y^2})$. If M is a maximal spacelike surface of revolution in $\mathbf{L}^3$, then M is a pseudo-sphere $z = \sinh^{-1}(\sqrt{x^2 + y^2})$.*

Proof. Case 1. M is timelike. In this case, $w^2 - 1 < 0$. From the equation (4.3), $1 - w^2 = (bC_1)^2$, i.e., $w^2 = 1 - (bC_1)^2$. Hence $\dfrac{db}{du} = \pm\sqrt{1 - (bC_1)^2}$. By the separation of variables,

$$\int \frac{1}{C_1}\frac{d(bC_1)}{\sqrt{1 - (bC_1)^2}} = \pm\int du.$$

Since $\sin^{-1}(bC_1) = \pm C_1(u + C_2)$,

$$b = \pm\frac{1}{C_1}\sin(C_1 u + C_1 C_2)$$

where C_2 is also constant. Then, we have a general solution

$$b = \pm c\sin\left(\frac{u}{c} + d\right)$$

where c, d are constants. We may also assume $d = 0$ by performing a reparametrization $\tilde{u} = u + cd$. Moreover, since $b(u) > 0$ by hypothesis, we take the positive sign for $0 < \dfrac{u}{c} < \pi$. Hence, M is a surface given by the coordinate patch

$$X(u, v) = \left(c \sin\left(\frac{u}{c}\right) \cos v, c \sin\left(\frac{u}{c}\right) \sin v, u \right)$$

with the timelike profile curve $\alpha(u) = \left(c \sin\left(\frac{u}{c}\right), 0, u \right)$. Scaling $X(u, v)$ by c, we have

$$X(u, v) = \left(\sin\left(\frac{u}{c}\right) \cos v, \sin\left(\frac{u}{c}\right) \sin v, \frac{u}{c} \right).$$

Thus, $x^2 + y^2 = \sin^2\left(\dfrac{u}{c}\right)$, or $\sqrt{x^2 + y^2} = \sin\left(\dfrac{u}{c}\right)$. Therefore, $z = \dfrac{u}{c} = \sin^{-1}(\sqrt{x^2 + y^2})$.

Case 2. M is spacelike. In this case, $w^2 - 1 > 0$. From the equation (4.3), $w^2 = 1 + (bC_1)^2$, or $\dfrac{db}{du} = \pm\sqrt{1 + (bC_1)^2}$. By the separation of variables,

$$\int \frac{1}{C_1} \frac{dbC_1}{\sqrt{1 + (bC_1)^2}} = \pm \int du.$$

Since $\sinh^{-1}(bC_1) = \pm C_1(u + C_2)$,

$$b = \pm \frac{1}{C_1} \sinh(C_1 u + C_1 C_2)$$

where C_2 is also constant. Then, we have a general solution

$$b = \pm c \sinh\left(\frac{u}{c} + d\right)$$

where c, d are constants. We may also assume $d = 0$ by performing a reparametrization $\tilde{u} = u + cd$. Moreover, since $b(u) > 0$ by hypothesis, we take the positive sign for $0 < \dfrac{u}{c}$. Hence, M is a surface given by the coordinate patch

$$X(u, v) = \left(c \sinh\left(\frac{u}{c}\right) \cos v, c \sinh\left(\frac{u}{c}\right) \sin v, u \right)$$

with the timelike profile curve $\alpha(u) = \left(c \sinh\left(\frac{u}{c}\right), 0, u \right)$. Scaling $X(u, v)$ by c, we have

$$X(u, v) = \left(\sinh\left(\frac{u}{c}\right) \cos v, \sinh\left(\frac{u}{c}\right) \sin v, \frac{u}{c} \right).$$

Thus, $x^2 + y^2 = \sinh^2\left(\dfrac{u}{c}\right)$, or $\sqrt{x^2 + y^2} = \sinh\left(\dfrac{u}{c}\right)$. Therefore, $z = \dfrac{u}{c} = \sinh^{-1}(\sqrt{x^2 + y^2})$. $\square$

