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ALMOST-PRIMES REPRESENTED BY $p + a^m$

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Abstract. Let $a \geqslant 2$ be a fixed integer in this paper. By using the method of Goldston, Pintz and Yıldırım, we will prove that there are infinitely many almost-primes which can be represented as $p + a^m$ in at least two different ways.

1. Introduction

In 1934, Romanoff [9] proved that the integers of the form $p+2^m$ have a positive density. Thereafter, many works have been done involving the so-called Romanoff's constant:

$$c = \liminf_{x\to\infty} \frac{\#\{n \leqslant x : n = p + 2^m\}}{x}.$$

For example, Chen and Sun [1] proved that $c > 0.0868$, this result is improved by Habsieger and Roblot [6] to 0.0933 and by Pintz [7] to 0.09368. Their works mainly based on studying the mean values involving $r(n)$, the number of different representations of n in the form $p + 2^m$.

Prachar [8] studied a more generalized problem. He proved that if $a > 1$ and (m_j) is a strictly increasing sequence of non-negative integers, then the number of distinct integers $\leqslant x$ which can be expressed in the form $p + a^{m_j}$ is

$$\gg \frac{x}{\log x}\#\{m_j : a^{m_j} \leqslant x\}.$$

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In this paper, we take interest in almost-primes with $r(n) \geqslant 2$. It is early in 1950 that Erdös [2] proved that there are infinitely many integers satisfying

$$r(n) \gg \log\log n,$$

but his method can not be applied to attack the problem on almost-primes. The main result of this paper is the following theorem:

Theorem 1.1. *Let $a \geqslant 2$ be an fixed integer. Then there exists a positive integer R, such that there are infinitely many integers n satisfying:*
(1) n has at most R distinct prime divisors;
(2) n can be represented as $p + a^m$ in at least two different ways.

We should mention to Friedlander and Iwaniec [4] who claimed: “We believe (although we did not check all details) that the method presented here can, when combined with the Fundamental Lemma, produce infinitely many almost-prime integers which have two different representations in the form $p + a^m$. ” Therefore, what we do in this paper is just to “check the details”.

Throughout the paper, we denote ε to be a sufficiently small positive real number, and write

$$\Lambda^\flat(n) = \begin{cases} \log n, & \text{if } n \text{ is a prime,} \\ 0, & \text{otherwise.} \end{cases}$$

As usual, $\tau_k(n)$ is the divisor function and $\varphi(n)$ is the Euler's function.

2. Basic Considerations

The proof of Theorem 1.1 is based on the lower-bound sieve and the method of Goldston, Pintz and Yıldırım (see eg. [4], [5] and [10]).

Let N be a sufficiently large integer, we write

$$\mathcal{M} = \Big\{a^m : 1 \leqslant m \leqslant \frac{\log N}{2\log a}\Big\}$$

and $\mathcal{H} = \{a^m : 1 \leqslant m \leqslant k\}$ a subset of $\mathcal{M}$. Let

$$Q(X) = \prod_{1\leqslant j\leqslant k} (X - a^j),$$

and $\omega(d)$ denote the number of solutions $n \pmod d$ of $Q(n) \equiv 0 \pmod d$. Note that if $p \mid a$, then $\omega(p) = 1$; if $p \nmid a$, then $\omega(p) < p$ since $Q(0) \not\equiv 0$

(mod p). Therefore, $\omega(p) < p$ for every prime p, in another word, $\mathcal{H}$ is "admissible".

We write

$$\det \mathcal{H} = \sum_{1\leqslant i<j\leqslant k} (a^j - a^i)^2 = a^{k(k-1)} \prod_{1\leqslant j\leqslant k-1} (a^j - 1)^{2(k-j)},$$

and let Δ be the product of all prime divisors of a and all primes p for which $a^j \equiv 1 \pmod p$ with some $1 \leqslant j \leqslant k$. Then we can easily check the following three things:

(i) Since $\mathcal{H}$ is admissible, Δ is divisible by all primes $p \leqslant k+1$. In practice, we shall choose k to be an even integer, therefore $k+2$ is not a prime.

(ii) If $p \nmid \Delta$, then $\omega(p) = k$.

(iii) For any $a^m \in \mathcal{M}$, we have $\Delta \mid Q(a^m)$ since

$$Q(a^m) = \prod_{1\leqslant j\leqslant k} \left(a^m - a^j\right) = a^{k(k+1)/2} \prod_{m-k\leqslant j\leqslant m-1} \left(a^j - 1\right).$$

Now we consider the sequence (a_n) supported on the dyadic segment $(\frac{N}{2}, N]$ as well as $(Q(n), \Delta) = 1$ with

$$a_n = \Bigg(\sum_{a^m\in\mathcal{M}} \Lambda^\flat(n - a^m) - \log N\Bigg)\Bigg(\sum_{\nu\mid Q(n)} \lambda_\nu\Bigg), \tag{2.1}$$

where (λ_ν) is an upper-bound sieve supported on squarefree numbers $\nu < D = N^{\frac{1}{2}-2\varepsilon}$, $(\nu, \Delta) = 1$, whence the summation over ν is non-negative. Here we choose (λ_ν) to be the Selberg's Λ^2-sieve, that is

$$\sum_{\nu\mid n} \lambda_\nu = \Bigg(\sum_{d\mid n} \rho_d\Bigg)^2$$

where (ρ_d) is a sequence of real numbers supported on squarefree numbers d with $d < \sqrt{D}$, $(d, \Delta) = 1$ which satisfies $\rho_1 = 1$ and

$$|\rho_d| \leqslant 1 \tag{2.2}$$

for all d (see Lemma 6.1). Thus

$$\lambda_\nu = \sum_{[d_1,d_2]=\nu} \rho_{d_1}\rho_{d_2} \tag{2.3}$$

and $|\lambda_\nu| \leqslant \tau_3(\nu)$ for all ν. If we can give a proper lower bound for the number of almost-primes n such that $a_n > 0$, we will prove Theorem 1.1. Therefore, we need to apply a lower-bound sieve to n.

Let

$$T = \{N/2 < n \leqslant N : (Q(n), \Delta) = 1\},$$
$$T_1 = T \cap \Big\{n : \sum_{a^m \in \mathcal{M}} \Lambda^\flat(n - a^m) - \log N > 0\Big\},$$
$$T_2 = T \cap \Big\{n : \sum_{a^m \in \mathcal{M}} \Lambda^\flat(n - a^m) - \log N \leqslant 0\Big\},$$

and $\mathcal{A} = (a_n)_{n \in T_1}$. We choose the sifting set $\mathcal{P} = \{p \geqslant k+2 : p \nmid a\}$ since it is easy to deduce $(n, a) = 1$ from $(Q(n), \Delta) = 1$, and as usual, denote

$$P(z) = \prod_{\substack{p<z \\ p \in \mathcal{P}}} p.$$

Let (λ'_d) be a lower-bound sieve of level $D' = N^\varepsilon$, then the sifting function

$$\begin{aligned} S(\mathcal{A}, \mathcal{P}, z) &= \sum_{\substack{n \in T_1 \\ (n, P(z))=1}} a_n \geqslant \sum_{n \in T_1} a_n \sum_{d | (n, P(z))} \lambda'_d \\ &= \sum_{n \in T} a_n \sum_{d | (n, P(z))} \lambda'_d - \sum_{n \in T_2} a_n \sum_{d | (n, P(z))} \lambda'_d = S_1 - S_2 \end{aligned} \tag{2.4}$$

say. If we can produce a positive lower bound of $S(\mathcal{A}, \mathcal{P}, z)$ for $z = D'^{\frac{1}{s}}$, we will deduce that there are infinitely many integers n which have at most $s\varepsilon^{-1} + k + 2$ distinct prime factors and satisfy $a_n > 0$.

Now we give a careful look at S_2, we write

$$T_{21} = \{n \in T_2 : n - a^m \text{ is not a prime for any } a^m \in \mathcal{M}\},$$
$$T_{22} = T_2 \setminus T_{21} = \{n \in T_2 : \exists\ a^m \in \mathcal{M}, \ \text{ such that } n - a^m \text{ is a prime}\}.$$

Then,

$$-\sum_{n\in T_{21}} a_n \sum_{d|(n,P(z))} \lambda'_d = (\log N)\sum_{n\in T_{21}}\Big(\sum_{\nu|Q(n)}\lambda_\nu\Big)\Big(\sum_{d|(n,P(z))}\lambda'_d\Big)$$

$$= (\log N)\sum_{\substack{N/2<n\leqslant N\\(Q(n),\Delta)=1}}\Big(\sum_{\nu|Q(n)}\lambda_\nu\Big)\Big(\sum_{d|(n,P(z))}\lambda'_d\Big)$$

$$-(\log N)\sum_{\substack{N/2<n\leqslant N\\(Q(n),\Delta)=1\\ \exists\ a^m\in\mathcal{M},\ \text{ s.t. } n-a^m \text{ is prime}}}\Big(\sum_{\nu|Q(n)}\lambda_\nu\Big)\Big(\sum_{d|(n,P(z))}\lambda'_d\Big)$$

$$\geqslant (\log N)\sum_{\substack{N/2<n\leqslant N\\(Q(n),\Delta)=1}}\Big(\sum_{\nu|Q(n)}\lambda_\nu\Big)\Big(\sum_{d|(n,P(z))}\lambda'_d\Big) - (\log N)\sum_{\substack{N/2<n\leqslant N\\(Q(n),\Delta)=1\\(n,P(z))=1}}\Big(\sum_{\nu|Q(n)}\lambda_\nu\Big).$$

Noticing that

$$S_1 = \sum_{\substack{N/2<n\leqslant N\\(Q(n),\Delta)=1}}\sum_{a^m\in\mathcal{M}}\Lambda^\flat(n-a^m)\Big(\sum_{\nu|Q(n)}\lambda_\nu\Big)\Big(\sum_{d|(n,P(z))}\lambda'_d\Big)$$
$$-(\log N)\sum_{\substack{N/2<n\leqslant N\\(Q(n),\Delta)=1}}\Big(\sum_{\nu|Q(n)}\lambda_\nu\Big)\Big(\sum_{d|(n,P(z))}\lambda'_d\Big),$$

we finally get from (2.4) that

(2.5)
$$S(\mathcal{A},\mathcal{P},z) \geqslant \sum_{\substack{N/2<n\leqslant N\\(Q(n),\Delta)=1}}\sum_{a^m\in\mathcal{M}}\Lambda^\flat(n-a^m)\Big(\sum_{\nu|Q(n)}\lambda_\nu\Big)\Big(\sum_{d|(n,P(z))}\lambda'_d\Big)$$
$$-(\log N)\sum_{\substack{N/2<n\leqslant N\\(Q(n),\Delta)=1\\(n,P(z))=1}}\Big(\sum_{\nu|Q(n)}\lambda_\nu\Big) - \sum_{n\in T_{22}} a_n\sum_{d|(n,P(z))}\lambda'_d$$
$$= S_3 - S_4 - S_5$$

say.

