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ITERATED ENTIRE FUNCTIONS AND THEIR GROWTH PROPERTIES ON THE BASIS OF (p, q)-TH ORDER

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Abstract. Entire functions have been investigated so popularly to have been divided into a large number of specialized subjects. Even the limited subject of entire functions is too vast to be dealt with in a single volume with any approach to completeness. Here, in this paper, we choose to investigate certain interesting results associated with the comparative growth properties of iterated entire functions using (p, q)-th order and (p, q) -th lower order, in a rather comprehensive and systematic manner.

1. Introduction, Definitions and Notations

Throughout this paper, let $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{C}$ be the sets of positive integers, positive real numbers and finite complex numbers, respectively. The following notations are used:

$$\log^{[k]} x := \begin{cases} \log\left(\log^{[k-1]} x\right) & (k \in \mathbb{N}) \\ x & (k = 0) \end{cases}$$

and

$$\exp^{[k]} x := \begin{cases} \exp\left(\exp^{[k-1]} x\right) & (k \in \mathbb{N}) \\ x & (k = 0). \end{cases}$$

Let f be analytic on the closed disk $|z| \leq r$ for some $r \in \mathbb{R}^+$ and let $M(r, f)$ be the maximum modulus of the function f on the closed disk

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$|z| \leq r$. Then it is easy to see from the maximum modulus theorem that

$$M(r,f) = \sup_{|z|=r} |f(z)|.$$

Alternatively, another growth indicator closely related to an entire function f is so-called Nevanlinna's characteristic function of f , which is denoted by $T(r,f)$, is defined as follows:

$$T(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta,$$

where $\log^+ x := \max\{\log x, 0\}$ $(x > 0)$. It is easy to see that

$$T(r,f) \leq \log^+ M(r,f)$$

for all entire functions f and all $r \in \mathbb{R}^+$.

The following definitions are recalled.

Definition 1.1. *The order ρ_f and the lower order λ_f of an entire function f are defined, respectively, as follows:*

$$\rho_f := \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f := \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 1.2. (*see* [10]). *Let $l \in \mathbb{N} \setminus \{1\}$. The generalized order $\rho_f^{[l]}$ and the generalized lower order $\lambda_f^{[l]}$ of an entire function f are, respectively, defined by*

$$\rho_f^{[l]} := \limsup_{r\to\infty} \frac{\log^{[l]} M(r,f)}{\log r} \quad \text{and} \quad \lambda_f^{[l]} := \liminf_{r\to\infty} \frac{\log^{[l]} M(r,f)}{\log r}.$$

It is obvious that the special case of Definition 1.2 when $l = 2$ coincides with Definition 1.1.

Definition 1.3. *A function $\rho_f^{[l]}(r)$ is called a generalized proximate order of a meromorphic function f relative to $T(r,f)$ if*

(i) *$\rho_f^{[l]}(r)$ is non-negative and continuous for $r \geqslant r_0$, say,*

(ii) *$\rho_f^{[l]}(r)$ is differentiable for $r \geqslant r_0$ except possibly at isolated points at which $\rho_f^{[l]\prime}(r+0)$ and $\rho_f^{[l]\prime}(r-0)$ exist,*

(iii) $\lim_{r\to\infty} \rho_f^{[l]}(r) = \rho_f^{[l]} < \infty,$

(iv) $\lim_{r\to\infty} \rho_f^{[l]\prime}(r) \prod_{j=0}^{l-1} \log^{[j]} r = 0$ *and*

(v) $\limsup_{r\to\infty} \frac{\log^{[l-2]} T(r,f)}{r^{\rho_f^{[l]}(r)}} = 1.$

It is noted that the existence of such a proximate order in Definition 1.3 is proved by Lahiri [8]. Similarly, a generalized lower proximate order of f can be defined in the following way.

Definition 1.4. *A function $\lambda_f^{[l]}(r)$ is defined as a generalized lower proximate order of a meromorphic function f relative to $T(r,f)$ if*

(i) $\lambda_f^{[l]}(r)$ *is non-negative and continuous for $r \geqslant r_0$, say,*

(ii) $\lambda_f^{[l]}(r)$ *is differentiable for $r \geqslant r_0$ except possibly at isolated points at which $\lambda_f^{[l]\prime}(r+0)$ and $\lambda_f^{[l]\prime}(r-0)$ exist,*

(iii) $\lim_{r\to\infty} \lambda_f^{[l]}(r) = \lambda_f^{[l]} < \infty,$

(iv) $\lim_{r\to\infty} \lambda_f^{[l]\prime}(r) \prod_{j=0}^{l-1} \log^{[j]} r = 0$ *and*

(v) $\liminf_{r\to\infty} \frac{\log^{[l-2]} T(r,f)}{r^{\lambda_f^{[l]}(r)}} = 1.$

Juneja *et al.* [7] defined the (p,q)-th order and (p,q)-th lower order of an entire function f, respectively, as follows:

(1.1)

$$\rho_f(p,q) := \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} \quad \text{and} \quad \lambda_f(p,q) := \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r},$$

where $p,\ q \in \mathbb{N}$ with $p \geq q$. For $p=2$ and $q=1$, we simplify to denote $\rho_f(2,1)$ and $\lambda_f(2,1)$ by ρ_f and λ_f, respectively (see Definition 1.1).

If $f(z)$ and $g(z)$ are entire functions, then the iteration of f with respect to g is defined as follows (see [10]):

$$f_1(z) := f(z);$$
$$f_2(z) := f(g(z)) = f(g_1(z));$$
$$f_3(z) := f(g(f(z))) = f(g(f_1(z))) = f(g_2(z));$$
$$\cdots$$
$$f_n(z) := f(g(f \cdots (h(z)) \cdots)) \quad (n \in \mathbb{N}),$$

where $h(z) = f(z)$ when n is odd and $h(z) = g(z)$ when n is even.

Similarly one defines

$$g_1(z) := g(z);$$
$$g_2(z) := g(f(z)) = g(f_1(z));$$
$$\cdots$$
$$g_n(z) := g(f(g_{n-2}(z))) = g(f_{n-1}(z)) \quad (n \in \mathbb{N}).$$

It is obvious that $f_n(z)$ and $g_n(z)$ $(n \in \mathbb{N})$ are all entire functions.

The growth properties of entire functions using (p, q)-th order and (p, q)-th lower order have been investigated in such works as (for example) [2, 3, 5, 7]. In the sequel of these works, in this paper, we aim at investigating further growth properties of iterated entire functions on the basis of (p, q)-th order and (p, q)-th lower order under the restriction p, $q \in \mathbb{N}$ with $p \geq q$, in a rather comprehensive and systematic manner.

2. Lemmas

Here we present certain required properties involving the definitions in Section 1. We begin by presenting a class of functions $A_l(r)$ $(l \in \mathbb{N})$ with $A_0(r)$ a constant such that

$$A_l(r) = \log\left\{B_{l-1} + \frac{A_{l-1}(r)}{f_{l-1}(r)}\right\}, \tag{2.1}$$

where B_{l-1} $(l \in \mathbb{N})$ is a constant and $f_{l-1}(r)$ $(l \in \mathbb{N} \setminus \{1\})$ is an increasing function of r with $f_0(r) = 1$. It is obvious that

$$\lim_{r\to\infty} \frac{A_l(r)}{F(r)} = 0$$

for all functions $F(r)$ satisfying the properties of Nevanlinna's characteristic function or maximum modulus function. It should be noted that in our subsequent discussion, $A_l(r)$ may be different for different values of l unless otherwise stated.

Lemma 2.1. (*see* [1]). *If f and g are any two entire functions, then, for all sufficiently large values of $r \in \mathbb{R}^+$,*

$$M(r, f \circ g) \leq M(M(r, g), f).$$

Lemma 2.2. *Let f and g be any two entire functions such that $\rho_f(p, q) < \infty$ and $\rho_g(a, b) < \infty$ where a, b, p, $q \in \mathbb{N}$ with $a \geq b$ and $p \geq q$. Then, for all sufficiently large values of r and any $\varepsilon \in \mathbb{R}^+$, the following inequalities hold true:*

• *For any even* $n \in \mathbb{N}$

(i) $(q < a,\ b < p)$

$$\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)\right]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r, g) + A_l(r);$$

(ii) $(p = b \geq q,\ a > q,\ n > 2)$

$$\log^{\left[p+\frac{n-2}{2}(a-q)\right]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon)(\rho_g(a,b)+\varepsilon) \log^{[q]} M(r, g) + A_l(r);$$

(iii) $(p > b,\ q = a)$

$$\log^{\left[p+\frac{n-2}{2}(p-b)\right]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r, g) + A_l(r);$$

(iv) $(a = b = p = q)$

$$\log^{[p]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon)^{\frac{n}{2}} (\rho_g(a,b)+\varepsilon)^{\frac{n-2}{2}} \log^{[q]} M(r, g);$$

(v) $(p = a > q = b)$

$$\log^{[p+(n-2)(p-q)]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r, g) + A_l(r).$$

(vi) $(p < b,\ q < a,\ b - p = a - q)$

$$\log^{[p+a-q]} M(r, f_n) \leq (\rho_g(a,b)+\varepsilon)^{\frac{n-2}{2}} \log^{[a]} M(r, g) + A_l(r);$$

(vii) $(p < b,\ q < a,\ b - p < a - q)$

$$\log^{\left[p+a+\frac{n-2}{2}(a-b)-q\right]} M(r, f_n) \leq \log^{[a]} M(r, g) + A_l(r);$$

(viii) $(p > b,\ q > a,\ q - a = p - b)$

$$\log^{[p]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon)^{\frac{n}{2}} \log^{[q]} M(r, g) + A_l(r);$$

(ix) $(p > b,\ q > a,\ q - a < p - b)$

$$\log^{\left[p+\frac{n-2}{2}(p+a-b-q)\right]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r, g) + A_l(r);$$

• *For any odd* $n \in \mathbb{N} \setminus \{1\}$

(x) $(q < a,\ b < p)$

$$\log^{\left[p+\frac{n-1}{2}(a-q)+\frac{n-3}{2}(p-b)\right]} M(r, f_n) \leq (\rho_g(a,b)+\varepsilon) \log^{[b]} M(r, f) + A_l(r);$$

(xi) $(p = b \geq q,\ a > q)$

$$\log^{\left[p+\frac{n-1}{2}(a-q)\right]} M(r, f_n) \leq (\rho_g(a,b)+\varepsilon) \log^{[b]} M(r, f) + A_l(r);$$

(xii) $(p > b,\ q = a)$

$$\log^{\left[p+\frac{n-3}{2}(p-b)\right]} M(r, f_n) \le (\rho_f(p,q)+\varepsilon)(\rho_g(a,b)+\varepsilon)\log^{[b]} M(r,f) + A_l(r);$$

(xiii) $(a = b = p = q)$

$$\log^{[p]} M(r, f_n) \le \left[(\rho_f(p,q)+\varepsilon)(\rho_g(a,b)+\varepsilon)\right]^{\frac{n-1}{2}} \log^{[b]} M(r,f);$$

(xiv) $(p = a > q = b)$

$$\log^{[p+(n-2)(p-q)]} M(r, f_n) \le (\rho_g(p,q)+\varepsilon)\log^{[q]} M(r,f) + A_l(r);$$

(xv) $(p < b,\ q < a,\ b-p = a-q)$

$$\log^{[p+a-q]} M(r, f_n) \le (\rho_g(a,b)+\varepsilon)^{\frac{n-1}{2}} \log^{[b]} M(r,f) + A_l(r);$$

(xvi) $(p < b,\ q < a,\ b-p < a-q)$

$$\log^{\left[p+a+\frac{n-3}{2}(a-b)-q\right]} M(r, f_n) \le (\rho_g(a,b)+\varepsilon)\log^{[b]} M(r,f) + A_l(r);$$

(xvii) $(p > b,\ q > a,\ q-a = p-b)$

$$\log^{[p]} M(r, f_n) \le (\rho_f(p,q)+\varepsilon)^{\frac{n-1}{2}} \log^{[p]} M(r,f) + A_l(r);$$

(xviii) $(p > b,\ q > a,\ q-a < p-b)$

$$\log^{\left[p+\frac{n-1}{2}(p+a-b-q)\right]} M(r, f_n) \le \log^{[p]} M(r,f) + A_l(r);$$

where A_l are given in (2.1).