5. Surfaces of revolution with constant Gaussian curvatures in $\mathbf{L}^3$

We have seen the Gaussian curvature K of the surface of revolution M with the unit speed profile curve $\alpha(u) = (b(u), 0, a(u))$ in Theorem 3.3. Hence if M is a timelike surface in $\mathbf{L}^3$, $b'(u)^2 - a'(u)^2 = -1$. And if M is a spacelike surface in $\mathbf{L}^3$, $b'(u)^2 - a'(u)^2 = 1$.

Now, we will construct the surfaces of positive, zero and negative constant Gaussian curvatures in $\mathbf{L}^3$. By scaling them, we use the Gaussian Curvatures $K = 1$, $K = 0$ and $K = -1$.

(I) $K = 1$. Since $K = \dfrac{b''(u)}{b(u)} = 1$, $b''(u) - b(u) = 0$. To solve this equation, set $w = b'$ as Case 1. Since $b''(u) = w\dfrac{dw}{db}$, $w\dfrac{dw}{db} - b(u) = 0$. Or,

$$\int w\,dw = \int b\,db.$$

Thus

(5.1) $$w^2 = b^2 + C_1,$$

where C_1 is constant. So, $\dfrac{db}{du} = \pm\sqrt{C_1 + b^2}$.
Integrating this equation,

$$\int \frac{db}{\sqrt{C_1 + b^2}} = \pm(u + C_2),$$

where C_2 is constant. We may take a positive constant C_1. Hence ,

$$\sinh^{-1}\frac{b}{\sqrt{C_1}} = \pm(u + C_2).$$

Thus we have

$$b = \pm\sqrt{C_1}\sinh(u + C_2).$$

By performing the reparametrization $\tilde{u} = \sqrt{C_1}u$,

$$b = \pm\sqrt{C_1}\sinh\left(\frac{\tilde{u}}{\sqrt{C_1}} + C_2\right).$$

Or, resetting constants $c = \sqrt{C_1}$ and $d = C_2$,

$$b = \pm c \sinh\left(\frac{\tilde{u}}{c} + d\right).$$

Again reparametrizing by $u = \tilde{u} + cd$, and taking the positive sign

$$b = c \sinh\left(\frac{u}{c}\right).$$

Theorem 5.1. *Let M be a surface of revolution with $K = 1$ in $\mathbf{L}^3$. If M is timelike, then M is a cone;*

$$z = \pm\frac{1}{c} \int \sqrt{x^2 + y^2 + 2} du.$$

If M is spacelike, then M is a paraboloid;

$$z = \pm\sqrt{x^2 + y^2 + 1}.$$

Proof. Case1. M is timelike. Since $0 = b'^2 - a'^2 + 1 = \cosh^2\left(\frac{u}{c}\right) - a'^2 + 1$, $a'^2 = \cosh^2\left(\frac{u}{c}\right) + 1$. Thus, $a'(u) = \pm\sqrt{\cosh^2\left(\frac{u}{c}\right) + 1}$ Hence, for $0 < \frac{u}{c}$,

$$a(u) = \pm \int \sqrt{\cosh^2\left(\frac{u}{c}\right) + 1} du.$$

Therefore, we have the profile curve $\alpha(u)$ for the timelike surface of revolution M by

$$\alpha(u) = \left(c \sinh\left(\frac{u}{c}\right), 0, \pm \int \sqrt{\cosh^2\left(\frac{u}{c}\right) + 1} du\right).$$

