On 2-Absorbing and Weakly 2-Absorbing Primary Ideals of a Commutative Semiring

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Abstract. Let $R$ be a commutative semiring. The purpose of this note is to investigate the concept of 2-absorbing (resp., weakly 2-absorbing) primary ideals generalizing of 2-absorbing (resp., weakly 2-absorbing) ideals of semirings. A proper ideal $I$ of $R$ said to be a 2-absorbing (resp., weakly 2-absorbing) primary ideal if whenever $a, b, c \in R$ such that $abc \in I$ (resp., $0 \neq abc \in I$), then either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Moreover, when $I$ is a $Q$-ideal and $P$ is a $k$-ideal of $R/I$ with $I \subseteq P$, it is shown that if $P$ is a 2-absorbing (resp., weakly 2-absorbing) primary ideal of $R$, then $P/I$ is a 2-absorbing (resp., weakly 2-absorbing) primary ideal of $R/I$ and it is also proved that if $I$ and $P/I$ are weakly 2-absorbing primary ideals, then $P$ is a weakly 2-absorbing primary ideal of $R$.

1. Introduction

We assume that all rings are commutative semiring with non-zero identity. The concept of semiring was studied by Vandive [17] in 1934. A none-empty set $R$ with two binary operations addition and multiplication is called semiring if:

1. $(R, +)$ is a commutative monoid with identity element 0.
2. $(R, \cdot)$ is a monoid with identity element $1 \neq 0$.
3. The multiplication both from left and right is distributes over addition.
4. $0.a = a.0 = 0$ for every $a \in R$.

If $(R, \cdot)$ is a commutative semigroup, so $R$ is a commutative semiring. The set $\mathbb{Z}_0^+$, which denotes the set of all non-negative integer, is a semiring under usual addition and multiplication of non-negative integer but it is not a ring. Semirings have got important structure in rings theory. A non-empty set $I$ is called an ideal if for every $a, b \in I$ and $r \in R$, then $a + b \in I$ and $ra \in I$. The ideal $I$ is called a $k$-ideal (subtractive ideal) if $a, a + b \in I$, then $b \in I$. By definition, every ideal of semiring

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\( R \) is a \( k \)-ideal of \( R \). An ideal \( I \) of semiring \( R \) is called strongly \( k \)-ideal, whenever \( a + b \in I \) for some \( a, b \in R \), then \( a \in I \) and \( b \in I \). Clearly, every strongly \( k \)-ideal is a \( k \)-ideal. Let \( I \) be an ideal of semiring \( R \). \( I \) is also called a \( Q \)-ideal (partitioning ideal) if there exists a subset \( Q \) of \( R \) such that

1. \( R = \bigcup \{ q + I | q \in Q \} \)
2. If \( q_1, q_2 \in Q \), then \( (q_1 + I) \cap (q_2 + I) \neq \emptyset \) if and only if \( q_1 = q_2 \).

Let \( I \) be a \( Q \)-ideal of \( R \) and \( R/I(Q) = \{ q + I | q \in Q \} \). Then \( R/I(Q) \) forms a semiring under the binary operations “\( \oplus \)” and “\( \odot \)” define as follows:

\[
(q_1 + I) \oplus (q_2 + I) = q_3 + I
\]

where \( q_4 \in Q \) is unique such that \( q_1 + q_2 + I \subseteq q_3 + I \).

\[
r \odot (q_1 + I) = q_4 + I
\]

where \( q_4 \in Q \) is unique such that \( rq_1 + I \subseteq q_4 + I \). This semiring \( R/I(Q) \) is said to be the quotient semiring of \( R \) by \( I \). By definition of \( Q \)-ideal, there exists a unique element \( qt \) such that \( 0 + I \subseteq qt + I \), so \( qt + I \) is a zero element of \( R/I \). Let \( R \) be a semiring, \( I \) be a \( Q \)-ideal and \( P \) be a \( k \)-ideal of \( R \) with \( I \subseteq P \). Then \( P/I = \{ q + I | q \in P \cap Q \} \) is a \( k \)-ideal of \( R/I \). If \( I \) is a \( Q \)-ideal of \( R \) and \( L \) a \( k \)-ideal of \( R/I \), then \( L = J/I \) where \( J = \{ r \in R : q_1 + I \subseteq L \} \) is a \( k \)-ideal of \( R \), [3]. If \( R \) and \( S \) are semirings, then a function \( \gamma : R \rightarrow S \) is a morphism of semiring if and only if (1) \( \gamma(0_R) = 0_S \); (2) \( \gamma(1_R) = 1_S \) and (3) \( \gamma(r + s) = \gamma(r) + \gamma(s) \) and \( \gamma(rs) = \gamma(r)\gamma(s) \) for all \( r, s \in R \).

A morphism of semirings which is both monomorphism and epimorphism is called isomorphism. In this case, we write \( R \cong S \). If \( \gamma : R \rightarrow S \) is a morphism of semirings and \( \rho \) is a congruence relation on \( S \), then the relation \( \rho' \) on \( R \) defines by \( r\rho' s \) if and only if \( \gamma(r)\rho\gamma(s) \), is a congruence relation on \( R \). In particular, each morphism of semirings \( \gamma : R \rightarrow S \) defines a congruence relation \( \equiv \gamma \) on \( R \) by setting \( r \equiv s \) if and only if \( \gamma(r) = \gamma(s) \). Let \( \gamma : R \rightarrow S \) be a morphism of semirings. If \( J \) is an ideal of \( S \), then \( \gamma^{-1}(J) \) is an ideal of \( R \). Moreover, if \( J \) is \( k \)-ideal, then so is \( \gamma^{-1}(J) \). If \( \gamma \) is an epimorphism and \( I \) is an ideal of \( R \), then \( \gamma(I) \) is an ideal of \( S \), [12, Proposition 9.46]. If \( \gamma : R \rightarrow S \) is a morphism of semirings, then \( \gamma^{-1}(0) \) is an ideal of \( R \). So it said to be the Kernel of \( \gamma \) and denoted by \( ker(\gamma) \). Therefor another congruence relation defined on \( R \) by \( \gamma \) is the relation \( \equiv_{ker(\gamma)} \). It is obviously true that \( r \equiv_{\gamma} s \) whenever \( r \equiv_{ker(\gamma)} s \). Notice that the converse is not necessary true. When the relation \( \equiv_{\gamma} \) and \( \equiv_{ker(\gamma)} \) coincide, then the morphism \( \gamma \) is called steady. A steady morphism \( \gamma : R \rightarrow S \) is monomorphism if and only if \( ker(\gamma) = \{0\} \), [12, Proposition 9.45].