Before doing further calculations, we should study the reduced composition of sieve-twisted sums.

3. Reduced Composition of Sieves

Let (λ_d) be a finite sequence supported on squarefree numbers and write

$$\theta_n = \sum_{d|n} \lambda_d.$$

For $g(d)$ a multiplicative function supported on finite set of squarefree numbers with $0 \leqslant g(p) < 1$, we denote $h(d)$ the multiplicative function supported on squarefree numbers with

$$h(p) = \frac{g(p)}{1 - g(p)}.$$

We call g a density function and h the relative density function of g. Now we consider the sieve-twisted sum

$$G = \sum_{d} \lambda_d g(d).$$

Lemma 3.1. *It holds that*

$$G = VG^*, \tag{3.1}$$

where

$$V = \prod_{p} (1 - g(p)) \qquad \text{and} \qquad G^* = \sum_{d} \theta_d h(d). \tag{3.2}$$

Proof. This is Lemma A.1 of [3]. □

Next, we consider the reduced composition of two sieve-twisted sums of the following type:

$$G' * G'' = \sum\sum_{(d_1,d_2)=1} \lambda'_{d_1} \lambda''_{d_2} g'(d_1) g''(d_2). \tag{3.3}$$

We have

Lemma 3.2.

$$G' * G'' = \sum\sum_{(b_1,b_2)=1} \theta'_{b_1}\theta''_{b_2} g'(b_1)g''(b_2) \prod_{p\nmid b_1 b_2} (1 - g'(p) - g''(p)). \tag{3.4}$$

Proof. This is Lemma A.2 of [3]. □

Now assume that (λ') is an upper-bound sieve (either from the beta-sieve or from the Selberg's sieve), (λ'') is a beta-sieve of level D'', while g'' is supported on the divisors of $P(z'') = \prod_{p<z''} p$ for some $z'' \leqslant D''$ and satisfying

$$\prod_{w\leqslant p<w'} (1 - g(p))^{-1} \leqslant \Big(\frac{\log w'}{\log w}\Big)^{\kappa} \Big(1 + O\Big(\frac{1}{\log w}\Big)\Big) \tag{3.5}$$

for some $\kappa > 0$ and any $0 < w < w'$. If we denote by $h^{(1)}(d)$ and $h^{(2)}(d)$ the multiplicative functions supported on squarefree numbers with

$$h^{(1)}(p) = \frac{g'(p)}{1 - g'(p) - g''(p)} \qquad \text{and} \qquad h^{(2)}(p) = \frac{g''(p)}{1 - g'(p) - g''(p)},$$

then we get (at primes)

$$g^{(1)} = \frac{h^{(1)}}{1 + h^{(1)}} = \frac{g'}{1 - g''} \qquad \text{and} \qquad g^{(2)} = \frac{h^{(2)}}{1 + h^{(2)}} = \frac{g''}{1 - g'} \tag{3.6}$$

respectively. Thus Lemma 3.2 indicates

$$\begin{aligned} G' * G'' &= \prod_p (1 - g'(p) - g''(p)) \sum\sum_{(b_1,b_2)=1} \theta'_{b_1}\theta''_{b_2} h^{(1)}(b_1)h^{(2)}(b_2) \\ &= \prod_p (1 - g'(p) - g''(p)) \sum_{b_1} \theta'_{b_1} h^{(1)}(b_1) \sum_{(b_2,b_1)=1} \theta''_{b_2} h^{(2)}(b_2). \end{aligned} \tag{3.7}$$

From Lemma 3.1 and the Fundamental Lemma of the sieve we know that

$$\sum_{(b_2,b_1)=1} \theta''_{b_2} h^{(2)}(b_2) = \prod_{p\nmid b_1} (1 - g^{(2)}(p))^{-1} \sum_{(d,b_1)=1} \lambda''_d g^{(2)}(d) = 1 + O(e^{-s''}),$$

provided that $s'' = \log D''/\log z''$ is sufficiently large. Inserting this into (3.7) and noticing that $\theta'_{b_1} \geqslant 0$, we obtain

$$\begin{aligned} G' * G'' &= (1 + O(e^{-s''})) \prod_p (1 - g'(p) - g''(p)) \Big(\sum_{b_1} \theta'_{b_1} h^{(1)}(b_1) \Big) \\ &= (1 + O(e^{-s''})) \prod_p (1 - g'(p) - g''(p))(1 - g^{(1)})^{-1} \Big(\sum_d \lambda'(d) g^{(1)}(d) \Big) \\ &= (1 + O(e^{-s''})) \prod_p (1 - g''(p)) \Big(\sum_d \lambda'(d) g^{(1)}(d) \Big). \end{aligned}$$

Therefore, we conclude:

PROPOSITION 3.3. *Suppose that (λ') is an upper-bound sieve, (λ'') is a beta-sieve of level D''. Let g'' be a density function supported on the divisors of $P(z'')$ for some $z'' \leqslant D''$. Then*

$$G' * G'' = (1 + O(e^{-s''})) V'' G^{(1)} \tag{3.8}$$

provided that $s'' = \log D''/\log z''$ is sufficiently large, where

$$V'' = \prod_p (1 - g''(p)), \qquad G^{(1)} = \sum_d \lambda'(d) g^{(1)}(d)$$

with $g^{(1)}$ defined in (3.6).

4. Estimation of S_5

From (2.5) we know that

$$\begin{aligned}
|S_5| &= \Bigg| \sum_{n\in T_{22}} \Big(\sum_{a^m\in\mathcal{M}} \Lambda^\flat(n-a^m) - \log N \Big) \Big(\sum_{\nu|Q(n)} \lambda_\nu \Big) \Big(\sum_{d|(n,P(z))} \lambda'_d \Big) \Bigg| \\
&\leqslant \sum_{n\in T_{22}} \log \frac{N}{\frac{N}{2}-\sqrt{N}} \Big(\sum_{\nu|Q(n)} \lambda_\nu \Big) \Big| \sum_{d|(n,P(z))} \lambda'_d \Big| \\
&\leqslant \Big(\log 2 + O\Big(\frac{1}{\sqrt{N}}\Big)\Big) \sum_{\substack{N/2<n\leqslant N\\ (Q(n),\Delta)=1}} \Big(\sum_{\nu|Q(n)} \lambda_\nu \Big) \Big| \sum_{d|(n,P(z))} \lambda'_d \Big| \\
&= \Big(\log 2 + O\Big(\frac{1}{\sqrt{N}}\Big)\Big) \Bigg[\sum_{\substack{N/2<n\leqslant N\\ (Q(n),\Delta)=1\\ (n,P(z))=1}} \sum_{\nu|Q(n)} \lambda_\nu \\
&\qquad\qquad - \sum_{\substack{N/2<n\leqslant N\\ (Q(n),\Delta)=1\\ (n,P(z))>1}} \Big(\sum_{\nu|Q(n)} \lambda_\nu \Big) \Big(\sum_{d|(n,P(z))} \lambda'_d \Big) \Bigg] \\
&= \Big(\log 2 + O\Big(\frac{1}{\sqrt{N}}\Big)\Big) \Bigg[2 \sum_{\substack{N/2<n\leqslant N\\ (Q(n),\Delta)=1\\ (n,P(z))=1}} \sum_{\nu|Q(n)} \lambda_\nu \\
&\qquad\qquad - \sum_{\substack{N/2<n\leqslant N\\ (Q(n),\Delta)=1}} \Big(\sum_{\nu|Q(n)} \lambda_\nu \Big) \Big(\sum_{d|(n,P(z))} \lambda'_d \Big) \Bigg] \\
&= \Big(\log 2 + O\Big(\frac{1}{\sqrt{N}}\Big)\Big) (2S_{51} - S_{52})
\end{aligned}$$

say. In order to estimate S_{51}, we introduce an upper-bound beta-sieve (λ'') of level D'. Then

$$S_{51} \leqslant \sum_{\substack{N/2<n\leqslant N\\(Q(n),\Delta)=1}} \Big(\sum_{\nu|Q(n)} \lambda_\nu\Big)\Big(\sum_{d|(n,P(z))} \lambda''_d\Big)$$
$$= \sum_{d|P(z)} \lambda''_d \sum_{(\nu,\Delta d)=1} \lambda_\nu \sum_{\substack{N/2<n\leqslant N\\(Q(n),\Delta)=1\\Q(n)\equiv 0\,(\mathrm{mod}\ \nu)\\n\equiv 0\,(\mathrm{mod}\ d)}} 1.$$