Thus, we obtain the coordinate patch for M given by

$$X(u,v) = \left(c \sinh\left(\frac{u}{c}\right) \cos v, c \sinh\left(\frac{u}{c}\right) \sin v, \pm \int \sqrt{\cosh^2\left(\frac{u}{c}\right) + 1} du\right),$$

or scaling by c,

$$X(u,v) = \left(\sinh\left(\frac{u}{c}\right) \cos v, \sinh\left(\frac{u}{c}\right) \sin v, \pm\frac{1}{c} \int \sqrt{\cosh^2\left(\frac{u}{c}\right) + 1} du\right).$$

Since $x^2 + y^2 = \sinh^2\left(\frac{u}{c}\right) = \cosh^2\left(\frac{u}{c}\right) - 1$,

$$z = \frac{1}{c} \int \sqrt{\cosh^2\left(\frac{u}{c}\right) + 1} du = \pm\frac{1}{c} \int \sqrt{x^2 + y^2 + 2} du.$$

Note that since we may estimate $a(u)$ approximately by Taylor expansion about $u = 0$, we have $a(u) = \pm\left(\sqrt{2}u + \dfrac{u^3}{6\sqrt{2}c^2} + \dfrac{u^5}{48\sqrt{2}c^4} + O(u^7)\right)$. Hence we can draw $X(u,v)$ by using a computer program for some value of c.

Case2. M is spacelike. Then $0 = b'^2 - a'^2 - 1 = \cosh^2\left(\dfrac{u}{c}\right) - a'^2 - 1$, or $a'^2 = \cosh^2\left(\dfrac{u}{c}\right) - 1 = \sinh^2\left(\dfrac{u}{c}\right)$. Hence, for $0 < \dfrac{u}{c}$,

$$a(u) = \pm c\cosh\left(\frac{u}{c}\right).$$

Therefore, we have the profile curve $\alpha(u)$ for the timelike surface of revolution M by

$$\alpha(u) = \left(c\sinh\left(\frac{u}{c}\right), 0, \pm c\cosh\left(\frac{u}{c}\right)\right).$$

Thus, we obtain the coordinate patch for M given by

$$X(u,v) = \left(c\sinh\left(\frac{u}{c}\right)\cos v, c\sinh\left(\frac{u}{c}\right)\sin v, \pm c\cosh\left(\frac{u}{c}\right)\right),$$

or scaling by c,

$$X(u,v) = \left(\sinh\left(\frac{u}{c}\right)\cos v, \sinh\left(\frac{u}{c}\right)\sin v, \pm\cosh\left(\frac{u}{c}\right)\right).$$

Since $x^2 + y^2 = \sinh^2\left(\dfrac{u}{c}\right)$,

$$z^2 = \cosh^2\left(\frac{u}{c}\right) = \left(\sinh^2\left(\frac{u}{c}\right) + 1\right) = x^2 + y^2 + 1.$$

Since we have

$$z = \pm\sqrt{x^2 + y^2 + 1},$$

which is a paraboloid in $\mathbf{L}^3$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

(II) $K = 0$. Since $K = \dfrac{b''(u)}{b(u)} = 0$, $b''(u) = 0$. Hence $b(u) = cu + d_1$ where c, d_1 are constants.
If the profile curve $\alpha(u)$ is timelike (or spacelike),

$$(b'(u))^2 - (a'(u))^2 = -1 \text{ (or } (b'(u))^2 - (a'(u))^2 = 1),$$

and hence

$$(a'(u))^2 = (b'(u))^2 + 1 = c + 1 \text{ (or } (a'(u))^2 = (b'(u))^2 - 1 = c - 1).$$

Thus we have

$$a(u) = \pm\sqrt{(c+1)}u + d_2 \text{ (or } a(u) = \pm\sqrt{(c-1)}u + d_2)$$

respectively. So, the profile curve is

$$\alpha(u) = \left(cu + d_1, \pm\sqrt{(c+1)}u + d_2 \right) \text{ (or } \alpha(u) = \left(cu + d_1, \pm\sqrt{(c-1)}u + d_2 \right)$$

respectively.