Proof. Let $n \in \mathbb{N}$ be an even integer for (i)-(ix). Then it follows from Lemma 2.1 that, for all sufficiently large values of r,

$$\log^{[p]} M(r, f_n) = M(r, f \circ g_{n-1}) \le \log^{[p]} M(M(r, g_{n-1}), f).$$

Therefore, in view of (1.1), for all sufficiently large values of r, we find

$$\log^{[p]} M(r, f_n) \le (\rho_f(p,q)+\varepsilon)\log^{[q]} M(r, g_{n-1}) \tag{2.2}$$

for any $\varepsilon \in \mathbb{R}^+$.

Case I. $q < a$ and $b < p$.

It is seen from (2.2) that, for all sufficiently large values of r,

$$\log^{[p+a-q]} M(r, f_n) \le \log^{[a]} M(r, g_{n-1}) + A_l(r).$$

Applying (2.2) to continue this process, we have

$$\log^{[p+a-q]} M(r, f_n) \le \log^{[a]} M(M(r, f_{n-2}), g) + A_l(r),$$

$$(2.3) \quad \log^{[p+a-q]} M(r, f_n) \le (\rho_g(a,b) + \varepsilon) \log^{[b]} M(r, f_{n-2}) + A_l(r),$$

$$\log^{[p+a-q+p-b]} M(r, f_n) \le \log^{[p]} M(M(r, g_{n-3}), f) + A_l(r),$$

and so on. We thus have that, for even $n \in \mathbb{N}$,

$$\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)\right]} M(r, f_n) \le (\rho_f(p,q) + \varepsilon) \log^{[q]} M(r, g) + A_l(r).$$

Similarly, we find that, for odd $n \in \mathbb{N} \setminus \{1\}$,

$$\log^{\left[p+\frac{n-1}{2}(a-q)+\frac{n-3}{2}(p-b)\right]} M(r, f_n) \le (\rho_g(a,b) + \varepsilon) \log^{[b]} M(r, f) + A_l(r).$$

Hence (i) and (x) of the lemma are established.

Case II. $p = b \ge q$ and $a > q$.

Here we find from (2.3) that, for all sufficiently large values of r,

$$\log^{[p+a-q]} M(r, f_n) \le (\rho_g(a,b) + \varepsilon) \log^{[p]} M(r, f_{n-2}) + A_l(r),$$

from which we have

$$\log^{[p+a-q]} M(r, f_n) \le (\rho_g(a,b) + \varepsilon) \log^{[p]} M(M(r, g_{n-3}), f) + A_l(r) \;.$$

Continuing this process to arrive at the following inequality: For even $n \in \mathbb{N} \setminus \{1, 2\}$,

$$\log^{\left[p+\frac{n-2}{2}(a-q)\right]} M(r, f_n) \le (\rho_f(p,q) + \varepsilon)(\rho_g(a,b) + \varepsilon) \log^{[q]} M(r, g) + A_l(r).$$

Likewise, for odd $n \in \mathbb{N} \setminus \{1\}$,

$$\log^{\left[p+\frac{n-1}{2}(a-q)\right]} M(r, f_n) \le (\rho_g(a,b) + \varepsilon) \log^{[b]} M(r, f) + A_l(r).$$

Hence (ii) and (xi) of the lemma are proved.

Case III. $p \ge b$ and $q = a$.

It follows from (2.2) that, for all sufficiently large values of r,

$$\log^{[p]} M(r, f_n) \le (\rho_f(p,q) + \varepsilon) \log^{[a]} M(r, g_{n-1}).$$

Continuing this process, we have

$$\begin{aligned}
\log^{[p]} M(r, f_n) &\le (\rho_f(p,q) + \varepsilon) \log^{[a]} M(M(r, f_{n-2}), g), \\
\log^{[p]} M(r, f_n) &\le (\rho_f(p,q) + \varepsilon)(\rho_g(a,b) + \varepsilon) \log^{[b]} M(r, f_{n-2}), \\
\log^{[p+p-b]} M(r, f_n) &\le \log^{[p]} M(M(r, g_{n-3}), f) + A_l(r),
\end{aligned}$$

and so on.

We finally arrive at the following inequality: For even $n \in \mathbb{N}$,

$$\log^{\left[p+\frac{n-2}{2}(p-b)\right]} M\left(r, f_n\right) \leq \left(\rho_f\left(p, q\right)+\varepsilon\right) \log^{[q]} M\left(r, g\right)+A_l\left(r\right).$$

Similarly, for odd $n \in \mathbb{N} \setminus \{1\}$, we find

$$\log^{\left[p+\frac{n-3}{2}(p-b)\right]} M\left(r, f_n\right) \leq \left(\rho_f\left(p, q\right)+\varepsilon\right)\left(\rho_g\left(a, b\right)+\varepsilon\right) \log^{[b]} M\left(r, f\right)+A_l\left(r\right).$$

Hence we prove (iii) and (xii) of the lemma.

Case IV. $a=b=p=q$.

We have from (2.2) that, for all sufficiently large values of r,

$$\log^{[p]} M\left(r, f_n\right) \quad \leq \quad \left(\rho_f\left(p, q\right)+\varepsilon\right) \log^{[a]} M\left(r, g_{n-1}\right).$$

Continuing this process, we have

$$\begin{aligned}
\log^{[p]} M\left(r, f_n\right) &\leq \left(\rho_f\left(p, q\right)+\varepsilon\right) \log^{[a]} M\left(M\left(r, f_{n-2}\right), g\right), \\
\log^{[p]} M\left(r, f_n\right) &\leq \left(\rho_f\left(p, q\right)+\varepsilon\right)\left(\rho_g\left(a, b\right)+\varepsilon\right) \log^{[b]} M\left(M\left(r, g_{n-3}\right), f\right), \\
\log^{[p]} M\left(r, f_n\right) &\leq \left(\rho_f\left(p, q\right)+\varepsilon\right)\left(\rho_g\left(a, b\right)+\varepsilon\right) \log^{[p]} M\left(M\left(r, g_{n-3}\right), f\right),
\end{aligned}$$

and so on.

We finally have the following inequality: For even $n \in \mathbb{N}$,

$$\log^{[p]} M\left(r, f_n\right) \leq \left(\rho_f\left(p, q\right)+\varepsilon\right)^{\frac{n}{2}}\left(\rho_g\left(a, b\right)+\varepsilon\right)^{\frac{n-2}{2}} \log^{[q]} M\left(r, g\right).$$

Likewise, for odd $n \in \mathbb{N} \setminus \{1\}$,

$$\log^{[p]} M\left(r, f_n\right) \leq \left[\left(\rho_f\left(p, q\right)+\varepsilon\right)\left(\rho_g\left(a, b\right)+\varepsilon\right)\right]^{\frac{n-1}{2}} \log^{[b]} M\left(r, f\right).$$

Hence (iv) and (xiii) of the lemma are established.

Case V. $p=a>q=b$.

We find from (2.2) that, for all sufficiently large values of r,

$$\log^{[p+p-q]} M\left(r, f_n\right) \leq \log^{[p]} M\left(r, g_{n-1}\right)+A_l\left(r\right).$$

Continuing this process, we have

$$\begin{aligned}
\log^{[p+p-q]} M\left(r, f_n\right) &\leq \log^{[p]} M\left(M\left(r, f_{n-2}\right), g\right)+A_l\left(r\right), \\
\log^{[p+p-q]} M\left(r, f_n\right) &\leq \left(\rho_g\left(p, q\right)+\varepsilon\right) \log^{[q]} M\left(M\left(r, g_{n-3}\right), f\right)+A_l\left(r\right), \\
\log^{[p+2(p-q)]} M\left(r, f_n\right) &\leq \log^{[p]} M\left(M\left(r, g_{n-3}\right), f\right)+A_l\left(r\right),
\end{aligned}$$

and so on.

We finally arrive at the following inequality: For even $n \in \mathbb{N}$,

$$\log^{[p+(n-2)(p-q)]} M\left(r, f_n\right) \leq \left(\rho_f\left(p, q\right)+\varepsilon\right) \log^{[q]} M\left(r, g\right)+A_l\left(r\right).$$

Similarly, for odd $n \in \mathbb{N} \setminus \{1\}$, we have

$$\log^{[p+(n-2)(p-q)]} M(r, f_n) \le (\rho_g(p,q) + \varepsilon) \log^{[q]} M(r, f) + A_l(r).$$

Hence we prove (v) and (xiv) of the lemma.

Case VI. $p < b$, $q < a$, $b - p = a - q$.

We see from (2.3) that, for all sufficiently large values of r,

$$\log^{[p+a-q]} M(r, f_n) \le (\rho_g(a,b) + \varepsilon) \log^{[b-p]} \left[\log^{[p]} M(r, f_{n-2})\right] + A_l(r).$$

Continuing this process, we have

$$\begin{aligned} &\log^{[p+a-q]} M(r, f_n) \\ &\le (\rho_g(a,b) + \varepsilon) \log^{[b-p]} \left[(\rho_f(p,q) + \varepsilon) \log^{[q]} M(r, g_{n-3})\right] + A_l(r), \end{aligned} \tag{2.4}$$

$$\log^{[p+a-q]} M(r, f_n) \le (\rho_g(a,b) + \varepsilon) \log^{[a]} M(M(r, f_{n-4}), g) + A_l(r),$$

and so on.

We thus have the following inequality: For even $n \in \mathbb{N}$,

$$\log^{[p+a-q]} M(r, f_n) \le (\rho_g(a,b) + \varepsilon)^{\frac{n-2}{2}} \log^{[a]} M(r, g) + A_l(r).$$

Likewise, for odd $n \in \mathbb{N} \setminus \{1\}$, we have

$$\log^{[p+a-q]} M(r, f_n) \le (\rho_g(a,b) + \varepsilon)^{\frac{n-1}{2}} \log^{[b]} M(r, f) + A_l(r).$$

Hence (vi) and (xv) of the lemma are proved.

Case VII. $p < b$, $q < a$, $b - p < a - q$.

It follows from (2.4) that, for all sufficiently large values of r,

$$\log^{[p+a-q+a-q-b+p]} M(r, f_n) \le \log^{[a]} M(M(r, f_{n-4}), g) + A_l(r).$$

Continuing this process to finally give the following inequality: For even $n \in \mathbb{N}$,

$$\log^{\left[p+a-q+\frac{n-2}{2}(a-q-b+q)\right]} M(r, f_n) \le \log^{[a]} M(r, g) + A_l(r).$$

Similarly, for odd $n \in \mathbb{N} \setminus \{1\}$, we have

$$\log^{\left[p+a-q+\frac{n-3}{2}(a-q-b+q)\right]} M(r, f_n) \le (\rho_g(a,b) + \varepsilon) \log^{[b]} M(r, f) + A_l(r).$$

Hence (vii) and (xvi) of the lemma are established.

Case VIII. $p > b$, $q > a$, $q - a = p - b$.

We find from (2.2) that, for all sufficiently large values of r,

$$\log^{[p]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r, g_{n-1}),$$
$$\log^{[p]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q-a]} \log^{[a]} M(r, g_{n-1}),$$
$$\log^{[p]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q-a]} \left[\log^{[a]} M(M(r, f_{n-2}), g)\right],$$

$$\log^{[p]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q-a]} \left[(\rho_g(a,b)+\varepsilon) \log^{[b]} M(r, f_{n-2})\right], \tag{2.5}$$

$$\log^{[p]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[p]} M(M(r, g_{n-3}), f) + A_l(r),$$

and so on.

We finally have the following inequality: For even $n \in \mathbb{N}$,

$$\log^{[p]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon)^{\frac{n}{2}} \log^{[q]} M(r, g) + A_l(r).$$

Likewise, for odd $n \in \mathbb{N} \setminus \{1\}$, we have

$$\log^{[p]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon)^{\frac{n-1}{2}} \log^{[p]} M(r, f) + A_l(r).$$

Hence (viii) and (xvii) of the lemma are proved.

Case IX. $p > b$, $q > a$, $q - a < p - b$.

We find from (2.5) that, for all sufficiently large values of r,

$$\log^{[p]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q-a+b]} M(r, f_{n-2}) + A_l(r),$$
$$\log^{[p+p+a-b-q]} M(r, f_n) \leq \log^{[p]} M(M(r, g_{n-3}), f) + A_l(r),$$

and so on.

We finally arrive at the following inequality: For even $n \in \mathbb{N}$,

$$\log^{\left[p+\frac{n-2}{2}(p+a-b-q)\right]} M(r, f_n) \leq (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r, g) + A_l(r).$$

Similarly, for odd $n \in \mathbb{N} \setminus \{1\}$, we have

$$\log^{\left[p+\frac{n-1}{2}(p+a-b-q)\right]} M(r, f_n) \leq \log^{[p]} M(r, f) + A_l(r).$$

Hence we prove (ix) and (xviii). This completes the proof of the lemma. □

Lemma 2.3. (*see* [4]). *Let g be an entire function. Then, for any $\delta > 0$, the function $r^{\lambda_g^{[l]}+\delta-\lambda_g^{[l]}(r)}$ is an increasing function of r.*

Lemma 2.4. (*see* [4]). *Let g be an entire function. Then, for any $\delta > 0$, the function $r^{\rho_g^{[l]}+\delta-\rho_g^{[l]}(r)}$ is an increasing function of r.*

3. Main Results

Here we state main results asserted by the following theorems.