Let \( R \) be a commutative semiring. Recall that an ideal \( I \) of semiring \( R \) is called proper if \( I \subset R \) and a proper ideal \( I \) of \( R \) is called prime (resp., weakly prime) ideal if whenever \( a, b \in R \) such that \( ab \in I \) (resp., \( 0 \neq ab \in I \)), then either \( a \in I \) or \( b \in I \). A proper ideal \( I \) of \( R \) is called primary (resp., weakly primary) ideal if whenever \( a, b \in R \) such that \( ab \in I \) (resp., \( 0 \neq ab \in I \)), then either \( a \in I \) or \( b^n \in I \).
for some positive integer $n$. In this case, if $I$ is a primary ideal of $R$ and $P := \sqrt{I}$ is a prime ideal of $R$, we call that $I$ is a $P$-primary ideal of $R$. The radical of an ideal $I$ denoted by $\sqrt{I}$ and defined as the set of all elements $a \in R$ such that $a^n \in I$ for some positive integer $n$, that is, $\sqrt{I} = \{a \in R | a^n \in I$ for some positive integer $n\}$. It is an ideal of $R$ containing $I$, and is the intersection of all prime ideals of $R$ containing $I$. It is easy to show that if an ideal $I$ is $k$-ideal, then $\sqrt{I}$ is a $k$-ideal. Furthermore, an element $a \in R$ said to be nilpotent whenever there exists positive integer $n$ such that $a^n = 0$. The set $\{a \in R | a^n = 0$ for some positive integer $n\}$ denoted by $\text{Nil}(R)$.

A. Badawi in [6] introduced a new generalization of prime ideals over a commutative ring. A proper ideal $I$ of a commutative ring $R$ with $I \neq 0$ is said to be a 2-absorbing ideal if whenever $a, b, c \in R$ such that $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. Clearly, every prime ideal is a 2-absorbing ideal. A 2-absorbing (resp., weakly 2-absorbing) ideal of a semiring was introduced by A. Yousefian Darani in [18]. He defined that a proper ideal $I$ of semiring $R$ said to be a 2-absorbing (resp., weakly 2-absorbing) ideal if whenever $a, b, c \in R$ such that $abc \in I$ (resp., $0 \neq abc \in I$), then either $ab \in I$ or $ac \in I$ or $bc \in I$. Recently, A. Badawi, U. Tekir and E. Yetkin in [8] have introduced the concept of 2-absorbing primary ideals over a commutative ring which is a generalization of primary ideals. A proper ideal $I$ of $R$ said to be a 2-absorbing primary ideal if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

In this paper, we will define the concept of 2-absorbing (resp., weakly 2-absorbing) primary ideal of a semiring. Let $R$ be a semiring and $I$ be an ideal of $R$. $I$ is a 2-absorbing (resp., weakly 2-absorbing) primary ideal if whenever $a, b, c \in R$ with $abc \in I$ (resp., $0 \neq abc \in I$), then $ab \in I$ or $ac \in I$ or $bc \in I$. We generalize the concept of strongly 2-absorbing primary ideal. Then a proper ideal $I$ of semiring $R$ calls strongly 2-absorbing primary ideal if whenever $I_1 I_2 I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $R$, then either $I_1 I_2 \subseteq I$ or $I_2 I_3 \subseteq I$ or $I_1 I_3 \subseteq I$. In fact, among the other things we prove that the radical of a 2-absorbing primary ideal of a semiring is a 2-absorbing ideal (Theorem 2.4). It is shown that if $I_1$ is a $P_1$-primary ideal of $R$ and $I_2$ is a $P_2$-primary ideal of $R$, then $I_1 I_2$, $I_1 \cap I_2$ and $P_1 P_2$ are 2-absorbing primary ideals of $R$ (Theorem 2.6). It is shown that if $\sqrt{I}$ is a proper ideal of semiring $R$ such that $I$ is a prime ideal, then $I$ is a 2-absorbing primary ideal of $R$ (Theorem 2.8). It is shown that if $I$ is a $Q$-ideal and $P$ a 2-absorbing primary $k$-ideal of $R/I$ with $I \subseteq P$, then $P/I$ is a 2-absorbing primary ideal of $R/I$ (Theorem 2.11). Let $R = R_1 \times R_2$ where $R_1, R_2$ be commutative semirings. It is shown that $I_1$ (resp., $I_2$) is a 2-absorbing primary ideal of $R_1$ (resp., $R_2$) if and only if $I_1 \times R_2$ (resp., $R_1 \times I_2$) is a 2-absorbing primary ideal of $R$ and $I = I_1 \times I_2$ is a 2-absorbing primary ideal of $R$ if and only if $I = I_1 \times R_2$ for some 2-absorbing primary ideal $I_1$ of $R_1$, or $I = R_1 \times I_2$ for some 2-absorbing primary ideal $I_2$ of $R_2$ or $I = I_1 \times I_2$ for some primary ideal $I_1$ of $R_1$ and for some primary ideal $I_2$ of $R_2$ (Theorems 2.16 and 2.17). It is shown that if $I$ is a proper strongly $k$-ideal of $R$, then $I$ is a 2-absorbing primary ideal if and only if $I_1 I_2 I_3 \subseteq I$ for some ideals $I_1, I_2$ and $I_3$ of $R$, then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq \sqrt{I}$ or $I_2 I_3 \subseteq \sqrt{I}$ (Theorem 2.20). In section 3, we
study the concept of weakly 2-absorbing primary ideal of commutative semirings. Indeed it is shown that if \( I \) is a weakly 2-absorbing primary \( k \)-ideal of \( R \), then either \( I \) is 2-absorbing primary or \( I^2 = 0 \) (Theorem 3.6). In the section 4, is got some characterizations in the semirings \( (\mathbb{Z}_0^+, \text{gcd}, \text{lcm}) \) and \( (\mathbb{Z}_0^+ \cup \{\infty\}, \text{max}, \text{min}) \).

2. 2-Absorbing Primary Ideals in Commutative Semirings

Definition 2.1. Let \( R \) be a semiring and \( I \) be a proper ideal. The ideal \( I \) said to be a 2-absorbing primary ideal if whenever \( a, b, c \in R \) with \( abc \in I \), then either \( ab \in I \) or \( bc \in I \) or \( ac \in I \).