Notice that the condition $(\nu, d) = 1$ is automatical since $d \mid n$, $\nu \mid Q(n)$ and $(d, a) = 1$. The innermost sum can be represented as

$$\sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\(Q(\alpha),\Delta)=1\\\alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))}} \sum_{\substack{\beta\,(\mathrm{mod}\ \nu)\\Q(\beta)\equiv 0\,(\mathrm{mod}\ \nu)}} \sum_{\substack{N/2<n\leqslant N\\n\equiv\alpha\,(\mathrm{mod}\ \Delta)\\n\equiv\beta\,(\mathrm{mod}\ \nu)\\n\equiv 0\,(\mathrm{mod}\ d)}} 1$$
$$= \sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\(Q(\alpha),\Delta)=1\\\alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))}} \sum_{\substack{\beta\,(\mathrm{mod}\ \nu)\\Q(\beta)\equiv 0\,(\mathrm{mod}\ \nu)}} \left(\frac{N/2}{\nu[\Delta,d]} + O(1)\right),$$

where

(4.1)
$$\sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\(Q(\alpha),\Delta)=1\\\alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))}} 1 = \sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\\alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))}} \sum_{\delta|(Q(\alpha),\Delta)} \mu(\delta) = \sum_{\substack{\delta|\Delta\\(\delta,d)=1}} \mu(\delta) \sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\\alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))\\Q(\alpha)\equiv 0\,(\mathrm{mod}\ \delta)}} 1$$
$$= \sum_{\substack{\delta|\Delta\\(\delta,d)=1}} \mu(\delta)\cdot\frac{\Delta}{(\Delta,d)\delta}\,\omega(\delta) = \frac{\Delta}{(\Delta,d)} \prod_{\substack{p|\Delta\\p\nmid d}} \left(1-\frac{\omega(p)}{p}\right)$$
$$= \frac{\Delta}{(\Delta,d)}\gamma(\mathcal{H}) \prod_{p|(\Delta,d)} \left(1-\frac{\omega(p)}{p}\right)^{-1}$$

with

$$\gamma(\mathcal{H}) = \prod_{p|\Delta} \left(1-\frac{\omega(p)}{p}\right).$$

Moreover, from $(\nu, \Delta) = 1$ we know that

$$\sum_{\substack{\beta \,(\mathrm{mod}\ \nu)\\ Q(\beta)\equiv 0 \,(\mathrm{mod}\ \nu)}} 1 = \tau_k(\nu).$$

Therefore, summing up the above four formulae we get

$$\begin{aligned} S_{51} \leqslant \gamma(\mathcal{H}) \sum_{d|P(z)} \sum_{(\nu,\Delta d)=1} \lambda_\nu \lambda''_d \frac{\Delta}{(\Delta, d)} \tau_k(\nu) \\ \times \left(\frac{N/2}{\nu[\Delta, d]} + O(1) \right) \prod_{p|(\Delta,d)} \left(1 - \frac{\omega(p)}{p}\right)^{-1}. \end{aligned}$$

Since $|\lambda_\nu(d)| \leqslant \tau_3(d)$ and $|\lambda''_d| \leqslant 1$, we have

$$\begin{aligned} S_{51} &\leqslant \frac{N}{2}\gamma(\mathcal{H}) \sum_{d|P(z)} \sum_{(\nu,\Delta d)=1} \lambda_\nu \lambda''_d \frac{\tau_k(\nu)}{d\nu} \prod_{p|(\Delta,d)} \left(1 - \frac{\omega(p)}{p}\right)^{-1} \\ &\quad + O\biggl(\gamma(\mathcal{H}) \sum_{d<D'} \sum_{\nu<D} \tau_3(\nu)\tau_k(\nu) \frac{\Delta}{(\Delta, d)} \prod_{p|(\Delta,d)} \left(1 - \frac{\omega(p)}{p}\right)^{-1}\biggr) \\ &= \frac{N}{2}\gamma(\mathcal{H}) \sum_{d|P(z)} \sum_{(\nu,\Delta d)=1} \lambda_\nu \lambda''_d \frac{\tau_k(\nu)}{d\nu} \prod_{p|(\Delta,d)} \frac{p}{p-\omega(p)} + O(\Delta D D' (\log D)^{3k-1}). \end{aligned}$$

From Proposition 3.3 we get for sufficiently large s that

$$S_{51} \leqslant (1 + O(e^{-s}))\frac{N}{2}\gamma(\mathcal{H}) G_1 V(z) + O(\Delta D D' (\log D)^{3k-1}), \tag{4.2}$$

where

$$V(z) = \prod_{\substack{p<z\\ p\nmid\Delta}} \left(1 - \frac{1}{p}\right) \prod_{\substack{k+2\leqslant p<z\\ p|\Delta, p\nmid a}} \left(1 - \frac{1}{p-\omega(p)}\right) \tag{4.3}$$

and

$$G_1 = \sum_{\substack{\nu<D\\ (\nu,\Delta)=1}} \lambda_\nu \frac{\tau_k(\nu)}{\varphi(\nu)}. \tag{4.4}$$

Analogously,

$$S_{52} = (1 + O(e^{-s}))\frac{N}{2}\gamma(\mathcal{H}) G_1 V(z) + O(\Delta D D' (\log D)^{3k-1}).$$

Therefore,

$$S_5 \ll N\gamma(\mathcal{H})G_1V(z) + N^{1-\frac{\varepsilon}{2}}. \tag{4.5}$$

5. Evaluation of S_3 and S_4

First, we mention that $S_4 = S_{51}\log N$, where S_{51} is defined in the previous section. Thus (4.2) implies that

$$S_4 \leqslant (1+O(e^{-s}))\frac{N\log N}{2}\gamma(\mathcal{H})G_1V(z) + O(\Delta DD'(\log N)^{3k}). \tag{5.1}$$

In order to calculate S_3, we change the order of summation to get

$$S_3 = \sum_{d|P(z)} \lambda'_d \sum_{a^m\in\mathcal{M}} U_d^{(m)}, \tag{5.2}$$

where

$$U_d^{(m)} = \sum_{\substack{N/2<n\leqslant N\\ (Q(n),\Delta)=1\\ n\equiv 0\,(\mathrm{mod}\ d)}} \Lambda^\flat(n-a^m)\Bigg(\sum_{\nu|Q(n)}\lambda_\nu\Bigg). \tag{5.3}$$

Next we come to the evaluation of $U_d^{(m)}$.

$$U_d^{(m)} = \sum_{(\nu,\Delta d)=1}\lambda_\nu \sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\ (Q(\alpha),\Delta)=1\\ \alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))}} \sum_{\substack{\beta\,(\mathrm{mod}\ \nu)\\ Q(\beta)\equiv 0\,(\mathrm{mod}\ \nu)}} \sum_{\substack{N/2<n\leqslant N\\ n\equiv\alpha\,(\mathrm{mod}\ \Delta)\\ n\equiv\beta\,(\mathrm{mod}\ \nu)\\ n\equiv 0\,(\mathrm{mod}\ d)}} \Lambda^\flat(n-a^m)$$

We write R_1 to be the summation with $(\beta - a^m, \nu) > 1$, then

$$\begin{aligned}
R_1 &= \sum_{(\nu,\Delta d)=1} \lambda_\nu \sum_{p|\nu} \log p \sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\ (Q(\alpha),\Delta)=1\\ \alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))\\ \alpha-a^m\equiv p\,(\mathrm{mod}\ \Delta)}} \sum_{\substack{\beta\,(\mathrm{mod}\ \nu)\\ Q(\beta)\equiv 0\,(\mathrm{mod}\ \nu)\\ (\beta-a^m,\nu)=p}} \sum_{\substack{N/2<n\leqslant N\\ n-a^m=p\\ n\equiv\beta\,(\mathrm{mod}\ \nu)\\ n\equiv 0\,(\mathrm{mod}\ d)}} 1\\
&\ll \sum_{\nu<D} \tau_3(\nu)\tau_k(\nu) \sum_{\substack{p|\nu\\ p\equiv -a^m\,(\mathrm{mod}\ d)}} \log p\\
&\ll \sum_{\substack{p<D\\ p\equiv -a^m\,(\mathrm{mod}\ d)}} \tau_3(p)\tau_k(p)\log p \sum_{\nu<D/p} \tau_3(\nu)\tau_k(\nu)\\
&\ll D(\log D)^{3k-1} \sum_{\substack{p<D\\ p\equiv -a^m\,(\mathrm{mod}\ d)}} \frac{\log p}{p} \ll \frac{D(\log D)^{3k}}{\varphi(d)},
\end{aligned}$$

where the implied constant depends only on k. If we denote by R_2 the summation with $(\beta - a^m, \nu) = 1$ and $(\alpha - a^m, \Delta) > 1$, then

$$\begin{aligned}
R_2 &= \sum_{(\nu,\Delta d)=1} \lambda_\nu \sum_{p|\Delta} \log p \sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\ (Q(\alpha),\Delta)=1\\ \alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))\\ (\alpha-a^m,\Delta)=p}} \sum_{\substack{\beta\,(\mathrm{mod}\ \nu)\\ Q(\beta)\equiv 0\,(\mathrm{mod}\ \nu)\\ (\beta-a^m,\nu)=1\\ \beta-a^m\equiv p\,(\mathrm{mod}\ \nu)}} \sum_{\substack{N/2<n\leqslant N\\ n-a^m=p\\ n\equiv\alpha\,(\mathrm{mod}\ \Delta)\\ n\equiv 0\,(\mathrm{mod}\ d)}} 1\\
&\ll \sum_{\nu<D} \tau_3(\nu) \sum_{\substack{p|\Delta\\ p\equiv -a^m\,(\mathrm{mod}\ d)}} \varphi\Big(\frac{\Delta}{p}\Big) \log p\\
&\ll D(\log D)^2\varphi(\Delta) \sum_{\substack{p|\Delta\\ p\equiv -a^m\,(\mathrm{mod}\ d)}} \frac{\log p}{p-1} \ll \Delta D(\log D)^3 \log\Delta.
\end{aligned}$$

Therefore we conclude that

(5.4)
$$\begin{aligned}
U_d^{(m)} &= \sum_{(\nu,\Delta d)=1} \lambda_\nu \sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\ ((\alpha-a^m)Q(\alpha),\Delta)=1\\ \alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))}} \sum_{\substack{\beta\,(\mathrm{mod}\ \nu)\\ Q(\beta)\equiv 0\,(\mathrm{mod}\ \nu)\\ (\beta-a^m,\nu)=1}} \sum_{\substack{N/2<n\leqslant N\\ n\equiv\alpha\,(\mathrm{mod}\ \Delta)\\ n\equiv\beta\,(\mathrm{mod}\ \nu)\\ n\equiv 0\,(\mathrm{mod}\ d)}} \Lambda^\flat(n-a^m)\\
&\quad + O\big(D(\log D)^{3k}\big),
\end{aligned}$$

where the implied constant depends only on k.