Theorem 3.1. *Suppose f and g are any two entire functions such that $\rho_f(p,q)$ and $\rho_g^{[l]}$ are both finite for p, q, $l \in \mathbb{N}$ with $p \geq q$ and $l \geq 2$. Then, for any even $n \in \mathbb{N}$, the following inequalities hold true:*

(i) $(p > 1,\ q = l > 2)$

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{n}{2}(p-1)+1\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \rho_f(p,q) \cdot 2^{\lambda_g^{[l]}};$$

(ii) $(p > 1,\ q = l = 2)$

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{n}{2}(p-1)+1\right]} M\left(r, f_n\right)}{\log M(r,g)} \leqslant 3 \cdot \rho_f(p,q) \cdot 2^{\lambda_g};$$

(iii) $(p > 1,\ q > l > 2,\ q - l = p - 1)$

$$\liminf_{r\to\infty} \frac{\log^{[p]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \left[\rho_f\left(p,q\right)\right]^{\frac{n}{2}} \cdot 2^{\lambda_g^{[l]}};$$

(iv) $(p > 1,\ q > l > 2,\ q - l < p - 1)$

$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p+l-1-q)\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \rho_f\left(p,q\right) \cdot 2^{\lambda_g^{[l]}};$$

(v) $(p > 1,\ q > l = 2,\ q - l < p - 1)$

$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p+1-q)\right]} M\left(r, f_n\right)}{\log M(r,g)} \leqslant 3 \cdot \rho_f\left(p,q\right) \cdot 2^{\lambda_g};$$

(vi) $(q < l - 1,\ 1 < p)$

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{n}{2}(l-q)+\frac{n}{2}(p-1)\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant 2^{\lambda_g^{[l]}};$$

(vii) $(p = q = 1,\ l - 1 > q,\ n > 2)$

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{n}{2}(l-1)\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant 2^{\lambda_g^{[l]}};$$

(viii) $(p = l,\, l-1 > q = 1)$

$$\liminf_{r\to\infty}\frac{\log^{[n(l-1)]} M(r, f_n)}{\log^{[l-1]} M(r,g)} \leqslant 2^{\lambda_g^{[l]}}.$$

Proof. We first consider the following two cases.

• $l > 2$.

Since $\liminf_{r\to\infty}\frac{\log^{[l-2]} T(r,g)}{r^{\lambda_g^{[l]}(r)}} = 1$, for given ε $(0 < \varepsilon < 1)$, we find that, for a sequence of values of $r \in \mathbb{R}^+$ tending to infinity,

$$\log^{[l-2]} T(r,g) < (1+\varepsilon) r^{\lambda_g^{[l]}(r)}$$

and, for all sufficiently large $r \in \mathbb{R}^+$,

$$\log^{[l-2]} T(r,g) > (1-\varepsilon) r^{\lambda_g^{[l]}(r)}.$$

Since $\log M(r,g) \le 3T(2r,g)$, for a sequence of values of r tending to infinity, we find that, for any $\delta > 0$,

$$\begin{aligned}
\frac{\log^{[l-1]} M(r,g)}{\log^{[l-2]} T(r,g)} &\le \frac{\log^{[l-2]}\{3\,T(2r,g)\}}{\log^{[l-2]} T(r,g)} = \frac{\log^{[l-2]}\, T(2r,g)}{\log^{[l-2]} T(r,g)} \\
&\le \frac{1+\varepsilon}{1-\varepsilon}\cdot\frac{(2r)^{\lambda_g^{[l]}+\delta}}{(2r)^{\lambda_g^{[l]}+\delta-\lambda_g^{[l]}(2r)}}\cdot\frac{1}{r^{\lambda_g^{[l]}(r)}} \\
&= \frac{1+\varepsilon}{1-\varepsilon}\cdot 2^{\lambda_g^{[l]}+\delta}\cdot\frac{r^{\lambda_g^{[l]}+\delta-\lambda_g^{[l]}(r)}}{(2r)^{\lambda_g^{[l]}+\delta-\lambda_g^{[l]}(2r)}}.
\end{aligned}$$

In view of Lemma 2.3, since $r^{\lambda_g^{[l]}+\delta-\lambda_g^{[l]}(r)}$ is an increasing function of r, we have

$$\frac{\log^{[l-1]} M(r,g)}{\log^{[l-2]} T(r,g)} \le \frac{1+\varepsilon}{1-\varepsilon}\cdot 2^{\lambda_g^{[l]}+\delta}.$$

Since both $\varepsilon > 0$ and $\delta > 0$ are arbitrary, we get

$$\liminf_{r\to\infty}\frac{\log^{[l-1]} M(r,g)}{\log^{[l-2]} T(r,g)} \le 2^{\lambda_g^{[l]}}. \tag{3.1}$$

• $l = 2$.

It follows from (v) of Definition 1.4 that

$$\liminf_{r\to\infty}\frac{T(r,g)}{r^{\lambda_g(r)}} = 1.$$

For given ε $(0 < \varepsilon < 1)$, we see that, for a sequence of values of $r \in \mathbb{R}^+$ tending to infinity,

$$T(r, g) < (1 + \varepsilon) r^{\lambda_g(r)}$$

and, for all sufficiently large values of $r \in \mathbb{R}^+$,

$$T(r, g) > (1 - \varepsilon) r^{\lambda_g(r)}.$$

Since $\log M(r, g) \leq 3T(2r, g)$, for a sequence of values of $r \in \mathbb{R}^+$ tending to infinity, we find that, for any $\delta > 0$,

$$\frac{\log M(r, g)}{T(r, g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)} \cdot 2^{\lambda_g + \delta} \cdot \frac{r^{\lambda_g + \delta - \lambda_g(r)}}{(2r)^{\lambda_g + \delta - \lambda_g(2r)}}.$$

In view of Lemma 2.3, since $r^{\lambda_g + \delta - \lambda_g(r)}$ is an increasing function of r, we have

$$\frac{\log M(r, g)}{T(r, g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)} \cdot 2^{\lambda_g + \delta} \tag{3.2}$$

Since both $\varepsilon > 0$ and $\delta > 0$ are arbitrary, we find from (3.2) that

$$\liminf_{r \to \infty} \frac{\log M(r, g)}{T(r, g)} \leq 3 \cdot 2^{\lambda_g}. \tag{3.3}$$

Next consider the following more specified cases.

Case I. $p > 1$ and $q = l > 2$.

We find from the third part of Lemma 2.2 that, for all sufficiently large values of $r \in \mathbb{R}^+$,

$$\log^{\left[\frac{n}{2}(p-1)+1\right]} M(r, f_n) \leqslant (\rho_f(p, q) + \varepsilon) \log^{[l-1]} M(r, g) + A_l(r).$$

Since $\varepsilon > 0$ is arbitrary, we thus obtain that

$$\liminf_{r \to \infty} \frac{\log^{\left[\frac{n}{2}(p-1)+1\right]} M(r, f_n)}{\log^{[l-1]} M(r, g)} \leqslant \rho_f(p, q) \liminf_{r \to \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r, g)}. \tag{3.4}$$

Case II. $p > 1$, $q > l > 2$ and $q - l = p - 1$.

We obtain from the eighth part of Lemma 2.2 that, for all sufficiently large values of $r \in \mathbb{R}^+$,

$$\log^{[p]} M(r, f_n) \leqslant (\rho_f(p, q) + \varepsilon)^{\frac{n}{2}} \log^{[l-1]} M(r, g) + A_l(r).$$

Since $\varepsilon > 0$ is arbitrary, we thus find that

$$\liminf_{r \to \infty} \frac{\log^{[p]} M(r, f_n)}{\log^{[l-1]} M(r, g)} \leqslant [\rho_f(p, q)]^{\frac{n}{2}} \liminf_{r \to \infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r, g)}. \tag{3.5}$$

Case III. $p > 1$, $q > l$ and $q - l < p - 1$.

We find from the ninth part of Lemma 2.2 that, for all sufficiently large values of $r \in \mathbb{R}^+$,

$$\log^{\left[p+\frac{n-2}{2}(p+l-1-q)\right]} M(r, f_n) \leqslant (\rho_f(p,q) + \varepsilon) \log^{[l-1]} M(r, g) + A_l(r).$$

Since $\varepsilon > 0$ is arbitrary, we thus have that

(3.6)
$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p+l-1-q)\right]} M(r, f_n)}{\log^{[l-1]} M(r,g)} \leqslant \rho_f(p,q) \liminf_{r\to\infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r,g)}.$$

Case IV. $q < l - 1$ and $1 < p$.

We find from the first part of Lemma 2.2 that, for all sufficiently large values of $r \in \mathbb{R}^+$,

$$\log^{[l-q-1]} \log^{\left[\frac{n-2}{2}(l-q)+\frac{n}{2}(p-1)+1\right]} M(r, f_n)$$
$$\leqslant \log^{[l-q-1]} \left\{ (\rho_f(p,q) + \varepsilon) \log^{[q]} M(r, g) + A_l(r) \right\}.$$

That is,

$$\log^{\left[\frac{n}{2}(l-q)+\frac{n}{2}(p-1)\right]} M(r, f_n) \leqslant \log^{[l-1]} M(r, g) + A_l(r).$$

So we have

$$\frac{\log^{\left[\frac{n}{2}(l-q)+\frac{n}{2}(p-1)\right]} M(r, f_n)}{\log^{[l-1]} M(r,g)} \leqslant \frac{\log^{[l-1]} M(r, g) + A_l(r)}{\log^{[l-2]} T(r,g)}.$$

We thus have

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{n}{2}(l-q)+\frac{n}{2}(p-1)\right]} M(r, f_n)}{\log^{[l-1]} M(r,g)} \leqslant \liminf_{r\to\infty} \frac{\log^{[l-1]} M(r, g)}{\log^{[l-2]} T(r,g)}. \tag{3.7}$$

Case V. $p = q = 1$, $l - 1 > q$ and $n > 2$.

We find from the second part of Lemma 2.2 that, for all sufficiently large values of $r \in \mathbb{R}^+$,

$$\log^{[l-2]} \log^{\left[\frac{n-2}{2}(l-1)+1\right]} M(r, f_n)$$
$$\leqslant \log^{[l-2]} \left\{ (\rho_f(1,1) + \varepsilon) \left(\rho_g^{[l]} + \varepsilon\right) \log M(r, g) \right\}.$$

That is,

$$\log^{\left[\frac{n}{2}(l-1)\right]} M(r, f_n) \leqslant \log^{[l-1]} M(r, g) + A_l(r).$$

So we have

$$\frac{\log^{\left[\frac{n}{2}(l-1)\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \frac{\log^{[l-1]} M\left(r, g\right)+A_l\left(r\right)}{\log^{[l-2]} T(r,g)}.$$

We thus have

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(l-1)\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \liminf_{r\to\infty}\frac{\log^{[l-1]} M\left(r, g\right)}{\log^{[l-2]} T(r,g)}. \tag{3.8}$$

Case VI. $p = l$ and $l-1 > q = 1$.

We find from the fifth part of Lemma 2.2 that, for all sufficiently large values of $r \in \mathbb{R}^+$,

$$\log^{[l-2]}\log^{[(n-1)(l-1)+1]} M\left(r, f_n\right)$$
$$\leqslant \log^{[l-2]}\left\{\left(\rho_f\left(p,q\right)+\varepsilon\right)\log M\left(r,g\right)+A_l\left(r\right)\right\}.$$

That is,

$$\log^{[n(l-1)]} M\left(r, f_n\right) \leqslant \log^{[l-1]} M\left(r, g\right)+A_l\left(r\right).$$

So we have

$$\frac{\log^{[n(l-1)]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \frac{\log^{[l-1]} M\left(r, g\right)+A_l\left(r\right)}{\log^{[l-2]} T(r,g)}.$$

We thus have

$$\liminf_{r\to\infty}\frac{\log^{[n(l-1)]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \liminf_{r\to\infty}\frac{\log^{[l-1]} M\left(r, g\right)}{\log^{[l-2]} T(r,g)}. \tag{3.9}$$

Now it follows from (3.4) and (3.1) that

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(p-1)+1\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \rho_f(p,q).2^{\lambda_g^{[l]}}.$$

This proves the first part of the theorem.