Therefore, if M is timelike (or spacelike), then the coordinate patch is

$$X(u, v) = \left((cu + d_1)\cos v, (cu + d_1)\sin v, \pm\sqrt{(c+1)}u + d_2 \right)$$

$$\text{(or } \alpha(u) = \left((cu + d_1)\cos v, (cu + d_1)\sin v, \pm\sqrt{(c-1)}u + d_2 \right))$$

respectively.

Theorem 5.2. *Let M be a surface of revolution with $K = 0$ in $\mathbf{L}^3$. If M is timelike, then M is a cone;*

$$z = A_1\sqrt{x^2 + y^2} + B_1,$$

where $|A_1| > 1$ and B_1 are constants. If M is spacelike, then M is a cone;

$$z = A_2\sqrt{x^2 + y^2} + B_2,$$

where $|A_2| < 1$ and B_2 are constants.

Proof. Case 1. If $\alpha(u)$ is timelike, $\sqrt{x^2 + y^2} = cu + d_1$.

Since $u = \dfrac{1}{c}\left(\sqrt{x^2 + y^2} - d_1 \right)$ and since $z = \pm\sqrt{c+1}u + d_2$, we obtain

$$z = \frac{\pm\sqrt{c+1}}{c}\left(\sqrt{x^2 + y^2} - d_1 \right) + d_2.$$

Hence

$$z = \frac{(c-1)}{c}\left(\sqrt{x^2 + y^2} - d_1 \right) + d_2.$$

To simplify this equation, we set

$$A_1 = \frac{\pm\sqrt{c+1}}{c} \text{ and } B_1 = \frac{\pm\sqrt{c+1}d_1}{c} + d^2.$$

Since $c > -1$, $c \neq 0$. Thus

$$z = A_1\sqrt{x^2 + y^2} + B_1,$$

where A_1 and B_1 are constants.

Case 2. If $\alpha(u)$ is spacelike, then

$$\sqrt{x^2 + y^2} = cu + d_1 \text{ and } z = \pm\sqrt{c-1}u + d_2.$$

Since $u = \dfrac{1}{c}\left(\sqrt{x^2 + y^2} - d_1\right)$ and since $z = \pm\sqrt{c-1}u + d_2$, we obtain

$$z = \frac{\pm\sqrt{c-1}}{c}\left(\sqrt{x^2+y^2} - d_1\right) + d_2.$$

Hence

$$z = \frac{\pm\sqrt{c-1}}{c}\left(\sqrt{x^2+y^2} - d_1\right) + d_2.$$

To simplify this equation, we set

$$A_2 = \frac{\pm\sqrt{c-1}}{c} \quad \text{and} \quad B_2 = \frac{\pm\sqrt{c-1}d_1}{c} + d^2.$$

Since $c > 1$. Thus

$$z = A_2\sqrt{x^2 + y^2} + B_2,$$

where A_2 and B_2 are constants.   □

(III) $K = -1$. Since $K = \dfrac{b''(u)}{b(u)} = -1$, $b''(u) + b(u) = 0$. To solve

this equation, set $w = b'$. Since $b''(u) = w\dfrac{dw}{db}$, $w\dfrac{dw}{db} + b(u) = 0$. Or,

$$\int w\, dw = -\int b\, db.$$

Thus

$$w^2 = -b^2 + C_1,$$

where C_1 is constant. So, $\dfrac{db}{du} = \pm\sqrt{C_1 - b^2}, C_1 > 0$.
Integrating this equation,

$$\int \frac{db}{\sqrt{C_1 - b^2}} = \pm(u + C_2),$$

where C_2 is constant. Hence ,

$$\sin^{-1}\frac{b}{\sqrt{C_1}} = \pm(u + C_2).$$

Thus we have

$$b = \pm\sqrt{C_1}\sin(u + C_2).$$

By performing the reparametrization $\tilde{u} = \sqrt{C_1}u$,

$$b = \pm\sqrt{C_1}\sin\left(\frac{\tilde{u}}{\sqrt{C_1}}\right) + C_2.$$

Or, resetting the variable $u = \tilde{u}$, constants $c = \sqrt{C_1}$ and $d_1 = C_2$,