For $(b,q)=1$, we write

$$E(x,q;b)=\sum_{\substack{n\leqslant x\\ n\equiv b\,(\mathrm{mod}\ q)}}\Lambda^{\flat}(n)-\frac{x}{\varphi(q)}$$

as usual. Then

$$\begin{aligned}U_d^{(m)}&=\sum_{(\nu,\Delta d)=1}\lambda_\nu\sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\((\alpha-a^m)Q(\alpha),\Delta)=1\\ \alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))}}\sum_{\substack{\beta\,(\mathrm{mod}\ \nu)\\ Q(\beta)\equiv 0\,(\mathrm{mod}\ \nu)\\ (\beta-a^m,\nu)=1}}\frac{N/2}{\varphi(\nu[\Delta,d])}+R_d^{(m)}\\&\quad+O\big(D(\log D)^{3k}\big),\end{aligned}\tag{5.5}$$

where

$$\begin{aligned}R_d^{(m)}&=\sum_{(\nu,\Delta d)=1}\lambda_\nu\\&\times\sum_{\substack{\alpha\,(\mathrm{mod}\ \Delta)\\((\alpha-a^m)Q(\alpha),\Delta)=1\\ \alpha\equiv 0\,(\mathrm{mod}\ (\Delta,d))}}\sum_{\substack{\beta\,(\mathrm{mod}\ \nu)\\ Q(\beta)\equiv 0\,(\mathrm{mod}\ \nu)\\ (\beta-a^m,\nu)=1}}\big(E(N,\nu[\Delta,d];b)-E(N/2,\nu[\Delta,d];b)\big)\end{aligned}\tag{5.6}$$

with b the residue class modulo $\nu[\Delta,d]$ satisfying $b\equiv\alpha-a^m\pmod{\Delta}$, $b\equiv\beta-a^m\pmod{\nu}$ and $b\equiv -a^m\pmod{d}$. Notice that we include in the error term a few terms for $\Lambda^{\flat}(n)$ with n in the intervals $(N/2,N/2+a^m]$ and $(N,N+a^m]$.

We can easily deduce that for every m

$$\sum_d\lambda_d' R_d^{(m)}\ll\Delta\sum_{q\leqslant\Delta DD'}\tau_{k+3}(q)\max_{(b,q)=1}\big(|E(N,q;b)|+|E(N/2,q;b)|\big),$$

while by Cauchy's inequality and $E(N,q;b)\ll N/\varphi(q)$

$$\begin{aligned}&\sum_{q\leqslant\Delta DD'}\tau_{k+3}(q)\max_{(b,q)=1}|E(N,q;b)|\\&\leqslant\Big(\sum_{q\leqslant\Delta DD'}\frac{\tau_{k+3}^2(q)N}{\varphi(q)}\Big)^{\frac12}\Big(\sum_{q\leqslant\Delta DD'}\max_{(b,q)=1}|E(N,q;b)|\Big)^{\frac12},\end{aligned}$$

and the Bombieri-Vinogradov theorem indicates

$$\sum_{q \leqslant \Delta DD'} \tau_{k+3}(q) \max_{(b,q)=1} |E(N,q;b)| \ll \frac{N}{(\log N)^{A+1}}$$

for any positive real number A. The same estimate holds for the sum involving $E(N/2,q;b)$. Therefore,

$$\sum_d \lambda'_d R_d^{(m)} \ll \frac{N}{(\log N)^{A+1}}. \tag{5.7}$$

In order to calculate the main term in (5.5), we need to evaluate the sum over α and β respectively. Since $\Delta \mid Q(a^m)$ for any m, we have

$$\sum_{\substack{\alpha \,(\mathrm{mod}\ \Delta) \\ ((\alpha - a^m)Q(\alpha),\Delta)=1 \\ \alpha \equiv 0 \,(\mathrm{mod}\ (\Delta,d))}} 1 = \sum_{\substack{\alpha \,(\mathrm{mod}\ \Delta) \\ (Q(\alpha),\Delta)=1 \\ \alpha \equiv 0 \,(\mathrm{mod}\ (\Delta,d))}} 1 = \frac{\Delta}{(\Delta,d)} \gamma(\mathcal{H}) \prod_{p \mid (\Delta,d)} \left(1 - \frac{\omega(p)}{p}\right)^{-1}$$

by (4.1). For squarefree number ν satisfying $(\nu,\Delta)=1$, we write

$$\tau_k^{(m)}(\nu) = \sum_{\substack{\beta \,(\mathrm{mod}\ \nu) \\ Q(\beta) \equiv 0 \,(\mathrm{mod}\ \nu) \\ (\beta - a^m,\nu)=1}} 1,$$

then $\tau_k^{(m)}(\nu) = \tau_k(\nu_1)\tau_{k-1}(\nu_2)$ where $\nu = \nu_1\nu_2$ with $(\nu_1, Q(a^m)) = 1$ and $\nu_2 \mid Q(a^m)$. Therefore the main term in (5.5) is equal to

$$\begin{aligned} &\frac{\Delta N \gamma(\mathcal{H})}{2\varphi([\Delta,d]) \cdot (\Delta,d)} \prod_{p \mid (\Delta,d)} \left(1 - \frac{\omega(p)}{p}\right)^{-1} \sum_{(\nu,\Delta d)=1} \lambda_\nu \frac{\tau_k^{(m)}(\nu)}{\varphi(\nu)} \\ &\quad = \frac{\Delta N \gamma(\mathcal{H})}{2\varphi(\Delta)} \cdot \frac{1}{\varphi(d)} \prod_{p \mid (\Delta,d)} \left(1 - \frac{1}{p}\right)\left(1 - \frac{\omega(p)}{p}\right)^{-1} \sum_{(\nu,\Delta d)=1} \lambda_\nu \frac{\tau_k^{(m)}(\nu)}{\varphi(\nu)}. \end{aligned}$$

Summing over d, we get from Proposition 3.3 that

(5.8)

$$\sum_d \lambda'_d U_d^{(m)} = (1+O(e^{-s}))\frac{\Delta N\gamma(\mathcal{H})}{2\varphi(\Delta)}G^{(m)}\prod_{\substack{p<z\\ p\nmid\Delta}}\left(1-\frac{1}{p-1}\right)$$

$$\times\prod_{\substack{p<z\\ p|\Delta, p\nmid a}}\left(1-\frac{1}{p-\omega(p)}\right)+O\left(\frac{N}{(\log N)^{A+1}}\right),$$

$$\geqslant (1+O(e^{-s}))\frac{\Delta N\gamma(\mathcal{H})}{2\varphi(\Delta)}G^{(m)}V(z)+O\left(\frac{N}{(\log N)^{A+1}}\right)$$

where

$$G^{(m)} = \sum_{\substack{\nu<D\\ (\nu,\Delta)=1}}\lambda_\nu\frac{\tau_k^{(m)}(\nu)}{f(\nu)}$$

with $f(\nu)$ the multiplicative function satisfying $f(p)=p-2$, and the error term mainly comes from (5.7).

Combining (5.1), (5.2) and (5.8) we finally arrive at

(5.9)

$$S_3 - S_4 \geqslant (1+O(e^{-s}))\frac{N\gamma(\mathcal{H})V(z)}{2}\left(\frac{\Delta}{\varphi(\Delta)}\sum_{a^m\in\mathcal{H}}G_2\right.$$

$$\left.+\frac{\Delta}{\varphi(\Delta)}\sum_{a^m\in\mathcal{M}\setminus\mathcal{H}}G_3 - G_1\log N\right)+O\left(\frac{N}{(\log N)^{A+1}}\right),$$

where $V(z)$ is given in (4.3) and

(5.10)

$$G_2 = \sum_{\substack{\nu<D\\ (\nu,\Delta)=1}}\lambda_\nu\frac{\tau_{k-1}(\nu)}{f(\nu)},\qquad G_3 = \sum_{\substack{\nu<D\\ (\nu,\Delta)=1}}\lambda_\nu\frac{\tau_k^{(m)}(\nu)}{f(\nu)}\quad (a^m\in\mathcal{M}\setminus\mathcal{H}).$$

6. Choosing the Sifting Weights

In this section, we will choose the parameters λ_ν and give asymptotic formulae for G_1 and G_2. We follow the way given in [4].