For $l = 2$, in view of (3.3) and (3.4), we obtain

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(p-1)+1\right]} M\left(r, f_n\right)}{\log M(r,g)} \leqslant 3.\rho_f(p,q).2^{\lambda_g}.$$

Thus the second part of the theorem follows.

We find from (3.5) and (3.1) that

$$\liminf_{r\to\infty}\frac{\log^{[p]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \left[\rho_f\left(p,q\right)\right]^{\frac{n}{2}}.2^{\lambda_g^{[l]}}.$$

This proves the third part of the theorem.

From (3.6) and (3.1), we have

$$\liminf_{r\to\infty}\frac{\log^{\left[p+\frac{n-2}{2}(p+l-1-q)\right]} M\left(r,f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \rho_f\left(p,q\right).2^{\lambda_g^{[l]}}.$$

This proves the fourth part of the theorem.

For $l=2$, in view of (3.3) and (3.6), we have

$$\liminf_{r\to\infty}\frac{\log^{\left[p+\frac{n-2}{2}(p+1-q)\right]} M\left(r,f_n\right)}{\log M(r,g)} \leqslant 3.\rho_f\left(p,q\right).2^{\lambda_g}.$$

Thus the fifth part of the theorem follows.

From (3.7) and (3.1), it follows that

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(l-q)+\frac{n}{2}(p-1)\right]} M\left(r,f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant 2^{\lambda_g^{[l]}}.$$

Thus the sixth part of the theorem is established.

We find from (3.8) and (3.1) that

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(l-1)\right]} M\left(r,f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant 2^{\lambda_g^{[l]}}.$$

Thus the seventh part of the theorem follows.

From (3.9) and (3.1), we obtain

$$\liminf_{r\to\infty}\frac{\log^{[n(l-1)]} M\left(r,f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant 2^{\lambda_g^{[l]}}.$$

Thus the eighth part of the theorem is established. □

Corollary 3.2. *Under the same conditions of* Theorem 3.1 *with* $l=2$*, we have*

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(p-1)+2\right]} M\left(r,f_n\right)}{\log^{[3]} M(r,g)} \leqslant 1 \quad (p>1,\ q=2)$$

and

$$\liminf_{r\to\infty}\frac{\log^{\left[p+\frac{n-2}{2}(p+1-q)+1\right]} M\left(r,f_n\right)}{\log^{[q+1]} M(r,g)} \leqslant 1 \quad (p>1,\ q>2,\ q-1<p).$$

Proof. We find from (3.2) that, for a sequence of $r \in \mathbb{R}^+$ tending to infinity,

$$\log M(r,g) \leq \left\{ \frac{3(1+\varepsilon)}{(1-\varepsilon)} \cdot 2^{\lambda_g+\delta} \right\} \cdot T(r,g).$$

That is,

$$\log^{[q+1]} M(r,g) \leq \log^{[q]} T(r,g) + A_l(r). \tag{3.10}$$

Then we consider the following cases.

Case I. $p > 1$ and $q = l$.

It follows from the third part of Lemma 2.2 that, for all sufficiently large values of $r \in \mathbb{R}^+$,

$$\log^{\left[\frac{n}{2}(p-1)+1\right]} M(r,f_n) \leqslant (\rho_f(p,q)+\varepsilon) \log^{[q]} M(r,g) + A_l(r).$$

That is,

$$\log^{\left[\frac{n}{2}(p-1)+2\right]} M(r,f_n) \leqslant \log^{[3]} M(r,g) + A_l(r). \tag{3.11}$$

Now combining (3.10) and (3.11), we find that, for a sequence of $r \in \mathbb{R}^+$ tending to infinity,

$$\log^{\left[\frac{n}{2}(p-1)+2\right]} M(r,f_n) \leqslant \log^{[2]} T(r,g) + A_l(r).$$

That is,

$$\frac{\log^{\left[\frac{n}{2}(p-1)+2\right]} M(r,f_n)}{\log^{[3]} M(r,g)} \leq 1 + \frac{A_l(r)}{\log^{[2]} T(r,g)}.$$

We thus have

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{n}{2}(p-1)+3\right]} M(r,f_n)}{\log^{[3]} M(r,g)} \leq 1.$$

Hence the first part of the corollary follows.

Case II. $p > 1$, $q > l$ and $q - l < p - 1$.

From the ninth part of Lemma 2.2, we find that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\log^{\left[p+\frac{n-2}{2}(p+1-q)+1\right]} M(r,f_n) \leqslant \log^{[q+1]} M(r,g) + A_l(r). \tag{3.12}$$

Then, combining (3.10) and (3.12), we find that, for a sequence of $r \in \mathbb{R}^+$ tending to infinity,

$$\log^{\left[p+\frac{n-2}{2}(p+1-q)+1\right]} M(r,f_n) \leqslant \log^{[q]} T(r,g) + A_l(r).$$

That is,

$$\frac{\log^{\left[p+\frac{n-2}{2}(p+1-q)+1\right]} M\left(r, f_n\right)}{\log^{[q+1]} M\left(r, g\right)} \leq 1+\frac{A_l\left(r\right)}{\log^{[q]} T\left(r, g\right)}.$$

We thus have

$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p+1-q)+1\right]} T\left(r, f_n\right)}{\log^{[q+1]} M\left(r, g\right)} \leq 1.$$

Hence the second part of the corollary follows. □

In parallel with Theorem 3.1, we state the following theorem without proof.

Theorem 3.3. *Let f and g be any two entire functions such that $\rho_g(a,b)$ and $\rho_f^{[l]}$ are both finite where a, b, $l \in \mathbb{N}$ with $a \geq b$ and $l \geq 2$. Let $n \in \mathbb{N} \setminus \{1\}$ be odd. Then the following inequalities hold true:*

(i) $(a>1,\, b<l-1)$

$$\liminf_{r\to\infty} \frac{\log^{\left[l+\frac{n-1}{2}(a-1)+\frac{n-2}{2}((l-b)-2)+1\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r, f)} \leqslant 2^{\lambda_f^{[l]}};$$

(ii) $(a=1,\, b<l-1)$

$$\liminf_{r\to\infty} \frac{\log^{\left[l+\frac{n-2}{2}((l-b)-2)+1\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r, f)} \leqslant 2^{\lambda_f^{[l]}};$$

(iii) $(l=a>1,\, l-1>b=1)$

$$\liminf_{r\to\infty} \frac{\log^{[n(l-1)-b+1]} M\left(r, f_n\right)}{\log^{[l-1]} M(r, f)} \leqslant 2^{\lambda_f^{[l]}};$$

(iv) $(l=b>2,\, a>1)$

$$\liminf_{r\to\infty} \frac{\log^{\left[l+\frac{n-1}{2}(a-1)\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r, f)} \leqslant \rho_g(a,b)\cdot 2^{\lambda_f^{[l]}};$$

(v) $(a=b=l=2)$

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{n-1}{2}+2\right]} M\left(r, f_n\right)}{\log M(r, f)} \leqslant 3\,\rho_g(2,2)\cdot 2^{\lambda_f};$$

(vi) $(a>1,\, b-l=a-1)$

$$\liminf_{r\to\infty} \frac{\log^{[l+a-1]} M\left(r, f_n\right)}{\log^{[l-1]} M(r, f)} \leqslant \left[\rho_g(a,b)\right]^{\frac{n-1}{2}}\cdot 2^{\lambda_f^{[l]}};$$

(vii) $(a > 1,\, b > l > 2,\, b - l < a - 1)$

$$\liminf_{r\to\infty} \frac{\log^{\left[l+a+\frac{n-3}{2}(a-b)-1\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,f)} \leqslant \rho_g(a,b)\cdot 2^{\lambda_f^{[l]}};$$

(viii) $(a > 1,\, b > l = 2,\, b - l < a - 1)$

$$\liminf_{r\to\infty} \frac{\log^{\left[a+\frac{n-3}{2}(a-b)+1\right]} M\left(r, f_n\right)}{\log M(r,f)} \leqslant 3\,\rho_g(a,b)\cdot 2^{\lambda_f}.$$

Corollary 3.4. *Under the same conditions of* Theorem 3.3 *with* $l =$ 2, *the following inequalities hold true:*

(i) $(a = b = l)$

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{n-1}{2}+2\right]} M\left(r, f_n\right)}{\log M(r,f)} \leqslant 1;$$

(ii) $(a > 1,\, b > l,\, b - l < a - 1)$

$$\liminf_{r\to\infty} \frac{\log^{\left[a+\frac{n-3}{2}(a-b)+2\right]} M\left(r, f_n\right)}{\log^{[b+1]} M(r,f)} \leqslant 1.$$

Proof. A similar argument as in the proof of Corollary 3.2 will establish the results here. So the details of proof are omitted. □

Theorem 3.5. *Let* f *and* g *be any two entire functions such that* $\rho_f(p,q)$ *and* $\rho_g^{[l]}$ *are both finite where* $p,\, q,\, l \in \mathbb{N}$ *with* $p \geq q$ *and* $l \geq 2$. *For any even* $n \in \mathbb{N}$, *the following inequalities hold true:*

(i) $(p > 1,\, q = l > 2)$

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{n}{2}(p-1)+1\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \rho_f(p,q)\cdot 2^{\rho_g^{[l]}};$$

(ii) $(p > 1,\, q = l = 2)$

$$\liminf_{r\to\infty} \frac{\log^{\left[\frac{n}{2}(p-1)+1\right]} M\left(r, f_n\right)}{\log M(r,g)} \leqslant 3\,\rho_f(p,q)\cdot 2^{\rho_g};$$

(iii) $(p > 1,\, q > l > 2,\, q - l = p - 1)$

$$\liminf_{r\to\infty} \frac{\log^{[p]} M\left(r, f_n\right)}{\log^{[l-1]} M(r,g)} \leqslant \left[\rho_f\left(p,q\right)\right]^{\frac{n}{2}}\cdot 2^{\rho_g^{[l]}};$$

(iv) $(p > 1,\ q > l > 2,\ q - l < p - 1)$

$$\liminf_{r\to\infty}\frac{\log^{\left[p+\frac{n-2}{2}(p+l-1-q)\right]} M(r, f_n)}{\log^{[l-1]} M(r,g)} \leqslant \rho_f(p,q)\cdot 2^{\rho_g^{[l]}};$$

(v) $(p > 1,\ q > l = 2,\ q - l < p - 1)$

$$(v)\ \liminf_{r\to\infty}\frac{\log^{\left[p+\frac{n-2}{2}(p+1-q)\right]} M(r, f_n)}{\log M(r,g)} \leqslant 3\,\rho_f(p,q)\cdot 2^{\rho_g};$$

(vi) $(q < l - 1,\ 1 < p)$

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(l-q)+\frac{n}{2}(p-1)\right]} M(r, f_n)}{\log^{[l-1]} M(r,g)} \leqslant 2^{\rho_g^{[l]}};$$

(vii) $(p = q = 1,\ l - 1 > q,\ n > 2)$

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(l-1)\right]} M(r, f_n)}{\log^{[l-1]} M(r,g)} \leqslant 2^{\rho_g^{[l]}};$$

(viii) $(p = l,\ l - 1 > q = 1)$

$$\liminf_{r\to\infty}\frac{\log^{[n(l-1)]} M(r, f_n)}{\log^{[l-1]} M(r,g)} \leqslant 2^{\rho_g^{[l]}}.$$

Proof. **Case I.** $l > 2$.
Since

$$\limsup_{r\to\infty}\frac{\log^{[l-2]} T(r,f)}{r^{\rho_f^{[l]}(r)}} = 1,$$

for given ε $(0 < \varepsilon < 1)$, we find that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\log^{[l-2]} T(r,g) < (1+\varepsilon) r^{\rho_g^{[l]}(r)},$$

and for a sequence of $r \in \mathbb{R}^+$ tending to infinity,

$$\log^{[l-2]} T(r,g) > (1-\varepsilon) r^{\rho_g^{[l]}(r)}.$$

Since $\log M(r,g) \le 3T(2r,g)$, for a sequence of $r \in \mathbb{R}^+$ tending to infinity, we see that, for any $\delta > 0$,

$$\begin{aligned}\frac{\log^{[l-1]} M(r,g)}{\log^{[l-2]} T(r,g)} &\le \frac{\log^{[l-2]} T(2r,g) + A_l(r)}{\log^{[l-2]} T(r,g)}\\ &\le \frac{(1+\varepsilon)}{(1-\varepsilon)}\cdot\frac{(2r)^{\rho_g^{[l]}+\delta}}{(2r)^{\rho_g^{[l]}+\delta-\rho_g^{[l]}(2r)}}\cdot\frac{1}{r^{\rho_g^{[l]}(r)}} + A_l(r)\\ &\le \frac{(1+\varepsilon)}{(1-\varepsilon)}.2^{\rho_g^{[l]}+\delta} + A_l(r),\end{aligned}$$

where the fact in Lemma 2.4 is used: $r^{\rho_g^{[l]}+\delta-\rho_g^{[l]}(r)}$ is an increasing function of r.