Lemma 2.2. Let \( R \) be a semiring. Then the following statements hold:

1. Every primary ideal is 2-absorbing primary;
2. Every 2-absorbing primary ideal is 2-absorbing primary.

Proposition 2.3. Let \( I \) and \( K \) be ideals of semiring \( R \). If \( I \) is a 2-absorbing primary strongly \( k \)-ideal of \( R \) and \( abK \subseteq I \) for some \( a, b \in R \), then \( ab \in I \) or \( aK \subseteq I \) for some positive integer \( k \).

Proof. Assume that \( ab \notin I \), \( aK \notin I \) and \( bK \notin I \). So there exists \( k_1, k_2 \in K \) such that \( ak_1 \notin I \) and \( bk_2 \notin I \). Since \( abk_1, abk_2 \in I \) and \( I \) is a 2-absorbing primary ideal of \( R \), we conclude that \( bk_1 \in \sqrt{I} \) and \( ak_2 \in \sqrt{I} \). Now since \( ab(k_1 + k_2) \in I \), \( ab \notin I \) and \( I \) is a 2-absorbing primary ideal of \( R \), we have \( a(k_1 + k_2) \in I \) or \( b(k_1 + k_2) \in I \). If \( a(k_1 + k_2) \in I \), since \( I \) is a strongly \( k \)-ideal and \( ak_2 \in \sqrt{I} \), we have \( ak_1 \in \sqrt{I} \), which is a contradiction. If \( b(k_1 + k_2) \in I \), by previous sense and as \( bk_1 \in I \), we conclude that \( bk_2 \in \sqrt{I} \), which is a contradiction. Therefore the result is true.

Theorem 2.4. Let \( R \) be a semiring and \( I \) be an ideal of \( R \). If \( I \) is a 2-absorbing primary ideal of \( R \), then \( \sqrt{I} \) is a 2-absorbing ideal of \( R \).

Proof. Let \( abc \in \sqrt{I} \) for some \( a, b, c \in R \) but \( ac \notin \sqrt{I} \) and \( bc \notin \sqrt{I} \). Then there exists a positive integer \( n \) such that \( (abc)^n = a^n b^n c^n \in I \). Since \( I \) is a 2-absorbing primary ideal of \( R \) and \( ac, bc \notin \sqrt{I} \), we have \( a^n b^n \in I \) and so \( ab \in \sqrt{I} \). Hence \( \sqrt{I} \) is a 2-absorbing ideal of \( R \).

Lemma 2.5. Let \( R \) be a commutative semiring. Then the following statements hold:

1. If \( I \) and \( J \) are ideals of \( R \), then \( \sqrt{IJ} = \sqrt{I} \cap \sqrt{J} = \sqrt{I} \cap \sqrt{J} \);
2. If \( P \) is a prime ideal of \( R \), then \( \sqrt{P} = P \). Moreover, \( \sqrt{P^n} = P \) for some positive integer \( n \).

Theorem 2.6. Let \( R \) be a commutative semiring, \( I_1, I_2 \) be ideals of \( R \) and \( P_1, P_2 \) be prime ideals of \( R \). Suppose that \( I_1 \) is a \( P_1 \)-primary ideal of \( R \) and \( I_2 \) is a \( P_2 \)-primary ideal of \( R \). Then the following statements hold:

1. \( I_1 I_2 \) is a 2-absorbing primary ideal of \( R \);
2. \( I_1 \cap I_2 \) is a 2-absorbing primary ideal of \( R \);
3. \( P_1 P_2 \) is a 2-absorbing primary ideal of \( R \).
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Let \( R \) be a commutative semiring and \( P_1, P_2 \) be prime ideals of \( R \). If \( P_1^n \) is a \( P_1 \)-primary ideal and \( P_2^m \) is a \( P_2 \)-primary ideal for every \( n, m \geq 1 \), then \( P_1^n P_2^m \) and \( P_1^n \cap P_2^m \) are 2-absorbing primary ideals of \( R \).

Example 2.10. Let \( I \) be a 2-absorbing primary principal ideal in semiring \( \mathbb{Z}_0^+ +.\. \). Then \( I = \{0\} \) or \( I = < p^n > \) where \( p \) is a prime number and positive integer \( n > 1 \) or \( I = < p_1^n p_2^m > = < d > \) where \( d = p_1^n p_2^m \) is the power factorization of \( d \) and some positive integer \( n, m > 1 \).

Theorem 2.11. Let \( R \) be a commutative semiring, \( I \) be a \( Q \)-ideal and \( P \) be a \( k \)-ideal of \( R/I \) with \( I \subseteq P \). If \( P \) is a 2-absorbing primary ideal of \( R \), then \( P/I \) is a 2-absorbing primary ideal of \( R/I \).
Proof. Let \( P \) be a 2-absorbing primary ideal of \( R \). Assume that \( q_1 + I, q_2 + I, q_3 + I \in R/I \) such that \( (q_1 + I) \cap (q_2 + I) \cap (q_3 + I) \in P/I \) where \( q_1, q_2, q_3 \in Q \). So there exists a unique element \( q_4 \in P \cap Q \) such that \( q_1q_2q_3 + I \subseteq q_4 + I \in P/I \), then \( q_1q_2q_3 \in P \). Since \( P \) is a 2-absorbing primary ideal, we have \( q_1q_2q_3 \in \sqrt{P} \) or \( q_1q_3 \in \sqrt{P} \). If \( q_1q_2 \in P \), then \( (q_1 + I) \cap (q_2 + I) = q_5 + I \) where \( q_5 \) is the unique element with \( q_1q_2 + I \subseteq q_5 + I \). Hence \( q_5q_3 + r = q_5q_3 + s \) for some \( r, s \in I \), as \( P \) is a \( k \)-ideal and \( q_5 \in P \cap Q \). So \( (q_1 + I) \cap (q_2 + I) \in P/I \). Now we assume that \( q_1q_3 \in \sqrt{P} \). Then there exists positive integer \( n \) such that \( (q_1q_3)^n = q_1^n \sqrt{q_3^n} \in P \). Since \( q_1q_3 \subseteq q_1q_3 + I \), we can conclude that \( (q_1q_3)^n \subseteq (q_1q_3 + I)^n \), thus \( (q_1q_3)^n \subseteq (q_1q_3 + I)^n \cap q_1^n q_3^n + I \), and it follows that \( (q_1q_3 + I)^n = q_1^n q_3^n + I \in P/I \), that is, \( (q_1 + I)^n \cap (q_3 + I)^n \in P/I \). By the similar way, we can show that \( (q_2 + I)^n \cap (q_3 + I)^n \in P/I \), hence \( P/I \) is a 2-absorbing primary ideal of \( R/I \).