Denote

$$g_1(\nu) = \frac{\tau_k(\nu)}{\varphi(\nu)}, \qquad g_2(\nu) = \frac{\tau_{k-1}(\nu)}{f(\nu)},$$

and

$$g_3(\nu) = \frac{\tau_k^{(m)}(\nu)}{f(\nu)} \quad (a^m \in \mathcal{M} \setminus \mathcal{H}),$$

let $h_i(\nu)$ be the relative density function of $g_i(\nu)$. It is well-known from the Selberg's Λ^2-sieve theory that

$$G_1 = \sum_{\substack{c<\sqrt{D} \\ (c,\Delta)=1}} h_1(c) y_c^2, \tag{6.1}$$

where

$$y_c = \frac{\mu(c)}{h_1(c)} \sum_{m \equiv 0 \,(\mathrm{mod}\ c)} \rho_m g_1(m).$$

Using the Möbius inversion formula on divisor-closed set we obtain

$$\rho_m = \frac{\mu(m)}{g_1(m)} \sum_{c \equiv 0 \,(\mathrm{mod}\ m)} h_1(c) y_c. \tag{6.2}$$

Therefore the initial condition $\rho_1 = 1$ is equivalent to

$$\sum_{\substack{c<\sqrt{D} \\ (c,\Delta)=1}} h_1(c) y_c = 1. \tag{6.3}$$

Now we choose

$$y_c = \frac{1}{Y} \left(\log \frac{\sqrt{D}}{c} \right)^{\ell}, \tag{6.4}$$

for squarefree $c \leqslant \sqrt{D}$, $(c, \Delta) = 1$ and $y_c = 0$ otherwise. Inserting this into (6.3) we find that

$$Y = {\sum_{\substack{c<\sqrt{D} \\ (c,\Delta)=1}}}^{\flat} h_1(c) \left(\log \frac{\sqrt{D}}{c} \right)^{\ell}, \tag{6.5}$$

where $\sum^{\flat}$ means the summation goes through squarefree integers. Before going further, we give a result involving the sieve weight constituents which verifies (2.2).

Lemma 6.1. *For any integer $m \geqslant 1$, we have $|\rho_m| \leqslant 1$.*

Proof. From (6.2) and (6.4) we know that

$$\rho_m = \frac{\mu(m)h_1(m)}{Yg_1(m)} \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}/m \\ (c,\Delta m)=1}} h_1(c)\Big(\log\frac{\sqrt{D}}{cm}\Big)^{\ell}.$$

Then the desired result follows from

$$\begin{aligned} Y &= \sum_{u|m} \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D} \\ (c,\Delta)=1 \\ (c,m)=u}} h_1(c)\Big(\log\frac{\sqrt{D}}{c}\Big)^{\ell} = \sum_{u|m} h_1(u) \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}/u \\ (c,\Delta m)=1}} h_1(c)\Big(\log\frac{\sqrt{D}}{cu}\Big)^{\ell} \\ &\geqslant \Big(\sum_{u|m} h_1(u)\Big) \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}/m \\ (c,\Delta m)=1}} h_1(c)\Big(\log\frac{\sqrt{D}}{cm}\Big)^{\ell} \\ &= \frac{h_1(m)}{g_1(m)} \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}/m \\ (c,\Delta m)=1}} h_1(c)\Big(\log\frac{\sqrt{D}}{cm}\Big)^{\ell}. \end{aligned}$$

□

In order to calculate the sum in (6.5), we introduce the following lemma.

LEMMA 6.2. *Let κ and ℓ be positive integers and assume g is a multiplicative function supported on squarefree numbers such that*

$$g(p) = \frac{\kappa}{p} + O\Big(\frac{1}{p^2}\Big), \qquad \kappa \geqslant 1. \tag{6.6}$$

Then, for $x \geqslant 2$,

$$\sideset{}{^\flat}\sum_{\substack{m\leqslant x \\ (m,\Delta)=1}} g(m)\Big(\log\frac{x}{m}\Big)^{\ell} = \mathfrak{S}\frac{\ell!}{(\ell+\kappa)!}(\log x)^{\ell+\kappa}\Big(1+O\Big(\frac{1}{\log x}\Big)\Big), \tag{6.7}$$

where

$$\mathfrak{S} = \prod_{p\nmid\Delta}\Big(1-\frac{1}{p}\Big)^{\kappa}(1+g(p))\prod_{p|\Delta}\Big(1-\frac{1}{p}\Big)^{\kappa}, \tag{6.8}$$

and the implied constant depends only on κ, ℓ, Δ and on the one in (6.6).

Proof. This is Corollary A.6 of [4]. □

Since $h_1(p) = k(p-k-1)^{-1}$ for $p \nmid \Delta$, we get from Lemma 6.2 that

$$Y = \mathfrak{S}(\Delta)\frac{\ell!}{(k+\ell)!}(\log\sqrt{D})^{k+\ell}\Big(1 + O\Big(\frac{1}{\log D}\Big)\Big), \tag{6.9}$$

where

$$\mathfrak{S}(\Delta) = \prod_{p\nmid\Delta}\Big(1-\frac{1}{p}\Big)^k\Big(1-\frac{k}{p-1}\Big)^{-1}\prod_{p\mid\Delta}\Big(1-\frac{1}{p}\Big)^k. \tag{6.10}$$

Analogously, (6.1) and (6.4) indicate that

$$\begin{aligned} Y^2G_1 &= \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}\\(c,\Delta)=1}} h_1(c)\Big(\log\frac{\sqrt{D}}{c}\Big)^{2\ell} \\ &= \mathfrak{S}(\Delta)\frac{(2\ell)!}{(k+2\ell)!}(\log\sqrt{D})^{k+2\ell}\Big(1+O\Big(\frac{1}{\log D}\Big)\Big). \end{aligned}$$

Applying (6.9) we obtain

$$G_1 = \mathfrak{S}(\Delta)^{-1}\frac{(2\ell)!(k+\ell)!^2}{(k+2\ell)!\ell!^2}(\log\sqrt{D})^{-k}\Big(1+O\Big(\frac{1}{\log D}\Big)\Big). \tag{6.11}$$

Next we calculate G_2. We have

$$G_2 = \sum_{\substack{c<\sqrt{D}\\(c,\Delta)=1}}\frac{1}{h_2(c)}\Big(\sum_{m\equiv 0\,(\mathrm{mod}\ c)}\rho_m g_2(m)\Big)^2 \tag{6.12}$$

Notice that ρ_m is given in (6.2), whence

$$\begin{aligned} \sum_{m\equiv 0\,(\mathrm{mod}\ c)}\rho_m g_2(m) &= \sum_{m\equiv 0\,(\mathrm{mod}\ c)}\frac{\mu(m)g_2(m)}{g_1(m)}\sum_{d\equiv 0\,(\mathrm{mod}\ m)}h_1(d)y_d \\ &= \frac{\mu(c)g_2(c)}{g_1(c)}\sum_{d\equiv 0\,(\mathrm{mod}\ c)}h_1(d)y_d\sum_{u\mid\frac{d}{c}}\frac{\mu(u)g_2(u)}{g_1(u)}. \end{aligned}$$

Since

$$\sum_{u\mid\frac{d}{c}}\frac{\mu(u)g_2(u)}{g_1(u)} = \prod_{p\mid\frac{d}{c}}\Big(1-\frac{(k-1)(p-1)}{k(p-2)}\Big) = \prod_{p\mid\frac{d}{c}}\frac{p-k-1}{k(p-2)} = \frac{1}{h_1(d/c)f(d/c)},$$

we conclude that

$$\sum_{m\equiv 0\,(\mathrm{mod}\ c)} \rho_m g_2(m) = \mu(c)\frac{\varphi(c)}{\tau_k(c)}\frac{\tau_{k-1}(c)}{f(c)} \sum_{d\equiv 0\,(\mathrm{mod}\ c)} \frac{h_1(d)y_d}{h_1(d/c)f(d/c)}$$
$$= \mu(c)\varphi(c)h_1(c)\frac{\tau_{k-1}(c)}{\tau_k(c)} \sum_{d\equiv 0\,(\mathrm{mod}\ c)} \frac{y_d}{f(d)}.$$

Inserting this into (6.12), we have

$$G_2 = \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}\\(c,\Delta)=1}} \frac{1}{h_2(c)}\Big(\varphi(c)h_1(c)\frac{\tau_{k-1}(c)}{\tau_k(c)} \sum_{d\equiv 0\,(\mathrm{mod}\ c)} \frac{y_d}{f(d)}\Big)^2$$
$$= \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}\\(c,\Delta)=1}} h_2(c)\varphi(c)^2\Big(\sum_{d\equiv 0\,(\mathrm{mod}\ c)} \frac{y_d}{f(d)}\Big)^2$$
$$= \frac{1}{Y^2}\sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}\\(c,\Delta)=1}} h_2(c)\frac{\varphi(c)^2}{f(c)^2}\Bigg[\sideset{}{^\flat}\sum_{\substack{d<\sqrt{D}/c\\(d,\Delta c)=1}} \frac{1}{f(d)}\Big(\log\frac{\sqrt{D}}{cd}\Big)^\ell\Bigg]^2.$$

Applying Lemma 6.2 we have

$$Y^2G_2 = \frac{\mathfrak{S}_1(\Delta)^2}{(\ell+1)^2}\sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}\\(c,\Delta)=1}} h_2(c)\Big(\log\frac{\sqrt{D}}{c}\Big)^{2\ell+2}\Big(1+O\Big(\frac{1}{\log\sqrt{D}/c}\Big)\Big),$$

where

$$\mathfrak{S}_1(\Delta) = \prod_{p\nmid\Delta}\Big(1-\frac{1}{p}\Big)\Big(1+\frac{1}{p-2}\Big)\prod_{p|\Delta}\Big(1-\frac{1}{p}\Big)$$