Since $\varepsilon > 0$ and $\delta > 0$ are both arbitrary, we thus have

$$\liminf_{r\to\infty}\frac{\log^{[l-1]} M(r,g)}{\log^{[l-2]} T(r,g)} \leq 2^{\rho_g^{[l]}}. \tag{3.13}$$

Case II. $l = 2$.

Since

$$\limsup_{r\to\infty}\frac{T(r,g)}{r^{\rho_g(r)}} = 1,$$

in view of (v) of Definition 1.3, for given ε $(0 < \varepsilon < 1)$, it follows that, for all sufficiently large $r \in \mathbb{R}^+$,

$$T(r,g) < (1+\varepsilon) r^{\rho_g(r)},$$

and for a sequence of $r \in \mathbb{R}^+$ tending to infinity,

$$T(r,g) > (1-\varepsilon) r^{\rho_g(r)}.$$

Since $\log M(r,g) \leq 3T(2r,g)$, for a sequence of $r \in \mathbb{R}^+$ tending to infinity, we get that, for any $\delta > 0$,

$$\frac{\log M(r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)} \cdot \frac{(2r)^{\rho_g+\delta}}{(2r)^{\rho_g+\delta-\rho_g(2r)}} \cdot \frac{1}{r^{\rho_g(r)}} + A_l(r).$$

In view of Lemma 2.4, since $r^{\rho_g+\delta-\rho_g(r)}$ is an increasing function of r, we have

$$\frac{\log M(r,g)}{T(r,g)} \leq \frac{3(1+\varepsilon)}{(1-\varepsilon)} \cdot 2^{\rho_g+\delta} + A_l(r).$$

Since $\varepsilon > 0$ and $\delta > 0$ are both arbitrary, we thus find that

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)} \leq 3 \cdot 2^{\rho_g}. \tag{3.14}$$

Hence it follows from (3.4) and (3.13) that

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(p-1)+1\right]} M(r,f_n)}{\log^{[l-1]} M(r,g)} \leqslant \rho_f(p,q) \cdot 2^{\rho_g^{[l]}}.$$

This proves the first part of the theorem.

For $l = 2$, in view of (3.4) and (3.14), we get

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(p-1)+1\right]} M(r,f_n)}{\log M(r,g)} \leqslant 3.\rho_f(p,q) \cdot 2^{\rho_g}.$$

Thus the second part of the theorem is established.

Similarly, from (3.7) and (3.13), we get that

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n}{2}(l-q)+\frac{n}{2}(p-1)\right]}M\left(r,f_n\right)}{\log^{[l-1]}M(r,g)}\leqslant 2^{\rho_g^{[l]}}.$$

Thus the seventh part of the theorem follows.

Now, a similar argument as in the proof of Theorem 3.1, the other parts of Theorem 3.5 can be established. So details of their proofs are omitted. □

Theorem 3.6. *Let f and g be any two entire functions such that $\rho_g(a,b)$ and $\rho_f^{[l]}$ are both finite where a, b, $l\in\mathbb{N}$ with $a\geq b$ and $l\geq 2$. For any odd $n\in\mathbb{N}\setminus\{1\}$, the following inequalities hold true:*

(i) $(a>1,\ b<l-1)$

$$\liminf_{r\to\infty}\frac{\log^{\left[l+\frac{n-1}{2}(a-1)+\frac{n-2}{2}(l-b)-1\right]}M\left(r,f_n\right)}{\log^{[l-1]}M(r,f)}\leqslant 2^{\rho_f^{[l]}};$$

(ii) $(a=1,\ b<l-1)$

$$\liminf_{r\to\infty}\frac{\log^{\left[l+\frac{n-2}{2}(l-b)-1\right]}M\left(r,f_n\right)}{\log^{[l-1]}M(r,f)}\leqslant 2^{\rho_f^{[l]}};$$

(iii) $(l=a>1,\ l-1>b=1)$

$$\liminf_{r\to\infty}\frac{\log^{[n(l-1)-b+1]}M\left(r,f_n\right)}{\log^{[l-1]}M(r,f)}\leqslant 2^{\rho_f^{[l]}};$$

(iv) $(l=b>2,\ a>1)$

$$\liminf_{r\to\infty}\frac{\log^{\left[l+\frac{n-1}{2}(a-1)\right]}M\left(r,f_n\right)}{\log^{[l-1]}M(r,f)}\leqslant \rho_g(a,b)\cdot 2^{\rho_f^{[l]}};$$

(v) $(a=b=l=2)$

$$\liminf_{r\to\infty}\frac{\log^{\left[\frac{n-1}{2}+2\right]}M\left(r,f_n\right)}{\log M(r,f)}\leqslant 3\,\rho_g(2,2)\cdot 2^{\rho_f};$$

(vi) $(a>1,\ b>l,\ b-l=a-1)$

$$\liminf_{r\to\infty}\frac{\log^{[l+a-1]}M\left(r,f_n\right)}{\log^{[l-1]}M(r,f)}\leqslant \left[\rho_g(a,b)\right]^{\frac{n-1}{2}}\cdot 2^{\rho_f^{[l]}};$$

(vii) $(a > 1,\, b > l > 2,\, b - l < a - 1)$

$$\liminf_{r\to\infty} \frac{\log^{\left[l+a+\frac{n-3}{2}(a-b)-1\right]} M\left(r, f_n\right)}{\log^{[l-1]} M(r, f)} \leqslant \rho_g(a, b) \cdot 2^{\rho_f^{[l]}} ;$$

(viii) $(a > 1,\, b > l = 2,\, b - l < a - 1)$

$$\liminf_{r\to\infty} \frac{\log^{\left[a+\frac{n-3}{2}(a-b)+1\right]} M\left(r, f_n\right)}{\log M(r, f)} \leqslant 3\, \rho_g(a, b) \cdot 2^{\rho_f} .$$

Proof. The proof here can be carried out in parallel with that of Theorem 3.5. So its detailed account is omitted. □

The following two theorems are stated without proofs, since the results can be established in line with those in Lemma 2.2.

Theorem 3.7. *Let f and g be any two entire functions such that $\rho_f\left(p, q\right) < \infty$ and $\rho_g\left(a, b\right) < \infty$ where $a,\ b,\ p,\ q \in \mathbb{N}$ with $a \geq b$ and $p \geq q$. For any even $n \in \mathbb{N}$, the following inequalities hold true:*

(i) $(q < a,\, b < p)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)\right]} M\left(r, f_n\right)}{\log^{[q]} M\left(r, g\right)} \leq \rho_f\left(p, q\right);$$

(ii) $(p = b \geq q,\, a > q,\, n > 2)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(a-q)\right]} M\left(r, f_n\right)}{\log^{[q]} M\left(r, g\right)} \leq \rho_f\left(p, q\right) \cdot \rho_g\left(a, b\right);$$

(iii) $(p > b,\, q = a)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p-b)\right]} M\left(r, f_n\right)}{\log^{[q]} M\left(r, g\right)} \leq \rho_f\left(p, q\right);$$

(iv) $(a = b = p = q)$

$$\limsup_{r\to\infty} \frac{\log^{[p]} M\left(r, f_n\right)}{\log^{[q]} M\left(r, g\right)} \leq \left[\rho_f\left(p, q\right)\right]^{\frac{n}{2}} \left[\rho_g\left(a, b\right)\right]^{\frac{n-2}{2}} ;$$

(v) $(p = a > q = b)$

$$\limsup_{r\to\infty} \frac{\log^{[p+(n-2)(p-q)]} M\left(r, f_n\right)}{\log^{[q]} M\left(r, g\right)} \leq \rho_f\left(p, q\right);$$

(vi) $(p < b,\ q < a,\ b - p = a - q)$

$$\limsup_{r\to\infty}\frac{\log^{[p+a-q]} M(r, f_n)}{\log^{[q]} M(r, g)} \le [\rho_g(a,b)]^{\frac{n-2}{2}};$$

(vii) $(p < b,\ q < a,\ b - p < a - q)$

$$\limsup_{r\to\infty}\frac{\log^{\left[p+a+\frac{n-2}{2}(a-b)-q\right]} M(r, f_n)}{\log^{[q]} M(r, g)} \le 1;$$

(viii) $(p > b,\ q > a,\ q - a = p - b)$

$$\limsup_{r\to\infty}\frac{\log^{[p]} M(r, f_n)}{\log^{[q]} M(r, g)} \le [\rho_f(p,q)]^{\frac{n}{2}};$$

(ix) $(p > b,\ q > a,\ q - a < p - b)$

$$\limsup_{r\to\infty}\frac{\log^{\left[p+\frac{n-2}{2}(p+a-b-q)\right]} M(r, f_n)}{\log^{[q]} M(r, g)} \le \rho_f(p,q).$$

Theorem 3.8. *Let f and g be any two entire functions such that $\rho_f(p,q)$ and $\rho_g(a,b)$ are finite where $a, b, p, q \in \mathbb{N}$ with $a \ge b$ and $p \ge q$. For any odd $n \in \mathbb{N}\setminus\{1\}$, the following inequalities hold true:*

(i) $(q < a,\ b < p)$

$$\limsup_{r\to\infty}\frac{\log^{\left[p+\frac{n-1}{2}(a-q)+\frac{n-3}{2}(p-b)\right]} M(r, f_n)}{\log^{[b]} M(r, f)} \le \rho_g(a,b);$$

(ii) $(p = b \ge q,\ a > q)$

$$\limsup_{r\to\infty}\frac{\log^{\left[p+\frac{n-1}{2}(a-q)\right]} M(r, f_n)}{\log^{[b]} M(r, f)} \le \rho_g(a,b);$$

(iii) $(p > b,\ q = a)$

$$\limsup_{r\to\infty}\frac{\log^{\left[p+\frac{n-3}{2}(p-b)\right]} M(r, f_n)}{\log^{[b]} M(r, f)} \le \rho_f(p,q)\cdot\rho_g(a,b);$$

(iv) $(a = b = p = q)$

$$\limsup_{r\to\infty}\frac{\log^{[p]} M(r, f_n)}{\log^{[b]} M(r, f)} \le [\rho_f(p,q)\cdot\rho_g(a,b)]^{\frac{n-1}{2}};$$

(v) $(p = a > q = b)$

$$\limsup_{r\to\infty}\frac{\log^{[p+(n-2)(p-q)]} M(r, f_n)}{\log^{[b]} M(r, f)} \le \rho_g(p,q);$$

(vi) $(p < b,\ q < a,\ b - p = a - q)$

$$\limsup_{r\to\infty} \frac{\log^{[p+a-q]} M(r, f_n)}{\log^{[b]} M(r, f)} \leq [\rho_g(a,b)]^{\frac{n-1}{2}};$$

(vii) $(p < b,\ q < a,\ b - p < a - q)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+a+\frac{n-3}{2}(a-b)-q\right]} M(r, f_n)}{\log^{[b]} M(r, f)} \leq \rho_g(a,b);$$

(viii) $(p > b,\ q > a,\ q - a = p - b)$

$$\limsup_{r\to\infty} \frac{\log^{[p]} M(r, f_n)}{\log^{[b]} M(r, f)} \leq [\rho_f(p,q)]^{\frac{n-1}{2}};$$

(ix) $(p > b,\ q > a,\ q - a < p - b)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+\frac{n-1}{2}(p+a-b-q)\right]} M(r, f_n)}{\log^{[b]} M(r, f)} \leq 1.$$

Theorem 3.9. *Let f and g be any two entire functions such that $\rho_f(p,q)$ and $\rho_g(a,b)$ are finite where $a, b, p, q \in \mathbb{N}$ with $a \geq b$ and $p \geq q$. For any even $n \in \mathbb{N}$, the following inequalities hold true:*

(i) $(q < a,\ b < p)$

$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)\right]} M(r, f_n)}{\log^{[q]} M(r, g)} \leq \lambda_f(p,q);$$

(ii) $(p = b \geq q,\ a > q,\ n > 2)$

$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(a-q)\right]} M(r, f_n)}{\log^{[q]} M(r, g)} \leq \lambda_f(p,q) \cdot \rho_g(a,b);$$

(iii) $(p > b,\ q = a)$

$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p-b)\right]} M(r, f_n)}{\log^{[q]} M(r, g)} \leq \lambda_f(p,q);$$