In the following we get some characterizations of 2-absorbing primary ideals in the morphisms of semirings.

**Theorem 2.12.** Let \( \gamma : R \to S \) be a morphism of commutative semirings. Then the following statements hold:

1. If \( I \) is a 2-absorbing primary ideal of \( S \), then \( \gamma^{-1}(I) \) is a 2-absorbing primary ideal of \( R \);
2. If \( I \) is a 2-absorbing primary \( k \)-ideal of \( R \) with \( \ker(\gamma) \subseteq I \) and \( \gamma \) is onto steady morphism, then \( \gamma(I) \) is a 2-absorbing primary \( k \)-ideal of \( S \).

**Proof.** (1) Assume that \( a, b, c \in R \) with \( abc \in \gamma^{-1}(J) \). Then \( \gamma(abc) = \gamma(a)\gamma(b)\gamma(c) \in J \). Since \( J \) is a 2-absorbing primary ideal of \( S \), we have \( \gamma(a)\gamma(b) \in J \) or \( \gamma(b)\gamma(c) \in \sqrt{J} \) or \( \gamma(a)\gamma(c) \in \sqrt{J} \). Hence \( ab \in \gamma^{-1}(J) \) or \( bc \in \gamma^{-1}(\sqrt{J}) \) or \( ac \in \gamma^{-1}(\sqrt{J}) \). Obviously \( a \gamma^{-1}(\sqrt{J}) = \sqrt{\gamma^{-1}(J)} \). Therefore \( \gamma^{-1}(J) \) is a 2-absorbing primary ideal of \( R \).

(2) Assume that \( I \) is a 2-absorbing primary ideal of \( R \) and \( \ker(\gamma) \subseteq I \). Clearly, \( \gamma(I) \) is a \( k \)-ideal of \( S \). Let \( abc \in \gamma(I) \) for some \( a, b, c \in S \). There exists \( x, y, z \in R \) such that \( \gamma(x) = a, \gamma(y) = b \) and \( \gamma(z) = c \). Then \( abc = \gamma(x)\gamma(y)\gamma(z) = \gamma(xyz) \subseteq \gamma(I) \) and so \( \gamma(xyz) = \gamma(r) \) for some \( r \in I \). Since \( \gamma \) is steady, \( xyz + s = r + t \) for some \( s, t \in I \). Hence \( xyz \in I \), as \( I \) is a \( k \)-ideal of \( R \) and \( \ker(\gamma) \subseteq I \). Since \( I \) is 2-absorbing primary, we have \( xyz \in I \) or \( yzx \in \sqrt{I} \) or \( xz \in \sqrt{I} \). Thus \( ab \in \gamma(I) \) or \( bc \in \gamma(\sqrt{I}) \subseteq \sqrt{\gamma(I)} \) or \( ac \in \gamma(\sqrt{I}) \subseteq \sqrt{\gamma(I)} \). Therefore \( \gamma(I) \) is 2-absorbing primary.

Let \( I \) and \( J \) be ideals of \( S \) with \( I \subseteq J \). Then \( J/I = \{a + I \mid a \in J \} \) is an ideal of \( R \). Moreover, if \( J \) is a \( k \)-ideal of \( R \), then \( J/I \) is a \( k \)-ideal of \( R/J \). [5, Lemma 2]. In the following we can use it to show next result.

**Corollary 2.13** Let \( R \) be a commutative semiring and \( J \) be an ideal of \( R \). If \( I \) is a 2-absorbing primary \( k \)-ideal of \( R \) with \( J \subseteq I \), then \( I/J \) is a 2-absorbing primary ideal of \( R/J \).

A non-empty subset \( S \) of a semiring \( R \) said to be multiplicatively closed subset whenever \( a, b \in S \) implies that \( ab \in S \). Let \( S \) be a multiplicatively closed subset of
a semiring $R$. The relation is defined on the set $R \times S$ by $(r, s) \sim (t, y) \iff u_{ry} = u_{ts}$ for some $u \in S$ is an equivalence relation and the equivalence class of $(r, s) \in R \times S$ denoted by $r/s$. The set of all equivalence classes of $R \times S$ under “~” is denoted by $S^{-1}R$. The addition and multiplication are defined $r/s + t/y = (ry + ts)/sy$ and $(r/s)(t/y) = rt/sy$. The semiring $S^{-1}R$ is called quotient semiring $R$ by $S$.

Suppose that $R$ is a commutative semiring, $S$ be a multiplicatively closed subset and $I$ be an ideal. The set $S^{-1}I = \{a/b| a \in I, b \in S}\}$ is an ideal of $S^{-1}R$. It is easy to show that if $I$ is a $k$-ideal, then $S^{-1}I$ is a $k$-ideal of $S^{-1}R$, (see [12, 14, 15]). Clearly, we get some results that follow by $(r/s) = (t/y) \iff u_{ry} = u_{ts}$ for some $u \in S$ and $r/s = ar/as$ for all $a \in R$ and $r, s \in S$; its zero element is $0/1$ and its multiplicative identity element is $1/1$.

**Theorem 2.14.** Let $R$ be a commutative semiring and $S$ be a multiplicatively closed subset and $I$ be a $k$-ideal of $R$. If $I$ is a $2$-absorbing primary ideal of $R$ with $I \cap S = \emptyset$, then $S^{-1}I$ is a $2$-absorbing primary ideal of $S^{-1}R$.

**Proof.** Assume that $a, b, c \in R$ and $s, t, r \in S$ with $(a/s)(b/t)(c/r) \in S^{-1}I$. Then there exists $u \in S$ such that $(ua)bc \in I$. As $I$ is a $2$-absorbing primary ideal of $R$, we conclude that $(ua)b \in I$ or $bc \in \sqrt{I}$ or $(ua)c \in \sqrt{I}$. Firstly, if $(ua)b \in I$, then $(a/s)(b/t) = uab/ust \in S^{-1}I$. If $bc \in \sqrt{I}$, then $(b/t)(c/r) \in S^{-1}(\sqrt{I}) = \sqrt{S^{-1}I}$. Finally, if $(ua)c \in \sqrt{I}$, then $(a/s)(c/r) = uac/urs \in \sqrt{S^{-1}I}$. Hence $S^{-1}I$ is a $2$-absorbing primary ideal of $S^{-1}R$. \hfill $\square$

**Proposition 2.15.** Let $R$ be a commutative Semiring and $P$ be a $2$-absorbing primary ideal of $S^{-1}R$. Then $P \cap R$ is a $2$-absorbing primary ideal of $R$.