Applying Lemma 6.2 again we get

$$Y^2G_2 = \frac{\mathfrak{S}_1(\Delta)^2}{(\ell+1)^2}\cdot\mathfrak{S}_2(\Delta)\frac{(2\ell+2)!}{(k+2\ell+1)!}(\log\sqrt{D})^{k+2\ell+1}\Big(1+O\Big(\frac{1}{\log D}\Big)\Big),$$

where

$$\mathfrak{S}_2(\Delta) = \prod_{p\nmid\Delta}\Big(1-\frac{1}{p}\Big)^{k-1}\Big(1+\frac{k-1}{p-k-1}\Big)\prod_{p|\Delta}\Big(1-\frac{1}{p}\Big)^{k-1}.$$

Combining with (6.9), we get

(6.13)
$$G_2 = \frac{\mathfrak{S}_1(\Delta)^2\mathfrak{S}_2(\Delta)}{\mathfrak{S}(\Delta)^2}\frac{(k+\ell)!^2(2\ell+2)!}{(\ell+1)!^2(k+2\ell+1)!}(\log\sqrt{D})^{1-k}\Big(1+O\Big(\frac{1}{\log D}\Big)\Big).$$

If we write

$$\mathfrak{S}'(\Delta) = \prod_{p\nmid\Delta}\Big(1-\frac{1}{p}\Big)\Big(1+\frac{1}{p+2}\Big), \tag{6.14}$$

then a modicum of calculation shows that

$$\frac{\mathfrak{S}_1(\Delta)^2\mathfrak{S}_2(\Delta)}{\mathfrak{S}(\Delta)} = \frac{\varphi(\Delta)}{\Delta}\mathfrak{S}'(\Delta).$$

Therefore, comparing (6.13) with (6.11) we finally get

$$\frac{\Delta}{\varphi(\Delta)}G_2 = \frac{(2\ell+1)\mathfrak{S}'(\Delta)G_1\log D}{(\ell+1)(k+2\ell+1)}\Big(1+O\Big(\frac{1}{\log D}\Big)\Big). \tag{6.15}$$

The last task is to evaluate G_3, we will complete it in the next section.

7. Asymptotics of G_3 and Proof of the Theorem

First we will give a more precise form of the error term in Lemma 6.2.

LEMMA 7.1. *Under the assumption of Lemma 6.2, we have*

(7.1)
$$\sideset{}{^\flat}\sum_{\substack{m\leqslant x\\(m,\Delta)=1}} g(m)\Big(\log\frac{x}{m}\Big)^{\ell} = \mathfrak{S}\frac{\ell!}{(\ell+\kappa)!}(\log x)^{\ell+\kappa}+O\big((\log x)^{\kappa+\ell-1}\log\log(\Delta+2)\big),$$

where $\mathfrak{S}$ is given in (6.8) and the implied constant depends only on κ, ℓ and on the one in (6.6).

Proof. First we introduce the following asymptotic formula

$$\sideset{}{^\flat}\sum_{\substack{m\leqslant x\\(m,\Delta)=1}} g(m) = \frac{\mathfrak{S}}{\kappa!}(\log x)^{\kappa} + O\big((\log x)^{\kappa-1}\log\log(\Delta+2)\big), \tag{7.2}$$

the proof is analogous to the one given for Theorem A.5 in [4], the only difference is that the condition (A.15) appeared in [4] should be replaced by

$$\sum_{\substack{p\leqslant x\\ p\nmid\Delta}} g(p)\log p = \kappa\log x + O(\log\log(\Delta+2))$$

since

$$\begin{aligned}\sum_{p\mid\Delta}\frac{\log p}{p} &= \sum_{\substack{p\mid\Delta\\ p\leqslant\log(\Delta+2)}}\frac{\log p}{p} + \sum_{\substack{p\mid\Delta\\ p>\log(\Delta+2)}}\frac{\log p}{p}\\ &\ll \log\log(\Delta+2) + \frac{\log\log(\Delta+2)}{\log(\Delta+2)}\sum_{p\mid\Delta}1\\ &\ll \log\log(\Delta+2).\end{aligned}\tag{7.3}$$

Then, using partial summation we can get (7.1) from (7.2). □

LEMMA 7.2. *Under the assumption of Lemma 6.2, we have*

$$\begin{aligned}&\sideset{}{^\flat}\sum_{\substack{m\leqslant x\\ (m,\Delta)=1\\ m\mid\Delta'}} g(m)\Big(\log\frac{x}{m}\Big)^{\kappa+\ell}\\ &=\bigg(1+O\bigg(\frac{(\log\log(\Delta\Delta'+2))^{\kappa+1}}{\mathfrak{S}\log x}\bigg)\bigg)(\log x)^{\kappa+\ell}\prod_{\substack{p\nmid\Delta\\ p\mid\Delta'}}(1+g(p)),\end{aligned}$$

where $\mathfrak{S}$ is given in (6.8), and the implied constant depends only on κ, ℓ and on the one in (6.6).

Proof. We have

$$\begin{aligned}\sideset{}{^\flat}\sum_{\substack{m\leqslant x\\ (m,\Delta)=1}} g(m)\Big(\log\frac{x}{m}\Big)^{\ell} &= \sideset{}{^\flat}\sum\sideset{}{^\flat}\sum_{\substack{m_1m_2\leqslant x\\ (m_1m_2,\Delta)=1\\ m_1\mid\Delta',\ (m_2,\Delta')=1}} g(m_1m_2)\Big(\log\frac{x}{m_1m_2}\Big)^{\ell}\\ &= \sideset{}{^\flat}\sum_{\substack{m_1\leqslant x\\ (m_1,\Delta)=1\\ m_1\mid\Delta'}} g(m_1)\sideset{}{^\flat}\sum_{\substack{m_2\leqslant x/m_1\\ (m_2,\Delta\Delta')=1}} g(m_2)\Big(\log\frac{x}{m_1m_2}\Big)^{\ell}\end{aligned}$$

$$(7.4) \qquad = \sideset{}{^\flat}\sum_{\substack{m_1 \leqslant x \\ (m_1,\Delta)=1 \\ m_1 \mid \Delta'}} g(m_1)\Big[\frac{\mathfrak{S}'\ell!}{(\kappa+\ell)!}\Big(\log\frac{x}{m_1}\Big)^{\kappa+\ell} + O\Big(\Big(\log\frac{x}{m_1}\Big)^{\kappa+\ell-1}\log\log(\Delta\Delta'+2)\Big)\Big]$$

where

$$\mathfrak{S}' = \prod_{p\nmid\Delta\Delta'}\Big(1-\frac{1}{p}\Big)^{\kappa}(1+g(p))\prod_{p\mid\Delta\Delta'}\Big(1-\frac{1}{p}\Big)^{\kappa}.$$

It is obvious that

$$\sum_{m_1\mid\Delta'} g(m_1) \leqslant \exp\Big(\sum_{p\mid\Delta'} g(p)\Big) \ll \exp\Big(\kappa\sum_{p\mid\Delta'}\frac{1}{p}\Big) \ll \big(\log\log(\Delta'+2)\big)^{\kappa},$$

where the last step is analogous to (7.3). Therefore, the error term in (7.4) is

$$O\big((\log x)^{\kappa+\ell-1}(\log\log(\Delta\Delta'+2))^{\kappa+1}\big).$$

Now, using Lemma 7.1 to calculate the left hand side of (7.4), we have

$$\sideset{}{^\flat}\sum_{\substack{m \leqslant x \\ (m,\Delta)=1 \\ m \mid \Delta'}} g(m)\Big(\log\frac{x}{m}\Big)^{\kappa+\ell} = \frac{\mathfrak{S}}{\mathfrak{S}'}(\log x)^{\kappa+\ell}\Big(1+O\Big(\frac{(\log\log(\Delta\Delta'+2))^{\kappa+1}}{\mathfrak{S}\log x}\Big)\Big),$$

where $\mathfrak{S}$ is given in (6.8). Since

$$\frac{\mathfrak{S}}{\mathfrak{S}'} = \prod_{\substack{p\nmid\Delta \\ p\mid\Delta'}}(1+g(p)),$$

the desired result is obtained. □

Now we begin to calculate G_3. As in section 6, we have

$$(7.5) \qquad G_3 = \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D} \\ (c,\Delta)=1}} \frac{1}{h_3(c)}\Big(\sum_{m\equiv 0 \,(\mathrm{mod}\ c)} \rho_m g_3(m)\Big)^2.$$

From (6.2) we know that

$$\begin{aligned}\sum_{m\equiv 0\,(\mathrm{mod}\ c)} \rho_m g_3(m) &= \sum_{m\equiv 0\,(\mathrm{mod}\ c)} g_3(m)\mu(m)\frac{\varphi(m)}{\tau_k(m)} \sum_{d\equiv 0\,(\mathrm{mod}\ m)} h_1(d)y_d \\ &= \frac{\mu(c)\varphi(c)g_3(c)}{\tau_k(c)} \sum_{d\equiv 0\,(\mathrm{mod}\ c)} h_1(d)y_d \sum_{u\mid \frac{d}{c}} \frac{\mu(u)\varphi(u)g_3(u)}{\tau_k(u)} \\ &= \frac{\mu(c)\varphi(c)g_3(c)}{\tau_k(c)} \sum_{d\equiv 0\,(\mathrm{mod}\ c)} h_1(d)y_d \prod_{p\mid \frac{d}{c}} \left(1 - \frac{(p-1)\tau_k^{(m)}(p)}{k(p-2)}\right) \\ &= \frac{\mu(c)\varphi(c)g_3(c)}{\tau_k(c)} \sum_{d\equiv 0\,(\mathrm{mod}\ c)} h_1(d)y_d \frac{f_1(b/c)}{\tau_k(b/c)f(b/c)},\end{aligned}$$

where f_1 is the multiplicative function with

$$f_1(p) = k(p-2) - (p-1)\tau_k^{(m)}(p).$$

Therefore,

$$\sum_{m\equiv 0\,(\mathrm{mod}\ c)} \rho_m g_3(m) = \frac{\mu(c)\varphi(c)g_3(c)h_1(c)}{\tau_k(c)} \sum_d \frac{h_1(d)f_1(d)}{\tau_k(d)f(d)} y_{dc}. \tag{7.6}$$