(iv) $(a = b = p = q)$

$$\liminf_{r\to\infty} \frac{\log^{[p]} M(r, f_n)}{\log^{[q]} M(r, g)} \leq [\rho_f(p,q)]^{\left(\frac{n}{2}-1\right)} \cdot \lambda_f(p,q) \cdot [\rho_g(a,b)]^{\frac{n-2}{2}};$$

(v) $(p = a > q = b)$

$$\liminf_{r\to\infty} \frac{\log^{[p+(n-2)(p-q)]} M(r, f_n)}{\log^{[q]} M(r, g)} \leq \lambda_f(p,q);$$

(vi) $(p > b,\ q > a,\ q - a = p - b)$

$$\liminf_{r\to\infty} \frac{\log^{[p]} M(r, f_n)}{\log^{[q]} M(r, g)} \le \lambda_f(p, q) \cdot [\rho_f(p, q)]^{\left(\frac{n}{2}-1\right)};$$

(vii) $(p > b,\ q > a,\ q - a < p - b)$

$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p+a-b-q)\right]} M(r, f_n)}{\log^{[q]} M(r, g)} \le \lambda_f(p, q).$$

Theorem 3.10. *Let f and g be any two entire functions such that $\rho_f(p, q)$ and $\rho_g(a, b)$ are finite where $a, b, p, q \in \mathbb{N}$ with $a \ge b$ and $p \ge q$. For any odd $n \in \mathbb{N} \setminus \{1\}$, the following inequalities hold true:*

(i) $(q < a,\ b < p)$

$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-1}{2}(a-q)+\frac{n-3}{2}(p-b)\right]} M(r, f_n)}{\log^{[b]} M(r, f)} \le \lambda_g(a, b);$$

(ii) $(p = b,\ a > q)$

$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-1}{2}(a-q)\right]} M(r, f_n)}{\log^{[b]} M(r, f)} \le \lambda_g(a, b);$$

(iii) $(p > b,\ q = a)$

$$\liminf_{r\to\infty} \frac{\log^{\left[p+\frac{n-3}{2}(p-b)\right]} M(r, f_n)}{\log^{[b]} M(r, f)} \le \rho_f(p, q) \cdot \lambda_g(a, b);$$

(iv) $(a = b = p = q)$

$$\liminf_{r\to\infty} \frac{\log^{[p]} M(r, f_n)}{\log^{[b]} M(r, f)} \le [\rho_f(p, q)]^{\frac{n-1}{2}} \cdot [\rho_g(a, b)]^{\frac{n-3}{2}} \cdot \lambda_g(a, b);$$

(v) $(p = a > q = b)$

$$\liminf_{r\to\infty} \frac{\log^{[p+(n-2)(p-q)]} M(r, f_n)}{\log^{[b]} M(r, f)} \le \lambda_g(p, q);$$

(vi) $(p < b,\ q < a,\ b - p = a - q)$

$$\liminf_{r\to\infty} \frac{\log^{[p+a-q]} M(r, f_n)}{\log^{[b]} M(r, f)} \le \lambda_g(a, b) \cdot [\rho_g(a, b)]^{\frac{n-3}{2}};$$

(vii) $(p < b,\ q < a,\ b - p < a - q)$

$$\liminf_{r\to\infty} \frac{\log^{\left[p+a+\frac{n-3}{2}(a-b)-q\right]} M(r, f_n)}{\log^{[b]} M(r, f)} \le \lambda_g(a, b).$$

Proof. The proofs of Theorems 3.9 and 3.10 can be carried out in line with those of Theorems 3.7 and 3.8, respectively. So the details of their proofs are omitted. □

Theorem 3.11. *Let f and g be any two entire functions such that $\rho_g(a,b) < \lambda_f(p,q) \leq \rho_f(p,q) < \infty$ where a, b, p, $q \in \mathbb{N}$ with $a \geq b$ and $p \geq q$. For any even $n \in \mathbb{N}$, the following inequalities hold true:*

(i) $(p > b,\ q = a)$

$$\lim_{r\to\infty} \frac{\log^{\left[(p-1)+\frac{n-2}{2}(p-b)+1\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0;$$

(ii) $(a = b = p = q)$ *or* $(p > b,\ q > a,\ q - a = p - b)$

$$\lim_{r\to\infty} \frac{\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0;$$

(iii) $(p > b,\ q > a,\ q - a < p - b)$

$$\lim_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p+a-b-q)\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0;$$

(iv) $(q < a,\ b < p)$

$$\lim_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0;$$

(v) $(p = b \geq q,\ a > q,\ n > 2)$

$$\lim_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(a-q)+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0;$$

(vi) $(p = a > q = b)$

$$\lim_{r\to\infty} \frac{\log^{[p+(n-2)(p-q)+a-q-1]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0;$$

(vii) $(p < b,\ q < a,\ b - p = a - q)$

$$\lim_{r\to\infty} \frac{\log^{[p+2a-2q-1]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0;$$

(viii) $(p < b,\ q < a,\ b - p < a - q)$

$$\lim_{r\to\infty} \frac{\log^{\left[(p-q)+(a+b)+\frac{n}{2}(a-b)-(q+2)+1\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0.$$

Proof. Since $\rho_g(m,n) < \lambda_f(p,q)$, we can choose $\varepsilon > 0$ small enough that

$$\rho_g(a,b) + \varepsilon < \lambda_f(p,q) - \varepsilon. \tag{3.15}$$

Now for all sufficiently large $r \in \mathbb{R}^+$,

$$\log^{[a]} M\left(\exp^{[b-1]} r, g\right) \leqslant (\rho_g(a,b) + \varepsilon) \log^{[b]} \exp^{[b-1]} r.$$

That is,

$$\log^{[a]} M\left(\exp^{[b-1]} r, g\right) \leqslant (\rho_g(a,b) + \varepsilon) \log r.$$

We thus have

$$\log^{[a]} M\left(\exp^{[b-1]} r, g\right) \leqslant \log r^{(\rho_g(a,b)+\varepsilon)}. \tag{3.16}$$

Or, equivalently,

$$\log^{[a-1]} M\left(\exp^{[b-1]} r, g\right) \leqslant r^{(\rho_g(a,b)+\varepsilon)}. \tag{3.17}$$

For all sufficiently large $r \in \mathbb{R}^+$, we also have

$$\log^{[p]} M(\exp^{[q-1]} r, f) \geqslant (\lambda_f(p,q) - \varepsilon) \log^{[q]} \exp^{[q-1]} r.$$

That is,

$$\log^{[p]} M(\exp^{[q-1]} r, f) \geqslant (\lambda_f(p,q) - \varepsilon) \log r. \tag{3.18}$$

We thus have

$$\log^{[p]} M(\exp^{[q-1]} r, f) \geqslant \log r^{(\lambda_f(p,q)-\varepsilon)}.$$

Or, equivalently,

$$\log^{[p-1]} M(\exp^{[q-1]} r, f) \geqslant r^{(\lambda_f(p,q)-\varepsilon)}. \tag{3.19}$$

Here we consider the following nine cases which may arise:

Case I. $p > b$ and $q = a$.

We have from the third part of Lemma 2.2 that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\begin{aligned} &\log^{\left[(p-1)+\frac{n-2}{2}(p-b)+1\right]} M\left(\exp^{[b-1]} r, f_n\right) \\ &\qquad \leqslant (\rho_f\,(p,q) + \varepsilon) \log^{[a-1]} M\left(\exp^{[b-1]} r, g\right). \end{aligned} \tag{3.20}$$

Then it follows from (3.17) and (3.20) that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\log^{\left[(p-1)+\frac{n-2}{2}(p-b)+1\right]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant (\rho_f\,(p,q) + \varepsilon)\, r^{(\rho_g(a,b)+\varepsilon)}. \tag{3.21}$$

Case II. $a = b = p = q$.

We find from the fourth part of Lemma 2.2 that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\begin{aligned}(3.22)\quad &\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right) \\ &\leqslant \left(\rho_f\left(p,q\right)+\varepsilon\right)^{\frac{n}{2}}\left(\rho_g\left(a,b\right)+\varepsilon\right)^{\frac{n-2}{2}}\log^{[a-1]} M\left(\exp^{[b-1]} r, g\right).\end{aligned}$$

Then, from (3.17) and (3.22), we obtain that, for all sufficiently large $r \in \mathbb{R}^+$,

(3.23)
$$\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \left(\rho_f\left(p,q\right)+\varepsilon\right)^{\frac{n}{2}}\left(\rho_g\left(a,b\right)+\varepsilon\right)^{\frac{n-2}{2}} r^{\left(\rho_g(a,b)+\varepsilon\right)}.$$

Case III. $p > b, q > a$ and $q - a = p - b$.

In view of the eighth part of Lemma 2.2, we see that, for all sufficiently large $r \in \mathbb{R}^+$,

(3.24)
$$\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \left(\rho_f\left(p,q\right)+\varepsilon\right)^{\frac{n}{2}}\log^{[a-1]} M\left(\exp^{[b-1]} r, g\right).$$

Then, (3.17) and (3.24), we have that, for all sufficiently large $r \in \mathbb{R}^+$,

$$(3.25)\qquad \log^{[p]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \left(\rho_f\left(p,q\right)+\varepsilon\right)^{\frac{n}{2}} r^{\left(\rho_g(a,b)+\varepsilon\right)}.$$

Case IV. $p > b, q > a$ and $q - a < p - b$.

We obtain from the ninth part of Lemma 2.2 that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\begin{aligned}(3.26)\quad &\log^{\left[p\frac{n-2}{2}(p+a-b-q)\right]} M\left(\exp^{[b-1]} r, f_n\right) \\ &\leqslant \left(\rho_f\left(p,q\right)+\varepsilon\right)\log^{[a-1]} M\left(\exp^{[b-1]} r, g\right).\end{aligned}$$

Then, from (3.17) and 3.26, we have that, for all sufficiently large $r \in \mathbb{R}^+$,

(3.27)
$$\log^{\left[p+\frac{n-2}{2}(p+a-b-q)\right]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \left(\rho_f\left(p,q\right)+\varepsilon\right) r^{\left(\rho_g(a,b)+\varepsilon\right)}.$$

Further, from (3.16), it follows that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\exp^{[a-q]} \log^{[a]} M\left(\exp^{[b-1]} r, g\right) \leqslant \exp^{[a-q]} \log r^{\rho_g(a,b)+\varepsilon}.$$

That is,

$$\exp^{[a-q]}\log^{[a]}M\left(\exp^{[b-1]}r,g\right)\leqslant\exp^{[a-q-1]}r^{\rho_g(a,b)+\varepsilon}. \tag{3.28}$$

Case V. $q<a$ and $b<p$.

In view of the first part of Lemma 2.2, we find that, for all sufficiently large $r\in\mathbb{R}^+$,

$$\begin{aligned}&\log^{\left[(p-1)+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)+1\right]}M\left(\exp^{[b-1]}r,f_n\right)\\&\quad\leqslant(\rho_f(p,q)+\varepsilon)\exp^{[a-q]}\log^{[a]}M\left(\exp^{[b-1]}r,g\right).\end{aligned} \tag{3.29}$$

Then, from (3.28) and (3.29), we find that, for all sufficiently large $r\in\mathbb{R}^+$,

$$\begin{aligned}&\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)\right]}M\left(\exp^{[b-1]}r,f_n\right)\\&\quad\leqslant(\rho_f(p,q)+\varepsilon)\exp^{[a-q-1]}r^{\rho_g(a,b)+\varepsilon}.\end{aligned}$$

That is,

$$\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)+1\right]}M\left(\exp^{[b-1]}r,f_n\right)\leqslant\exp^{[a-q-2]}r^{\rho_g(a,b)+\varepsilon}.$$

We thus have

$$\begin{aligned}&\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)+a-q-1\right]}M\left(\exp^{[b-1]}r,f_n\right)\\&\quad\leqslant\log^{[a-q-2]}\exp^{[a-q-2]}r^{\rho_g(a,b)+\varepsilon}.\end{aligned}$$

Or, equivalently,

$$\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)+a-q-1\right]}M\left(\exp^{[b-1]}r,f_n\right)\leqslant r^{\rho_g(a,b)+\varepsilon}. \tag{3.30}$$

Case VI. $p=b\geq q$, $a>q$ and $n>2$.