**Proof.** Assume that $a, b, c \in R$ with $abc \in P \cap R$. Then $(a/1)(b/1)(c/1) \in P \cap R$. Since $P$ is a $2$-absorbing primary ideal of $S^{-1}R$, we have $(a/1)(b/1) \in P$ or $(b/1)(c/1) \in \sqrt{P}$ or $(a/1)(c/1) \in \sqrt{P}$. Hence $ab \in P \cap R$ or $bc \in \sqrt{P \cap R}$ or $ac \in \sqrt{P \cap R}$. Therefore $P \cap R$ is a $2$-absorbing primary ideal of $R$. \hfill $\square$

**Theorem 2.16.** Let $R = R_1 \times R_2$ where $R_1, R_2$ be commutative semirings. Then the following statements hold:

1. $I_1$ is a $2$-absorbing primary ideal of $R_1$ if and only if $I_1 \times I_2$ is a $2$-absorbing primary ideal of $R$;
2. $I_2$ is a $2$-absorbing primary ideal of $R_2$ if and only if $R_1 \times I_2$ is a $2$-absorbing primary ideal of $R$.

**Proof.** (1) Let $I_1$ be a $2$-absorbing primary ideal of $R_1$. Assume that $(a, 1)(b, 1)\ (c, 1) = (abc, 1) \in I_1 \times R_2$ such that $a, b, c \in R_1$. Then $abc \in I_1$ and so we conclude that $ab \in I_1$ or $bc \in \sqrt{I_1}$ or $ac \in \sqrt{I_1}$. Hence $(ab, 1) \in I_1 \times R_2$ or $(bc, 1) \in \sqrt{I_1 \times R_2} = \sqrt{I_1} \times R_2$ or $(ac, 1) \in \sqrt{I_1 \times R_2} = \sqrt{I_1} \times R_2$. Therefore $I_1 \times R_2$ is a $2$-absorbing primary ideal of $R$. Conversely, the proof is trivial.

(2) The proof is similar (1). \hfill $\square$

**Theorem 2.17.** Let $R = R_1 \times R_2$ where $R_1, R_2$ be commutative semirings and $I = I_1 \times I_2$ be an ideal of $R$ such that $I_1$ and $I_2$ are ideals of $R_1$ and $R_2$ respectively. Then the following statements are equivalent:
(1) $I$ is a 2-absorbing primary ideal of $R$;
(2) $I = I_1 \times I_2$ for some 2-absorbing primary ideal $I_1$ of $R_1$ or $I = R_1 \times I_2$ for some 2-absorbing primary ideal $I_2$ of $R_2$ or $I = I_1 \times I_2$ for some primary ideal $I_1$ of $R_1$ and for some primary ideal $I_2$ of $R_2$.

Proof. (1) $\Rightarrow$ (2) Assume that $I$ is a 2-absorbing primary ideal of $R$. If $I_2 = R_2$, then $I$ is 2-absorbing primary, by Theorem 2.16. If $I_1 = R_1$, then $I$ is 2-absorbing primary, by Theorem 2.16. Suppose that $I_2 \neq R_2$ and $I_1 \neq R_1$. On the other hand $\sqrt{I} = \sqrt{T_1} \times \sqrt{T_2}$. Assume that $I_1$ is not a primary ideal of $R_1$. So there exists $a, b \in R_1$ such that $ab \in I_1$ but $a \notin I_1$ and $b \notin \sqrt{T_1}$. Let $x = (a, 1), y = (1, 0)$ and $z = (b, 1)$. Hence $xyz = (a, 1)(1, 0)(b, 1) = (ab, 0) \in I$ but neither $(a, 1)(1, 0) \in I$ nor $(1, 0)(b, 1) \in \sqrt{T}$ nor $(a, 1)(b, 1) \in \sqrt{T}$, which is a contradiction. Then $I_1$ is a primary ideal of $R_1$. Now assume that $I_2$ is not a primary ideal of $R_2$. Then there are $c, d \in R_2$ such that $cd \in I_2$ but $c \notin I_2$ and $d \notin \sqrt{T_2}$. Let $x = (1, c), y = (0, 1)$ and $z = (1, d)$. Hence $xyz = (1, c)(0, 1)(1, d) = (0, cd) \in I$ but neither $(1, c)(0, 1) \in I$ nor $(0, 1)(1, d) \in \sqrt{T}$ nor $(1, c)(1, d) \in \sqrt{T}$, which is a contradiction. Hence $I_2$ is a primary ideal of $R_2$.

(2) $\Rightarrow$ (1) If $I_2 = R_2$ and $I_1$ is a 2-absorbing primary ideal of $R_1$, then $I = I_1 \times R_2$ is a 2-absorbing primary ideal of $R$, by Theorem 2.16. Similarly, if $I_1 = R_1$ and $I_2$ is a 2-absorbing primary ideal of $R_2$, then $R_1 \times I_2$ is a 2-absorbing primary ideal of $R$. Now assume that $I_1$ and $I_2$ are primary ideals of $R_1$ and $R_2$ respectively. Suppose that $(a_1, b_1)(a_2, b_2)(a_3, b_3) \in I_1 \times I_2$ for some $a_1, a_2, a_3 \in R_1$ and $b_1, b_2, b_3 \in R_2$. Since $I_1$ and $I_2$ are primary ideals, we may assume that one of $a_i$’s is in $I_1$, say $a_1$ and one of $b_i$’s is in $I_2$, say $b_2$. Hence $(a_1, b_1)(a_2, b_2) \in I_1 \times I_2$. Consequently, $I_1 \times I_2$ is a 2-absorbing primary ideal of $R$.

Example 2.18. Let $R = \mathbb{Z}_0^+ \times \mathbb{Z}_0^+$ be a semiring.

(1) We consider $I_1 = 12\mathbb{Z}$ and $I_2 = 6\mathbb{Z}$ which are 2-absorbing primary ideals of $\mathbb{Z}_0^+$. Then $I = 12\mathbb{Z} \times 6\mathbb{Z}$ is a 2-absorbing primary ideal. However, they are not primary ideals.

(2) Assume that $J = 4\mathbb{Z} \times 6\mathbb{Z}$ is an ideal of $R$. As we know that $4\mathbb{Z}$ is a primary ideal and $6\mathbb{Z}$ is not a primary ideal. Although it is a 2-absorbing primary ideal. Then it is easy to see that $J = 4\mathbb{Z} \times 6\mathbb{Z}$ is a 2-absorbing primary ideal of $R$.