$$b = \pm c \sin\left(\frac{u}{c}\right) + d_1.$$

Theorem 5.3. *Let M be a surface of revolution with $K = -1$ in $\mathbf{L}^3$. If M is spacelike, then M is no longer a surface. If M is timelike, then M is a cone;*

$$z = \frac{1}{c} \int \sqrt{2c^2 - x^2 - y^2}\, du + d^2$$

for $d_1 = 0$, where d_2 is constant.

Proof. Case 1. If M is spacelike, then

$$a'(u) = \pm\sqrt{b'(u)^2 - 1} = \pm\sqrt{\cos^2\left(\frac{u}{c}\right) - 1}.$$

It works when $\dfrac{u}{c} = 0, \pi, -\pi$. Then $a(u)$ is a point, say z_0. Hence

$$X(u, v) = (d_1 \cos v, d_1 \sin v, z_0),$$

which is no longer a surface.

Case 2. If M is timelike, since $a'(u) = \pm\sqrt{b'(u)^2 + 1}$, we have

$$a'(u) = \pm\sqrt{\cos^2\left(\frac{u}{c}\right) + 1}.$$

Thus

$$a(u) = \pm \int \sqrt{\cos^2\left(\frac{u}{c}\right) + 1}\, du + d_2,$$

where d_2 is constant. So, we have the profile curve of M by

$$\alpha(u) = \left(c \sin\left(\frac{u}{c}\right) + d_1, \pm \int \sqrt{\cos^2\left(\frac{u}{c}\right) + 1}\, du + d_2\right).$$

Therefore, we obtain a coordinate patch for the timelike surface of revolution M in $\mathbf{L}^3$ as follows;

$$X(u, v) = \left(\left(c \sin\left(\frac{u}{c}\right) + d_1\right) \cos v, \left(c \sin\left(\frac{u}{c}\right) + d_1\right) \sin v,\right.$$
$$\left. \pm \int \sqrt{\cos^2\left(\frac{u}{c}\right) + 1}\, du + d_2\right).$$

Since $x^2 + y^2 = \left(c \sin \left(\dfrac{u}{c} \right) + d_1 \right)^2$, $\pm \sqrt{x^2 + y^2} = c \sin \left(\dfrac{u}{c} \right) + d_1$. Taking the positive sign, $\sin \left(\dfrac{u}{c} \right) = \dfrac{1}{c} \left(\sqrt{x^2 + y^2} - d_1 \right)$. Thus,

$$z = \pm \int \sqrt{\cos^2 \left(\dfrac{u}{c} \right) + 1} \, du + d_2$$

$$= \pm \int \sqrt{2 - \left(\dfrac{1}{c} \left(\sqrt{x^2 + y^2} - d_1 \right) \right)^2} \, du + d_2.$$

If $d_1 = 0$,

$$z = \pm \dfrac{1}{c} \int \sqrt{2c^2 - x^2 - y^2} \, du + d^2.$$

$\square$

Remark 5.4. *When M is a timelike surface, it is hard to solve the integration of $a(u) = \pm \displaystyle\int \sqrt{\cos^2 \left(\dfrac{u}{c} \right) + 1} \, du + d_2$ while Maple program may show the figure. Thus, we may try to approximate this integration using the Taylor series.*

As the Case1 in Theorem 5.1, since we may estimate $a(u)$ approximately by Taylor expansion about $u = 0$, we have

$$a(u) = \pm \left(\sqrt{2} u - \dfrac{u^3}{6\sqrt{2}c^2} + \dfrac{u^5}{48\sqrt{2}c^4} + O(u^7) \right).$$

Hence we can also draw $X(u, v)$ by using a computer program for some value of c.

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