It follows from the second part of Lemma 2.2 that, for all sufficiently large $r\in\mathbb{R}^+$,

$$\begin{aligned}&\log^{\left[p+\frac{n-2}{2}(a-q)\right]}M\left(\exp^{[b-1]}r,f_n\right)\\&\quad\leqslant(\rho_f(p,q)+\varepsilon)(\rho_g(a,b)+\varepsilon)\exp^{[a-q]}\log^{[a]}M\left(\exp^{[b-1]}r,g\right).\end{aligned} \tag{3.31}$$

Then, from (3.28) and (3.31), we have that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\log^{\left[p+\frac{n-2}{2}(a-q)\right]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \left(\rho_f(p,q)+\varepsilon\right)\left(\rho_g(a,b)+\varepsilon\right)\exp^{[a-q-1]} r^{\rho_g(a,b)+\varepsilon}.$$

That is,

$$\log^{\left[p+\frac{n-2}{2}(a-q)+1\right]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \exp^{[a-q-2]} r^{\rho_g(a,b)+\varepsilon}.$$

We thus have

$$\log^{\left[p+\frac{n-2}{2}(a-q)+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \log^{[a-q-2]}\exp^{[a-q-2]} r^{\rho_g(a,b)+\varepsilon}.$$

Or, equivalently,

$$\log^{\left[p+\frac{n-2}{2}(a-q)+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant r^{\rho_g(a,b)+\varepsilon}. \tag{3.32}$$

Case VII. $p = a > q = b$.

In view of the fifth part of Lemma 2.2, we find that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\log^{[p+(n-2)(p-q)]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \left(\rho_f(p,q)+\varepsilon\right)\exp^{[a-q]}\log^{[a]} M\left(\exp^{[b-1]} r, g\right). \tag{3.33}$$

Then, from (3.28) and (3.33), we have that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\log^{[p+(n-2)(p-q)]} M\left(\exp^{[b-1]} r, f_n\right) \leq \left(\rho_f(p,q)+\varepsilon\right)\exp^{[a-q-1]} r^{\rho_g(a,b)+\varepsilon}.$$

That is,

$$\log^{[p+(n-2)(p-q)+1]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \exp^{[a-q-2]} r^{\rho_g(a,b)+\varepsilon}.$$

We thus have

$$\log^{[p+(n-2)(p-q)+a-q-1]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \log^{[a-q-2]}\exp^{[a-q-2]} r^{\rho_g(a,b)+\varepsilon}.$$

Or, equivalently,

$$\log^{[p+(n-2)(p-q)+a-q-1]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant r^{\rho_g(a,b)+\varepsilon}. \tag{3.34}$$

Case VIII. $p < b$, $q < a$ and $b - p = a - q$.

It follows from the sixth part of Lemma 2.2 that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\begin{aligned}\log^{[p+a-q]} M\left(\exp^{[b-1]} r, f_n\right) \\ \leqslant \left(\rho_g(a,b)+\varepsilon\right)^{\frac{n-2}{2}} \exp^{[a-q]} \log^{[a]} M\left(\exp^{[b-1]} r, g\right).\end{aligned} \tag{3.35}$$

Then, from (3.28) and (3.24), we get that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\log^{[p+a-q]} M\left(\exp^{[b-1]} r, f_n\right) \le \left(\rho_g(a,b)+\varepsilon\right)^{\frac{n-2}{2}} \exp^{[a-q-1]} r^{\rho_g(a,b)+\varepsilon}.$$

That is,

$$\log^{[p+a-q+1]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \exp^{[a-q-2]} r^{\rho_g(a,b)+\varepsilon}.$$

We thus have

$$\log^{[p+a-q+a-q-1]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \log^{[a-q-2]} \exp^{[a-q-2]} r^{\rho_g(a,b)+\varepsilon}.$$

Or, equivalently,

$$\log^{[p+2a-2q-1]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant r^{\rho_g(a,b)+\varepsilon}. \tag{3.36}$$

Case IX. $p < b$, $q < a$ and $b - p < a - q$.

We find from the seventh part of Lemma 2.2 that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\begin{aligned}\log^{\left[p+a+\frac{n-2}{2}(a-b)-q\right]} M\left(\exp^{[b-1]} r, f_n\right) \\ \leqslant \exp^{[a-q]} \log^{[a]} M\left(\exp^{[b-1]} r, g\right).\end{aligned} \tag{3.37}$$

Then, from (3.28) and (3.37), we have that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\log^{\left[p+a+\frac{n-2}{2}(a-b)-q\right]} M\left(\exp^{[b-1]} r, f_n\right) \le \exp^{[a-q-1]} r^{\rho_g(a,b)+\varepsilon}.$$

That is,

$$\log^{\left[p+a+\frac{n-2}{2}(a-b)-q+1\right]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant \exp^{[a-q-2]} r^{\rho_g(a,b)+\varepsilon}.$$

We thus have

$$\begin{aligned}\log^{\left[p+a+\frac{n-2}{2}(a-b)-q+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right) \\ \leqslant \log^{[a-q-2]} \exp^{[a-q-2]} r^{\rho_g(a,b)+\varepsilon}.\end{aligned}$$

Or, equivalently,

$$\log^{\left[p+a+\frac{n-2}{2}(a-b)-q+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant r^{\rho_g(a,b)+\varepsilon}. \tag{3.38}$$

Now, combining (3.21) and (3.19), we get that, for all sufficiently large $r \in \mathbb{R}^+$,

$$\frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{\left[p+\frac{n-2}{2}(p-b)\right]} M\left(\exp^{[b-1]} r, f_n\right)} \geq \frac{r^{(\lambda_f(p,q)-\varepsilon)}}{\left(\rho_f\left(p,q\right)+\varepsilon\right) r^{(\rho_g(a,b)+\varepsilon)}}. \tag{3.39}$$

Then, in view of (3.15), it follows from (3.39) that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{\left[p+\frac{n-2}{2}(p-b)\right]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty.$$

Therefore

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{\left[p+\frac{n-2}{2}(p-b)\right]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty.$$

We thus have

$$\lim_{r\to\infty} \frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{\left[p+\frac{n-2}{2}(p-b)\right]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty.$$

That is,

$$\lim_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p-b)\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0.$$

This proves the first part of the theorem.

Similarly, combining (3.15), (3.23) of Case II and (3.19), we obtain that, for all sufficiently large values of r,

$$\frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right)} \geq \frac{r^{(\lambda_f(p,q)-\varepsilon)}}{\left(\rho_f\left(p,q\right)+\varepsilon\right)^{\frac{n}{2}} \left(\rho_g\left(a,b\right)+\varepsilon\right)^{\frac{n-2}{2}} r^{(\rho_g(a,b)+\varepsilon)}},$$

from which we find

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty.$$

We thus have

$$\lim_{r\to\infty} \frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty.$$

Or, equivalently,

$$\lim_{r\to\infty}\frac{\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0. \tag{3.40}$$

Further combining (3.25) of Case III and (3.19), in view of (3.15), we see that, for all sufficiently large values of r,

$$\frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right)} \geq \frac{r^{(\lambda_f(p,q)-\varepsilon)}}{(\rho_f\,(p,q)+\varepsilon)^{\frac{n}{2}}\, r^{(\rho_g(a,b)+\varepsilon)}},$$

from which we have

$$\liminf_{r\to\infty}\frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty.$$

We thus have

$$\lim_{r\to\infty}\frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty.$$

Or, equivalently,

$$\lim_{r\to\infty}\frac{\log^{[p]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0. \tag{3.41}$$

Hence the second part of the theorem is easily seen to follow from (3.40) and (3.41).

Again combining (3.27) of Case IV and (3.19), we get that, for all sufficiently large values of r,

$$\frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{\left[p+\frac{n-2}{2}(p+a-b-q)\right]} M\left(\exp^{[b-1]} r, f_n\right)} \geq \frac{r^{(\lambda_f(p,q)-\varepsilon)}}{(\rho_f\,(p,q)+\varepsilon)\, r^{(\rho_g(a,b)+\varepsilon)}}. \tag{3.42}$$

Therefore, in view of (3.15), we find from (3.42) that

$$\lim_{r\to\infty}\frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{\left[p+\frac{n-2}{2}(p+a-b-q)\right]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty,$$

from which we have

$$\lim_{r\to\infty}\frac{\log^{\left[p+\frac{n-2}{2}(p+a-b-q)\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0.$$

This proves the third part of the theorem.

Similarly, combining (3.30) of Case V and (3.19), in view of (3.15), we obtain that, for all sufficiently large values of r,

$$\frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right)} \geq \frac{r^{(\lambda_f(p,q)-\varepsilon)}}{r^{\rho_g(a,b)+\varepsilon}}.$$

We thus have

$$\lim_{r\to\infty} \frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty.$$

Or, equivalently,

$$\lim_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0.$$

This establishes the fourth part of the theorem.

Analogously, in view of (3.15), (3.32) of Case VI and (3.19), we find that, for all sufficiently large values of r,

$$\frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{\left[p+\frac{n-2}{2}(a-q)+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right)} \geq \frac{r^{(\lambda_f(p,q)-\varepsilon)}}{r^{\rho_g(a,b)+\varepsilon}}.$$

We thus have

$$\lim_{r\to\infty} \frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{\left[p+\frac{n-2}{2}(a-q)+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty.$$

Or, equivalently,

$$\lim_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(a-q)+a-q-1\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} = 0.$$

Hence the fifth part of the theorem is proved.

Again, combining (3.15), (3.34) of Case VII and (3.19), we obtain that, for all sufficiently large values of r,

$$\frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{[p+(n-2)(p-q)+a-q-1]} M\left(\exp^{[b-1]} r, f_n\right)} \geq \frac{r^{(\lambda_f(p,q)-\varepsilon)}}{r^{\rho_g(a,b)+\varepsilon}},$$

from which we find

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M(\exp^{[q-1]} r, f)}{\log^{[p+(n-2)(p-q)+a-q-1]} M\left(\exp^{[b-1]} r, f_n\right)} = \infty.$$

We thus have

$$\lim_{r\to\infty}\frac{\log^{[p-1]}M(\exp^{[q-1]}r,f)}{\log^{[p+(n-2)(p-q)+a-q-1]}M\left(\exp^{[b-1]}r,f_n\right)}=\infty.$$

Or, equivalently,

$$\lim_{r\to\infty}\frac{\log^{[p+(n-2)(p-q)+a-q-1]}M\left(\exp^{[b-1]}r,f_n\right)}{\log^{[p-1]}M(\exp^{[q-1]}r,f)}=0.$$

This proves the sixth part of the theorem.

Further in view of (3.15), (3.36) of Case VIII and (3.19), we get that, for all sufficiently large values of r,

$$\frac{\log^{[p-1]}M(\exp^{[q-1]}r,f)}{\log^{[p+2a-2q-1]}M\left(\exp^{[b-1]}r,f_n\right)}\geq\frac{r^{(\lambda_f(p,q)-\varepsilon)}}{r^{\rho_g(a,b)+\varepsilon}}.$$

We thus have

$$\lim_{r\to\infty}\frac{\log^{[p-1]}M(\exp^{[q-1]}r,f)}{\log^{[p+2a-2q-1]}M\left(\exp^{[b-1]}r,f_n\right)}=\infty.$$

Or, equivalently,

$$\lim_{r\to\infty}\frac{\log^{[p+2a-2q-1]}M\left(\exp^{[b-1]}r,f_n\right)}{\log^{[p-1]}M(\exp^{[q-1]}r,f)}=0.$$

This proves the seventh part of the theorem.

Similarly, combining (3.15), (3.38) of Case IX and (3.19), we find that, for all sufficiently large values of r,

$$\frac{\log^{[p-1]}M(\exp^{[q-1]}r,f)}{\log^{\left[p+a+\frac{n-2}{2}(a-b)-q+a-q-1\right]}M\left(\exp^{[b-1]}r,f_n\right)}\geq\frac{r^{(\lambda_f(p,q)-\varepsilon)}}{r^{\rho_g(a,b)+\varepsilon}}.$$

We thus have

$$\lim_{r\to\infty}\frac{\log^{[p-1]}M(\exp^{[q-1]}r,f)}{\log^{\left[p+a+\frac{n-2}{2}(a-b)-q+a-q-1\right]}M\left(\exp^{[b-1]}r,f_n\right)}=\infty.$$

Or, equivalently,

$$\lim_{r\to\infty}\frac{\log^{\left[(p-q)+(a+b)+\frac{n}{2}(a-b)-(q+1)\right]}M\left(\exp^{[b-1]}r,f_n\right)}{\log^{[p-1]}M(\exp^{[q-1]}r,f)}=0.$$

This establishes the eighth part of the theorem. □

Remark 3.12. *An account of the conditions in* Theorem 3.11 *is given.*

(i) *The condition* $\rho_g(a,b) < \lambda_f(p,q)$ *in* Theorem 3.11 *being essential can be shown by letting* $f = g = \exp z$, $p = a = n = 2$ *and* $q = b = 1$. *Then we have*

$$\rho_g(a,b) = \lambda_f(p,q) = \rho_f(p,q) = 1.$$

We also see that

$$\log M(r, \exp^{[2]} z) \geq T(r, \exp^{[2]} z) + O(1) \sim \frac{\exp(r)}{(2\pi^3 r)^{\frac{1}{2}}} + O(1) \quad (r \to \infty)$$

and

$$\log M(r,f) = \log M(r, \exp z) = r.$$

We thus have

$$\begin{aligned}\lim_{r\to\infty} \frac{\log^{[p+m-q-1]} M\left(\exp^{[n-1]} r, f \circ g\right)}{\log^{[p-1]} M(\exp^{[q-1]} r, f)} &= \lim_{r\to\infty} \frac{\log^{[2]} M(r, f\circ g)}{\log M(r,f)} \\ &\geq \lim_{r\to\infty} \frac{r - \log r + O(1)}{r} = 1 \neq 0,\end{aligned}$$

which is contrary to the results in Theorem 3.11.