Definition 2.19. Let $R$ be a commutative semiring and $I$ be a proper ideal of $R$. The ideal $I$ is said to be a strongly 2-absorbing primary ideal if whenever $I_1I_2I_3 \subseteq I$ for some ideals $I_1, I_2, I_3$ of $R$, then either $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{T}$ or $I_1I_3 \subseteq \sqrt{T}$.

Theorem 2.20. Let $R$ be a commutative semiring and $I$ be a proper strongly k-ideal of $R$. Then the following statements are equivalent:

(1) $I$ is a 2-absorbing primary ideal;
(2) If $I_1I_2I_3 \subseteq I$ for some ideals $I_1, I_2$ and $I_3$ of $R$, then $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{T}$ or $I_1I_3 \subseteq \sqrt{T}$.

Proof. (1) $\Rightarrow$ (2) Assume that $I$ is a 2-absorbing primary ideal of $R$ and $I_1I_2I_3 \subseteq I$ for some ideals $I_1, I_2$ and $I_3$ of $R$. Let $I_1I_2 \notin I$, $I_2I_3 \notin \sqrt{T}$ and $I_1I_3 \notin \sqrt{T}$. Then there exists $i_1 \in I_1$ and $i_2 \in I_2$ such that $i_1i_2I_3 \subseteq I$ and $i_1I_2 \notin \sqrt{T}$ and $i_2I_3 \notin \sqrt{T}$.
Hence $i_1i_2 \subseteq I$, by Proposition 2.3. Since $I_1I_2 \nsubseteq I$, there exists $a \in I_1$ and $b \in I_2$ such that $ab \notin I$. By Proposition 2.3 and since $abI_3 \subseteq I$ and $I$ is 2-absorbing primary, we have $aI_3 \subseteq \sqrt{I}$ or $bI_3 \subseteq \sqrt{I}$. Now we have three cases:

**Case I:** We assume that $aI_3 \subseteq \sqrt{I}$ but $bI_3 \nsubseteq \sqrt{I}$. Since $i_1bI_3 \subseteq I$ but $bI_3 \nsubseteq \sqrt{I}$ and $i_1I_3 \nsubseteq \sqrt{I}$, we have $i_1b \in I$, by Proposition 2.3. We have $aI_3 \subseteq \sqrt{I}$ but $i_1I_3 \nsubseteq \sqrt{I}$, then $(a + i_1)I_3 \nsubseteq \sqrt{I}$. Since $I$ is a strongly $k$-ideal. On the other hand, $(a + i)bI_3 \subseteq I$, $bI_3 \nsubseteq \sqrt{I}$ and $(a + i_1)I_3 \nsubseteq \sqrt{I}$, we conclude that $(a + i)b \in I$, by Proposition 2.3. Then $ab \in I$ as $I$ is a strongly $k$-ideal, which is a contradiction.

**Case II:** We assume that $aI_3 \nsubseteq \sqrt{I}$ but $bI_3 \subseteq \sqrt{I}$. Hence the complete proof is the same way by Case I.

**Case III:** We assume that $aI_3 \subseteq \sqrt{I}$ and $bI_3 \subseteq \sqrt{I}$. At the first we consider $bI_3 \subseteq \sqrt{I}$. Since $i_2I_3 \nsubseteq \sqrt{I}$ and $I$ is a strongly $k$-ideal, we can conclude that $(b + i_2)I_3 \nsubseteq \sqrt{I}$. Since $i_1(b + i_2)I_3 \subseteq I$ but $i_1I_3 \nsubseteq \sqrt{I}$ and $(b + i_2)I_3 \nsubseteq \sqrt{I}$, we have $i_1(b + i_2) \notin I$. Then $i_1b \in I$ and $i_2i_2 \in I$. Since $I$ is a strongly $k$-ideal. Now we consider $aI_3 \subseteq \sqrt{I}$ but $i_1I_3 \nsubseteq \sqrt{I}$, so $(a + i_1)I_3 \nsubseteq \sqrt{I}$. As $(a + i_1)i_2I_3 \subseteq I$ but $(a + i_1)I_3 \nsubseteq \sqrt{I}$ and $i_2I_3 \nsubseteq \sqrt{I}$, we conclude that $(a + i_1)i_2 \notin I$. Then $ai_2 \in I$ and $i_2i_2 \in I$. Now as $(a + i_1)(b + i_2)I_3 \subseteq I$ but $(a + i_1)I_3 \nsubseteq \sqrt{I}$ and $(b + i_2)I_3 \nsubseteq \sqrt{I}$, we can conclude that $(a + i_1)(b + i_2) = ab + c \in I$ and so $ab \in I$, which is a contradiction. Therefore $I_1I_2 \subseteq I$ or $I_2I_3 \subseteq \sqrt{I}$ or $I_1I_3 \subseteq \sqrt{I}$.

(2) $\Rightarrow$ (1) The proof is straightforward.

One of the main sense, that is generalized for semirings, is the concept of primary decomposition. Let $R$ be a commutative semiring and $I$ be a proper ideal of $R$. A primary decomposition of $I$ is an epithet for $I$ as an intersection of finitely many primary ideals of $R$. On the other words a primary decomposition of $I$ is $I = I_1 \cap \cdots \cap I_r$ where each $I_i$ is $P_i$-primary ideal in semiring $R$. It is easy to show that if $R$ is a Noetherian semiring, then every proper $k$-ideal is a finite intersection of primary $k$-ideals. Since every primary ideal of a semiring $R$ is a 2-absorbing primary ideal, we claim that every proper ideal of $R$ has a 2-absorbing primary decomposition. In the next, we define the concept of $P$-2-absorbing primary ideal that is generalization of the concept of $P$-primary ideal in semirings.

**Definition 2.21.** Let $R$ be a semiring and $I$ be a 2-absorbing primary ideal of $R$. If $\sqrt{I} = P$ is a 2-absorbing ideal of $R$, then $I$ is called $P$-2-absorbing primary ideal of $R$.

The following theorem gives a characterization of $P$-2-absorbing primary ideals of semiring $R$.

**Theorem 2.22.** Let $I_1, \cdots, I_r$ be $P$-2-absorbing primary ideals of semiring $R$ where $P$ is a 2-absorbing ideal of $R$. Then $I = \bigcap_{i=1}^{r} I_i$ is a $P$-2-absorbing primary ideal of $R$.