Recalling the definition of y_{dc}, we can express the summation over d as

$$\frac{1}{Y} \sideset{}{^\flat}\sum_{\substack{d<\sqrt{D}/c \\ (d,\Delta c)=1}} \frac{h_1(d)f_1(d)}{\tau_k(d)f(d)} \left(\log\frac{\sqrt{D}}{dc}\right)^{\ell}.$$

Since

$$\frac{h_1(p)f_1(p)}{\tau_k(p)f(p)} = \begin{cases} \dfrac{1}{f(p)} & \text{if } p \mid Q(a^m), \\[2ex] \dfrac{\mu(p)h_1(p)}{f(p)} & \text{if } p \nmid Q(a^m), \end{cases}$$

we have (note that $\Delta \mid Q(a^m)$)
(7.7)

$$\sum_d \frac{h_1(d)f_1(d)}{\tau_k(d)f(d)} y_{dc} = \frac{1}{Y} \sideset{}{^\flat}\sum\sideset{}{^\flat}\sum_{\substack{uv<\sqrt{D}/c\\ (uv,\Delta c)=1\\ u|Q(a^m),\ (v,Q(a^m))=1}} \frac{1}{f(u)} \cdot \frac{\mu(v)h_1(v)}{f(v)} \Big(\log\frac{\sqrt{D}}{uvc}\Big)^\ell$$

$$= \frac{1}{Y} \sideset{}{^\flat}\sum_{\substack{u<\sqrt{D}/c\\ (u,\Delta c)=1\\ u|Q(a^m)}} \frac{1}{f(u)} \sideset{}{^\flat}\sum_{\substack{v<\sqrt{D}/(uc)\\ (v,Q(a^m)c)=1}} \frac{\mu(v)h_1(v)}{f(v)} \Big(\log\frac{\sqrt{D}}{uvc}\Big)^\ell$$

It is obvious that $\mu(v)h_1(v)/f(v) \ll v^{\varepsilon-2}$, therefore writing

$$\Big(\log\frac{\sqrt{D}}{uvc}\Big)^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} \Big(\log\frac{\sqrt{D}}{uc}\Big)^{\ell-j} \log^j v,$$

we get

$$\sum_v = \Big(\log\frac{\sqrt{D}}{uc}\Big)^\ell \sideset{}{^\flat}\sum_{\substack{v<\sqrt{D}/(uc)\\ (v,Q(a^m)c)=1}} \frac{\mu(v)h_1(v)}{f(v)} + O\bigg(\Big(\log\frac{\sqrt{D}}{uc}\Big)^{\ell-1}\bigg)$$

$$= \Big(\log\frac{\sqrt{D}}{uc}\Big)^\ell \prod_{p\nmid Q(a^m)c} \Big(1 - \frac{k}{(p-k-1)(p-2)}\Big) + O\bigg(\Big(\log\frac{\sqrt{D}}{uc}\Big)^{\ell-1}\bigg).$$

Inserting this into (7.7) and making use of Lemma 7.2 we have

$$\sum_d \frac{h_1(d)f_1(d)}{\tau_k(d)f(d)} y_{dc} = \frac{1}{Y}\Big(\log\frac{\sqrt{D}}{c}\Big)^\ell \bigg(1 + O\bigg(\frac{\varphi(c)}{f(c)}\Big(\log\frac{\sqrt{D}}{c}\Big)^{-1} (\log\log N)^2\bigg)\bigg)$$
$$\times \prod_{p\nmid Q(a^m)c} \Big(1 - \frac{k}{(p-k-1)(p-2)}\Big) \prod_{\substack{p\nmid \Delta c\\ p|Q(a^m)}} \Big(1 + \frac{1}{p-2}\Big),$$

combining with (7.5) and (7.6) we have

$$
\begin{aligned}
&Y^2G_3\\
&= \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}\\(c,\Delta)=1}} \frac{1}{h_3(c)}\frac{\varphi(c)^2g_3(c)^2h_1(c)^2}{\tau_k(c)^2}\Big(\log\frac{\sqrt{D}}{c}\Big)^{2\ell}\bigg(1+O\Big(\frac{\varphi(c)^2(\log\log N)^4}{f(c)^2\log(\sqrt{D}/c)}\Big)\bigg)\\
&\qquad\times\prod_{p\nmid Q(a^m)c}\Big(1-\frac{k}{(p-k-1)(p-2)}\Big)^2\prod_{\substack{p\nmid\Delta c\\p|Q(a^m)}}\Big(1+\frac{1}{p-2}\Big)^2\\
&= \sideset{}{^\flat}\sum_{\substack{c<\sqrt{D}\\(c,\Delta)=1}} \frac{\mathfrak{S}_1^{(m)}}{h_3(c)}\frac{\varphi(c)^2g_3(c)^2h_1(c)^2}{\tau_k(c)^2}\Big(\log\frac{\sqrt{D}}{c}\Big)^{2\ell}\bigg(1+O\Big(\frac{\varphi(c)^2(\log\log N)^4}{f(c)^2\log(\sqrt{D}/c)}\Big)\bigg)\\
&\qquad\times\prod_{\substack{p\nmid Q(a^m)\\p|c}}\Big(1-\frac{k}{(p-k-1)(p-2)}\Big)^{-2}\prod_{p|(c,Q(a^m))}\Big(1+\frac{1}{p-2}\Big)^{-2}\\
&= \sideset{}{^\flat}\sum\sideset{}{^\flat}\sum_{\substack{uv<\sqrt{D}\\(uv,\Delta)=1\\u|Q(a^m),\ (v,Q(a^m))=1}} \frac{\mathfrak{S}_1^{(m)}}{h_3(uv)}\frac{\varphi(uv)^2g_3(uv)^2h_1(uv)^2}{\tau_k(uv)^2}\Big(\log\frac{\sqrt{D}}{uv}\Big)^{2\ell}\prod_{p|u}\Big(1+\frac{1}{p-2}\Big)^{-2}\\
&\qquad\times\prod_{p|v}\Big(1-\frac{k}{(p-k-1)(p-2)}\Big)^{-2}\bigg(1+O\Big(\frac{\varphi(uv)^2(\log\log N)^4}{f(uv)^2\log(\sqrt{D}/uv)}\Big)\bigg),
\end{aligned}
$$

where

$$
\mathfrak{S}_1^{(m)}=\prod_{p\nmid Q(a^m)}\Big(1-\frac{k}{(p-k-1)(p-2)}\Big)^2\prod_{\substack{p\nmid\Delta\\p|Q(a^m)}}\Big(1+\frac{1}{p-2}\Big)^2. \tag{7.8}
$$

It is easy to verify that

$$
\frac{1}{h_3(p)}\frac{\varphi(p)^2g_3(p)^2h_1(p)^2}{\tau_k(p)^2}\Big(1+\frac{1}{p-2}\Big)^{-2}=h_3(p)
$$

for $p\mid Q(a^m)$ and also

$$
\frac{1}{h_3(p)}\frac{\varphi(p)^2g_3(p)^2h_1(p)^2}{\tau_k(p)^2}\Big(1-\frac{k}{(p-k-1)(p-2)}\Big)^{-2}=h_3(p)
$$

for $p \nmid Q(a^m)$. Thus

$$Y^2 G_3 = \mathfrak{S}_1^{(m)} \sideset{}{^\flat}\sum_{\substack{u<\sqrt{D}\\ (u,\Delta)=1\\ u\mid Q(a^m)}} h_3(u) \sideset{}{^\flat}\sum_{\substack{v<\sqrt{D}/u\\ (v,Q(a^m))=1}} h_3(v)\Big(\log\frac{\sqrt{D}}{uv}\Big)^{2\ell} \times\bigg(1+O\bigg(\frac{\varphi(uv)^2(\log\log N)^4}{f(uv)^2\log(\sqrt{D}/uv)}\bigg)\bigg).$$

Making use of Lemma 7.1 we obtain

$$Y^2 G_3 = \mathfrak{S}_1^{(m)} \sideset{}{^\flat}\sum_{\substack{u<\sqrt{D}\\ (u,\Delta)=1\\ u\mid Q(a^m)}} h_3(u)\bigg[\mathfrak{S}_2^{(m)}\frac{(2\ell)!}{(k+2\ell)!}\Big(\log\frac{\sqrt{D}}{u}\Big)^{k+2\ell} + O\bigg(\frac{\varphi(u)^2}{f(u)^2}\Big(\log\frac{\sqrt{D}}{u}\Big)^{k+2\ell-1}(\log\log N)^4\bigg)\bigg],$$

where

$$\mathfrak{S}_2^{(m)} = \prod_{p\nmid Q(a^m)}\Big(1-\frac{1}{p}\Big)^k\Big(1+\frac{k}{p-k-2}\Big)\prod_{p\mid Q(a^m)}\Big(1-\frac{1}{p}\Big)^k. \tag{7.9}$$