(ii) *The results in* Theorem 3.11 *remain valid with use of* limit inferior *instead of* limit *and the condition* $\rho_g(a,b) < \lambda_f(p,q) \leq \rho_f(p,q) < \infty$ *replaced by* $\lambda_g(a,b) < \lambda_f(p,q) \leq \rho_f(p,q) < \infty$. *Here the condition* $\lambda_g(a,b) < \lambda_f(p,q)$ *is also essential as easily shown in the case:* $f = g = \exp z$, $p = a = n = 2$ *and* $q = b = 1$.

In line with the results in Theorem 3.11, we present the following statements without their proof.

Theorem 3.13. *Let* f *and* g *be any two entire functions such that* $\rho_f(p,q) < \lambda_g(a,b) \leq \rho_g(a,b) < \infty$ *where* a, b, p, $q \in \mathbb{N}$ *with* $a \geq b$ *and* $p \geq q$. *Then each of the following statements holds true: For any odd* $n \in \mathbb{N} \setminus \{1\}$,

(i) $(p > b,\ q = a)$

$$\lim_{r\to\infty} \frac{\log^{\left[(p-1)+\frac{n-3}{2}(p-b)+1\right]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a-1]} M(\exp^{[b-1]} r, g)} = 0;$$

(ii) $(a = b = p = q)$ *or* $(p > b,\ q > a,\ q - a = p - b)$

$$\lim_{r\to\infty} \frac{\log^{[p]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a-1]} M(\exp^{[b-1]} r, g)} = 0;$$

(iii) $(p > b,\ q > a,\ q - a < p - b)$

$$\lim_{r\to\infty} \frac{\log^{\left[p+\frac{n-1}{2}(p+a-b-q)\right]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a-1]} M(\exp^{[b-1]} r, g)} = 0;$$

(iv) $(q < a,\ b < p)$

$$\lim_{r\to\infty} \frac{\log^{\left[2p+\frac{n-1}{2}(a-q)+\frac{n-2}{2}(p-b)-b-1\right]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a-1]} M(\exp^{[b-1]} r, g)} = 0;$$

(v) $(p = b \geq q,\ a > q)$

$$\lim_{r\to\infty} \frac{\log^{\left[2p+\frac{n-1}{2}(a-q)-b-1\right]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a-1]} M(\exp^{[b-1]} r, g)} = 0;$$

(vi) $(p = a > q = b)$

$$\lim_{r\to\infty} \frac{\log^{[2p+(n-2)(p-q)-b-1]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a-1]} M(\exp^{[b-1]} r, g)} = 0;$$

(vii) $(p < b,\ q < a,\ b - p = a - q)$

$$\lim_{r\to\infty} \frac{\log^{[2p+a-q-b-1]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a-1]} M(\exp^{[b-1]} r, g)} = 0;$$

(viii) $(p < b,\ q < a,\ b - p < a - q)$

$$\lim_{r\to\infty} \frac{\log^{\left[2p+a+\frac{n-3}{2}(a-b)-q-b-1\right]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a-1]} M(\exp^{[b-1]} r, g)} = 0.$$

Remark 3.14. *The results in* Theorem 3.13 *remain valid with use of limit inferior instead of limit and the condition* $\rho_f(p, q) < \lambda_g(a, b) \leq \rho_g(a, b) < \infty$ *replaced by* $\lambda_f(p, q) < \lambda_g(a, b) \leq \rho_g(a, b) < \infty$.

Theorem 3.15. *Let f and g be any two entire functions such that $\lambda_f(p,q) \le \rho_f(p,q) < \infty$ and $\rho_g(a,b) < \infty$ where a, b, p, $q \in \mathbb{N}$ with $a \ge b$ and $p \ge q$. Then each of the following statements holds true: For any even $n \in \mathbb{N}$,*

(i) $(p > b,\ q = a)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p-b)+1\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)} = 0;$$

(ii) $(a = b = p = q)$ *or* $(p > b,\ q > a,\ q - a = p - b)$

$$\limsup_{r\to\infty} \frac{\log^{[p+1]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)};$$

(iii) $(p > b,\ q > a,\ q - a < p - b)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(p+a-b-q)+1\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)};$$

(iv) $(q < a,\ b < p)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(a-q)+\frac{n-2}{2}(p-b)+a-q\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)};$$

(v) $(p = b \ge q,\ a > q,\ n > 2)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+\frac{n-2}{2}(a-q)+a-q\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)};$$

(vi) $(p = a > q = b)$

$$\limsup_{r\to\infty} \frac{\log^{[p+(n-2)(p-q)+a-q]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)};$$

(vii) $(p < b,\ q < a,\ b - p = a - q)$

$$\limsup_{r\to\infty} \frac{\log^{[p+2a-2q]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)};$$

(viii) $(p < b,\ q < a,\ b - p < a - q)$

$$\limsup_{r\to\infty}\frac{\log^{\left[p+a+\frac{n-2}{2}(a-b)-q+a-q\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)}.$$

Proof. For (i), let $p > b$ and $q = a$. It follows from (3.21) that, for all sufficiently large values of r,

(3.43)
$$\log^{\left[p+\frac{n-2}{2}(p-b)+1\right]} M\left(\exp^{[b-1]} r, f_n\right) \leqslant (\rho_g(a,b)+\varepsilon)\log r + A_l\,(r)\,.$$

Then, combining (3.18) and (3.43), we find that, for all sufficiently large values of r,

$$\frac{\log^{\left[p+\frac{n-2}{2}(p-b)+1\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{(\rho_g(a,b)+\varepsilon)\log r + A_l\,(r)}{(\lambda_f(p,q)-\varepsilon)\log r},$$

from which we have

$$\limsup_{r\to\infty}\frac{\log^{\left[p+\frac{n-2}{2}(p-b)+1\right]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)}.$$

This proves the first part of the theorem.

For (ii), we first suppose that $a = b = p = q$. Then, from (3.18) and (3.23), we obtain that, for all sufficiently large values of r,

$$\frac{\log^{[p+1]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{(\rho_g(a,b)+\varepsilon)\log r + A_l\,(r)}{(\lambda_f(p,q)-\varepsilon)\log r},$$

from which we find

$$\limsup_{r\to\infty}\frac{\log^{[p+1]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)}. \tag{3.44}$$

We next assume that $p > b$, $q > a$ and $q - a = p - b$. Then, combining (3.18) and (3.25), we have that, for all sufficiently large values of r,

$$\frac{\log^{[p+1]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{(\rho_g(a,b)+\varepsilon)\log r + A_l\,(r)}{(\lambda_f(p,q)-\varepsilon)\log r},$$

from which we obtain

$$\limsup_{r\to\infty}\frac{\log^{[p+1]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} \le \frac{\rho_g(a,b)}{\lambda_f(p,q)}. \tag{3.45}$$

Hence the second part of the theorem is seen to follow from (3.44) and (3.45).

A similar argument as in the proof of (i) and (ii) will prove the remaining parts (iii) to (viii) by considering (3.27), (3.30), (3.32), (3.34), (3.36), and (3.38), respectively, with the help of the inequality (3.18). The detailed account of their proofs is omitted. □

Remark 3.16. *Some comments related to* Theorem 3.15 *are given:*

(i) *The condition* $\rho_g(a,b) < \infty$ *in* Theorem 3.15 *is necessary as shown in the following example: Let* $f = \exp z$, $g = \exp^{[2]} z$ *and* $p = a = n = 2$, *and* $q = b = 1$. *Then we have*

$$\lambda_f(p,q) = \rho_f(p,q) = 1 \text{ and } \rho_g(a,b) = \infty.$$

Also we see

$$\log^{[3]} M(r, f \circ g) = \log^{[3]} \exp^{[3]} r = r,$$

from which we have

$$\log^{[2]} M(r,f) = \log r.$$

We thus have

$$\limsup_{r\to\infty} \frac{\log^{[p+(n-2)(p-q)+a-q]} M\left(\exp^{[b-1]} r, f_n\right)}{\log^{[p]} M(\exp^{[q-1]} r, f)} = \limsup_{r\to\infty} \frac{\log^{[3]} M(r, f\circ g)}{\log^{[2]} M(r,f)} = \limsup_{r\to\infty} \frac{r}{\log r} = \infty.$$

(ii) *The results in* Theorem 3.15 *remain valid with use of* limit inferior *instead of* limit superior *and* $\rho_g(a,b)$ *replaced by* $\lambda_g(a,b)$. *Here the condition* $\lambda_g(a,b) < \infty$ *is required, which can be easily seen by taking* $f = \exp z$, $g = \exp^{[2]} z$, $p = a = n = 2$, *and* $q = b = 1$.

We conclude this paper by giving one more theorem.

Theorem 3.17. *Let* f *and* g *be any two entire functions such that* $\lambda_g(a,b) \le \rho_g(a,b) < \infty$ *and* $\rho_f(p,q) < \infty$ *where* $a, b, p, q \in \mathbb{N}$ *with* $a \ge b$ *and* $p \ge q$. *Then each of the following statements holds true: For any odd* $n \in \mathbb{N} \setminus \{1\}$,

(i) $(p > b,\ q = a)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+\frac{n-3}{2}(p-b)+1\right]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a]} M(\exp^{[b-1]} r, g)} \le \frac{\rho_f(p,q)}{\lambda_g(a,b)};$$

(ii) $(a = b = p = q \quad or \quad (p > b,\ q > a,\ q - a = p - b)$

$$\limsup_{r\to\infty} \frac{\log^{[p+1]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a]} M(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p,q)}{\lambda_g(a,b)};$$

(iii) $(p > b,\ q > a,\ q - a < p - b)$

$$\limsup_{r\to\infty} \frac{\log^{\left[p+\frac{n-1}{2}(p+a-b-q)+1\right]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a]} M(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p,q)}{\lambda_g(a,b)};$$

(iv) $(q < a,\ b < p)$

$$\limsup_{r\to\infty} \frac{\log^{\left[2p+\frac{n-1}{2}(a-q)+\frac{n-2}{2}(p-b)-b\right]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a]} M(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p,q)}{\lambda_g(a,b)};$$

(v) $(p = b \geq q,\ a > q)$

$$\limsup_{r\to\infty} \frac{\log^{\left[2p+\frac{n-1}{2}(a-q)-b\right]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a]} M(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p,q)}{\lambda_g(a,b)};$$

(vi) $(p = a > q = b)$

$$\limsup_{r\to\infty} \frac{\log^{[2p+(n-2)(p-q)-b]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a]} M(\exp^{[b-1]} r, g)} = \frac{\rho_f(p,q)}{\lambda_g(a,b)};$$

(vii) $(p < b,\ q < a,\ b - p = a - q)$

$$\limsup_{r\to\infty} \frac{\log^{[2p+a-q-b]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a]} M(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p,q)}{\lambda_g(a,b)};$$

(viii) $(p < b,\ q < a,\ b - p < a - q)$

$$\limsup_{r\to\infty} \frac{\log^{\left[2p+a+\frac{n-3}{2}(a-b)-q-b\right]} M\left(\exp^{[q-1]} r, f_n\right)}{\log^{[a]} M(\exp^{[b-1]} r, g)} \leq \frac{\rho_f(p,q)}{\lambda_g(a,b)}.$$

Proof. A similar argument as in the proof of previous theorems will prove the statements here. So the details of proof are omitted. □

Remark 3.18. *The results in* Theorem 3.17 *in which* limit inferior *is used in place of* limit superior *are seen to remain valid with* $\rho_f(p,q)$ *replaced by* $\lambda_f(p,q)$ *and the other conditions* unaltered.

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