**Proof.** Assume that $abc \in I$ for some $a, b, c \in R$ and $ab \notin I$. Then $ab \notin I_i$ for some $1 \leq i \leq n$. Since every $I_i$ is $P$-2-absorbing primary ideal, we can conclude that $ac \in \sqrt{I_i} = P$ or $bc \in \sqrt{I_i} = P$. Therefore $I$ is a $P$-2-absorbing primary ideal of
3. Weakly 2-Absorbing Primary Ideals in Commutative Semirings

In this section we define the concept of weakly 2-absorbing primary ideal of a commutative semiring and generalize some basic results in semirings.

**Definition 3.1.** Let $R$ be a semiring and $I$ be a proper ideal. The ideal $I$ is said to be a **weakly 2-absorbing primary ideal** if whenever $a,b,c \in R$ with $0 \neq abc \in I$, then either $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$.

**Lemma 3.2.** Let $R$ be a semiring. Then the following statements hold:
1. Every weakly primary ideal is weakly 2-absorbing primary;
2. Every 2-absorbing primary ideal is weakly 2-absorbing primary.

**Theorem 3.3.** Let $R$ be a commutative semiring, $I$ be a Q-ideal and $P$ be a k-ideal of $R/I$ with $I \subseteq P$. Then the following statements hold:
1. If $P$ is a weakly 2-absorbing primary ideal of $R$, then $P/I$ is a weakly 2-absorbing primary ideal of $R/I$;
2. If $I$ and $P/I$ are weakly 2-absorbing primary ideals, then $P$ is a weakly 2-absorbing primary ideal of $R$.

*Proof.* (1) Assume that $q_1 + I, q_2 + I, q_3 + I \in R/I$ such that $0 \neq (q_1 + I) \circ (q_2 + I) \circ (q_3 + I) \in P/I$ where $q_1, q_2, q_3 \in Q$ and $0 \neq q_1 q_2 q_3 \in I$. Now this part proves completely similar Theorem 2.11.

(2) Assume that $I$ and $P/I$ are weakly 2-absorbing primary ideals. Let $0 \neq abc \in P$ for some $a,b,c \in R$. If $abc \in I$, then $ab \in I \subseteq P$ or $(bc)^n \in I \subseteq P$ or $(ac)^n \in I \subseteq P$. Since $I$ is a weakly 2-absorbing primary ideal. So we can assume that $abc \notin I$. Then there exists $q_1, q_2, q_3 \in Q$ such that $a \in q_1 + I$, $b \in q_2 + I$ and $c \in q_3 + I$. Hence $a = q_1 + e$, $b = q_2 + f$ and $c = q_3 + g$ for some $e, f, g \in I$. Since $abc = (q_1 + e)(q_2 + f)(q_3 + g) = q_1 q_2 q_3 + q_1 q_3 f + q_2 q_3 e + q_1 e f + q_2 q_3 g + q_3 q_2 g + q_2 e g + q_3 f g$ and $P$ is a k-ideal, we have $q_1 q_2 q_3 \in P$. Assume that $q$ is the unique element in $Q$ such that $(q_1 + I) \circ (q_2 + I) \circ (q_3 + I) = q + I$ where $q_1 q_2 q_3 + I \subseteq q + I$. Then $q_1 q_2 q_3 + i = q + h$ for some $i, h \in I$ and so $q \in P \cap Q$ and $q + I \in P/I$. Now suppose that $q + I$ is the unique element such that $q' + I$ is the zero element in $R/I$. If $(q_1 + I) \circ (q_2 + I) \circ (q_3 + I) = q' + I$, then $q_1 q_2 q_3 + j = q' + l$ for some $j, l \in I$. As $I$ is a Q-ideal of $R$, it is a k-ideal by [16, Corollary 2]. Thus $q_1 q_2 q_3 \in I$ and so $abc \in I$, which is a contradiction. Hence $(q_1 + I) \circ (q_2 + I) \circ (q_3 + I) \in P/I$. Since $P/I$ is a weakly 2-absorbing primary ideal, we conclude $q_1 q_2 + I \in P/I$ or $(q_2 q_3 + I)^n \in P/I$ or $(q_1 q_3 + I)^n \in P/I$ for some $n$. If $q_1 q_2 + I \in P/I$, then $ab = q_1 q_2 + ef \in P$. If $(q_1 q_3 + I)^n = q_1 q_3 q_2 + I \in P/I$, then it follows that $(ac)^n \in P$. In a similar way, we can show that $(bc)^n \in P$. Then it follows that either $q_1 q_2 \in P$ or $q_2 q_3 \in \sqrt{P}$ or $q_1 q_3 \in \sqrt{P}$. Hence $ab \in P$ or $bc \in \sqrt{P}$ or $ac \in \sqrt{P}$. Therefore $P$ is a weakly 2-absorbing primary ideal of semiring $R$. □

**Example 3.4.** Let $R = Z_{12}$ be a commutative semiring and $I = \{0\}$ be an ideal of $R$. Then $I$ is a weakly 2-absorbing primary ideal of $R$, by definition. Now we
consider $2.2.3 \in I$ but neither $2.2 \in I$ nor $2.3 \in \sqrt{I}$. So $I$ is not a 2-absorbing primary ideal. It is noticeable that every 2-absorbing primary ideal is a weakly 2-absorbing ideal by Lemma 3.2, but a weakly 2-absorbing primary ideal need not to be a 2-absorbing primary ideal. In the following result we show that provided which conditions it can be possible.

**Lemma 3.5.** Let $R$ be a commutative semiring and $I$ be a $k$-ideal of $R$. If $a \in I$ and $a + b \in \sqrt{I}$ for some $a, b \in R$, then $b \in \sqrt{I}$.

**Theorem 3.6.** Let $R$ be a commutative semiring and $I$ be an ideal of $R$. If $I$ is a weakly 2-absorbing primary $k$-ideal of $R$, then either $I$ is 2-absorbing primary or $I^3 = 0$.