Therefore, Lemma 7.2 implies

$$\begin{aligned} Y^2 G_3 = {}& \frac{(2\ell)!\,\mathfrak{S}_1^{(m)}\mathfrak{S}_2^{(m)}}{(k+2\ell)!}\prod_{\substack{p\nmid\Delta\\ p\mid Q(a^m)}}\Big(1+\frac{k-1}{p-k-1}\Big)\cdot(\log\sqrt{D})^{k+2\ell} \\ &\times\bigg(1+O\bigg(\frac{(\log\log N)^k}{\log D}\bigg)\bigg) \\ &+O\bigg(\mathfrak{S}_1^{(m)}\prod_{\substack{p\nmid\Delta\\ p\mid Q(a^m)}}\Big(1+\frac{k-1}{p-k-1}\frac{(p-1)^2}{(p-2)^2}\Big)\cdot(\log\sqrt{D})^{k+2\ell-1}\bigg). \end{aligned} \tag{7.10}$$

The error term can be disposed in the following way:

$$\mathfrak{S}_1^{(m)} \prod_{\substack{p\nmid\Delta \\ p|Q(a^m)}} \left(1+\frac{k-1}{p-k-1}\frac{(p-1)^2}{(p-2)^2}\right)$$
$$\ll \prod_{\substack{p\nmid\Delta \\ p|Q(a^m)}} \left(\frac{p-1}{p-2}\right)^2 \left(1+\frac{k-1}{p-k-1}\frac{(p-1)^2}{(p-2)^2}\right),$$

the product over primes $p \leqslant \log N$ is $O\big((\log\log N)^{k+1}\big)$, while for $p > \log N$ we have

$$\left(\frac{p-1}{p-2}\right)^2 \left(1+\frac{k-1}{p-k-1}\frac{(p-1)^2}{(p-2)^2}\right) \leqslant 1+\frac{2k}{\log N},$$

therefore the corresponding product is

$$\leqslant \left(1+\frac{2k}{\log N}\right)^{\omega(Q(a^m))} \ll \left(1+\frac{2k}{\log N}\right)^{\frac{k\log N}{\log\log N}} = 1+O\Big(\frac{1}{\log\log N}\Big),$$

since $Q(a^m) \leqslant N^{\frac{k}{2}}$. Whence the last O-term in (7.10) is $O\big((\log D)^{k+2\ell-1}(\log\log N)^{k+1}\big)$. Analogously, If we denote by

$$\mathfrak{S}^{(m)} = \frac{\mathfrak{S}_1^{(m)}\mathfrak{S}_2^{(m)}}{\mathfrak{S}(\Delta)} \prod_{\substack{p\nmid\Delta \\ p|Q(a^m)}} \left(1+\frac{k-1}{p-k-1}\right),$$

then it is easy to show that

$$\mathfrak{S}^{(m)} = \prod_{p\nmid\Delta}\left(1-\frac{k}{(p-k-1)(p-2)}\right) \prod_{\substack{p\nmid\Delta \\ p|Q(a^m)}} \left(1+\frac{1}{p-k-2}\right). \tag{7.11}$$

Therefore,

$$\mathfrak{S}_1^{(m)}\mathfrak{S}_2^{(m)} \prod_{\substack{p\nmid\Delta \\ p|Q(a^m)}} \left(1+\frac{k-1}{p-k-1}\right) \ll \log\log N.$$

Hence the total error in (7.10) is $O\big((\log D)^{k+2\ell-1}(\log\log N)^{k+1}\big)$.

Recalling the asymptotic formula of Y and G_1 in (6.9) and (6.11) respectively, we can deduce from (7.10) that

$$G_3 = \mathfrak{S}^{(m)} G_1 + O\left(\frac{G_1 (\log\log N)^{k+1}}{\log D}\right). \tag{7.12}$$

Now we come to the proof of Theorem 1.1. Inserting (6.15) and (7.12) into (5.9), we have

$$\begin{aligned}
&S_3 - S_4 \\
&\geqslant (1 + O(e^{-s}))\frac{N\gamma(\mathcal{H})V(z)G_1}{2}\left[\frac{k(2\ell+1)\mathfrak{S}'(\Delta)\log D}{(\ell+1)(k+2\ell+1)}\left(1 + O\left(\frac{1}{\log D}\right)\right)\right. \\
&\quad \left. + \frac{\Delta}{\varphi(\Delta)}\sum_{a^m \in \mathcal{M}\setminus\mathcal{H}} \mathfrak{S}^{(m)} - \log N + O\big((\log\log N)^{k+1}\big)\right] + O\left(\frac{N}{(\log N)^{A+1}}\right) \\
&\geqslant (1 + O(e^{-s}))\frac{N\gamma(\mathcal{H})V(z)G_1 \log N}{2}\left[\frac{k(2\ell+1)\mathfrak{S}'(\Delta)}{2(\ell+1)(k+2\ell+1)}(1 - 4\varepsilon)\right. \\
&\quad \left. + \frac{1}{2\log a}\prod_{p\nmid\Delta}\left(1 - \frac{k}{(p-k-1)(p-2)}\right) - 1 + O\left(\frac{(\log\log N)^{k+1}}{\log N}\right)\right] \\
&\quad + O\left(\frac{N}{(\log N)^{A+1}}\right).
\end{aligned}$$

Combining with (2.5) and (4.5) we get

$$\begin{aligned}
&S(\mathcal{A}, \mathcal{P}, z) \\
&\geqslant (1 + O(e^{-s}))\frac{N\gamma(\mathcal{H})V(z)G_1 \log N}{2}\left[\frac{k(2\ell+1)\mathfrak{S}'(\Delta)}{2(\ell+1)(k+2\ell+1)}(1 - 4\varepsilon)\right. \\
&\quad \left. + \frac{1}{2\log a}\prod_{p\nmid\Delta}\left(1 - \frac{k}{(p-k-1)(p-2)}\right) - 1 + O\left(\frac{(\log\log N)^{k+1}}{\log N}\right)\right] \\
&\quad + O\left(\frac{N}{(\log N)^{A+1}}\right).
\end{aligned} \tag{7.13}$$

Therefore, $S(\mathcal{A},\mathcal{P},z)$ has a positive lower bound provided that

(7.14)
$$\frac{k(2\ell+1)\mathfrak{S}'(\Delta)}{2(\ell+1)(k+2\ell+1)}(1-4\varepsilon)+\frac{1}{2\log a}\prod_{p\nmid\Delta}\left(1-\frac{k}{(p-k-1)(p-2)}\right)-1>0,$$

we verify this in the following way.

Firstly, (6.14) implies that

$$\mathfrak{S}'(\Delta)=\prod_{p\nmid\Delta}\left(1-\frac{3}{p(p+2)}\right)\geqslant\prod_{n\geqslant k+1}\left(1-\frac{3}{n(n+2)}\right)=\frac{k}{k+3}.$$

Secondly, we have

$$\prod_{p\nmid\Delta}\left(1-\frac{k}{(p-k-1)(p-2)}\right)\geqslant\gamma_k,$$

where

(7.15)
$$\gamma_k=\prod_{p>k+2}\left(1-\frac{k}{(p-k-1)(p-2)}\right)=\prod_{p>k+2}\left(1+\frac{k}{(p-k-2)(p-1)}\right)^{-1}.$$

We can prove that there are infinitely many k such that γ_k has absolute lower-bound by studying the mean value.

Lemma 7.3. *It holds for any $K\geqslant 1$ that*

$$\frac{1}{K}\sum_{\substack{K<k\leqslant 2K\\ 2|k}}\log\frac{1}{\gamma_k}\ll 1,$$

where the implied constant is absolute.

Proof. It is sufficient to prove that

$$\frac{1}{K}\sum_{K<k\leqslant 2K}\ \sum_{k+2<p\leqslant 4K}\frac{k}{(p-k-2)(p-1)}\ll 1.$$

The left hand side is equal to

$$\begin{aligned}
&\frac{1}{K}\sum_{K+2<p\leqslant 4K}\frac{1}{p-1}\sum_{K<k\leqslant \min(2K,p-3)}\frac{k}{p-k-2}\\
&=\frac{1}{K}\Bigg(\sum_{K+2<p\leqslant 2K+3}\frac{1}{p-1}\sum_{K<k\leqslant p-3}\frac{k}{p-k-2}\\
&\qquad\qquad+\sum_{2K+3<p\leqslant 4K}\frac{1}{p-1}\sum_{K<k\leqslant 2K}\frac{k}{p-k-2}\Bigg)\\
&=\frac{1}{K}(\mathcal{K}_1+\mathcal{K}_2)
\end{aligned}$$

say, where

$$\begin{aligned}
\mathcal{K}_1 &\ll \sum_{K+2<p\leqslant 2K+3}\ \sum_{K<k\leqslant p-3}\frac{1}{p-k-2} = \sum_{K+2<p\leqslant 2K+3}\ \sum_{k<p-K-2}\frac{1}{k}\\
&\ll \sum_{K+2<p\leqslant 2K+3}\log p \ll K.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{K}_2 &\ll \sum_{2K+3<p\leqslant 4K}\ \sum_{K<k\leqslant 2K}\frac{1}{p-k-2} = \sum_{2K+3<p\leqslant 4K}\left(\log\frac{p-K-2}{p-2K-2}+O(1)\right)\\
&= -\sum_{2K+3<p\leqslant 4K}\log\left(1-\frac{K}{p-K-2}\right)+O(K)\\
&\ll \sum_{2K+3<p\leqslant 4K}\frac{K}{p-K-2}+K \ll K.
\end{aligned}$$

The desired result is obtained. □

It follows from Lemma 7.3 that γ_k is bounded below by a positive absolute constant for some even number k in any dyadic segment. Choosing such a k, sufficiently large in terms of ε and a, and choosing $\ell = [\sqrt{k}/2]$, we find that the left hand side of (7.14) is positive. This completes the proof of Theorem 1.1.

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