**Proof.** Assume that $I^3 \neq 0$. We show that $I$ is a 2-absorbing primary ideal of $R$. Let $a, b, c \in R$ such that $abc \in I$. If $abc \neq 0$, then $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$, that is, $I$ is a 2-absorbing primary ideal of $R$. So we suppose that $abc = 0$. At the first, we assume that $abI \neq 0$ and say $abI = 0$ for some $r_0 \in I$. Then $0 \neq abI = ab(c + r_0) \in I$. As $I$ is weakly 2-absorbing primary, we get that $ab \in I$ or $b(c + r_0) \in \sqrt{I}$ or $a(c + r_0) \in \sqrt{I}$. Then $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$, by Lemma 3.5. So we can suppose that $abI = 0$. Likewise we can assume that $acI = 0$ and $bcI = 0$. Since $I^3 \neq 0$, there exists $a_0, b_0, c_0 \in I$ with $a_0b_0c_0 \neq 0$. If $a_0b_0c_0 \neq 0$, then $a(b + b_0)(c + c_0) \in I$, so it implies that $a(b + b_0) \in I$ or $(b + b_0)(c + c_0) \in \sqrt{I}$ or $(c + c_0) \in \sqrt{I}$. Hence $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$, by Lemma 3.5. So we can also assume that $a_0b_0c_0 = 0$. Likewise we can consider $a_0bc_0 = 0$ and $a_0b_0c = 0$. Now we can conclude that $0 \neq a_0b_0c_0 = (a + a_0)(b + b_0)(c + c_0) \in I$, so we get $(a + a_0)(b + b_0) \in I$ or $(b + b_0)(c + c_0) \in \sqrt{I}$ or $(a + a_0)(c + c_0) \in \sqrt{I}$. By Lemma 3.5, $ab \in I$ or $bc \in \sqrt{I}$ or $ac \in \sqrt{I}$. Hence $I$ is a 2-absorbing primary ideal of $R$. □

We can now use Theorem 3.6 to characterize weakly 2-absorbing primary ideals in semirings.

**Corollary 3.7.** Let $R$ be a commutative semiring and $I$ be a weakly 2-absorbing primary $k$-ideal of $R$. If $I$ is not a 2-absorbing primary ideal, then $\sqrt{I} = \sqrt{0}$.

**Proof.** Clearly $\sqrt{0} \subseteq \sqrt{I}$. By Theorem 3.6, $I^3 = 0$. So we get $I \subseteq \sqrt{0}$, then $\sqrt{I} \subseteq \sqrt{0}$. Hence $\sqrt{I} = \sqrt{0}$. □

**Corollary 3.8.** Let $R$ be a commutative semiring. If $I$ is a weakly 2-absorbing primary $k$-ideal of $R$ that is not 2-absorbing primary ideal, then $I$ is nilpotent.

**Proposition 3.9.** Let $R$ be a commutative Semiring and $P$ be a weakly 2-absorbing primary ideal of $S^{-1}R$. Then $P \cap R$ is a weakly 2-absorbing primary ideal of $R$.

**Proof.** This following from Proposition 2.15. □

**Theorem 3.10.** Let $R = R_1 \times R_2$ where $R_1, R_2$ be commutative semirings and $I = I_1 \times I_2$ be an ideal of $R$ such that $I_1$ and $I_2$ are ideals of $R_1$ and $R_2$ respectively. If $I$ is a weakly 2-absorbing primary ideal of $R$, then either $I = 0$ or $I$ is 2-absorbing primary.
4. Properties of 2-Absorbing Primary Ideals in Semiring $\mathbb{Z}_0^+$

In this section we give characterizations of 2-absorbing primary ideal in semiring $\mathbb{Z}_0^+$. In the following theorems we get that some results in semiring $(\mathbb{Z}_0^+, \gcd, \text{lcm})$ where $a \otimes b = \gcd\{a, b\}$ and $a \otimes b = \text{lcm}\{a, b\}$ for $a, b \in \mathbb{N}$. $a \otimes 0 = a$ and $a \otimes 0 = 0$ for all $a \in \mathbb{Z}_0^+$.

**Theorem 4.1.** A non-zero ideal $I$ of semiring $(\mathbb{Z}_0^+, \gcd, \text{lcm})$ is 2-absorbing primary ideal if and only if $I$ is a 2-absorbing ideal.

**Proof.** Assume that $I = I_1 \times I_2$ is a weakly 2-absorbing primary and $I \neq 0$. We show that $I$ is 2-absorbing primary. Let $(a, b) \in I = I_1 \times I_2$ such that $(a, b) \neq (0, 0)$. Then (0, 0) $\neq (a, 1)(1, 1)(1, b) \in I$. So either $(a, 1)(1, 1) \in I$ or $(1, 1)(1, b) \in \sqrt{I}$ or $(a, 1)(1, b) \in \sqrt{I}$. If $(a, 1) \in I$, then $(a, 1) \in I_1 \times R_2$. We show that $I_1$ is a 2-absorbing primary ideal of $R_1$. Let $x, y, z \in R_1$ such that $xyz \in I_1$. Then (0, 0) $\neq (x, 1)(y, 1)(z, 1) \in I$. Since $I$ is weakly 2-absorbing primary, we have $(x, 1)(y, 1) \in I_1 \times R_2$ or $(y, 1)(z, 1) \in \sqrt{I_1 \times R_2} = \sqrt{I_1 \times R_2} = \sqrt{I_1 \times R_2}$ and so $xyz \in I_1$ or $yz \in \sqrt{I_1}$ or $xz \in \sqrt{I_1}$. Then $I_1 \times R_2$ is a 2-absorbing primary ideal of $R_1$, by Theorem 2.16. If $(1, b) \in \sqrt{I_1 \times I_2}$, then $(1, b^n) \in I_1 \times I_2$ for some positive integer $n$ and so $I = R_1 \times I_2$. By similar way, $R_1 \times I_2$ is a 2-absorbing primary ideal. Now if $(a, 1)(1, b) \in \sqrt{I_1 \times I_2}$, we have $(a^n, b^n) \in I_1 \times I_2$ for some positive integer $n$. We show that $I_1$ and $I_2$ are primary ideals. Suppose that $R_2 \neq I_2$. Let $a, b \in R_2$ such that $ab \in I_2$ and $0 \neq i_1 \in I_1$. Then (0, 0) $\neq (i_1, 1)(1, a)(1, b) = (i_1, ab) \in I_1 \times I_2$. Since $(1, a)(1, b) \notin \sqrt{I_1 \times I_2}$, we can conclude that $(i_1, 1)(1, a) \in I_1 \times I_2$ or $(1, i_1)(1, b) \in \sqrt{I_1 \times I_2}$. Then $a \in I_2$ or $b \in \sqrt{I_2}$, that is, $I_1$ is a primary ideal. Similarly, we assume that $c, d \in R_1$ such that $cd \in I_1$ and $0 \neq i_2 \in I_2$. Hence (0, 0) $\neq (1, i_2)(c, 1)(d, 1) = (cd, i_2) \in I_1 \times I_2$ and as $R_1 \neq I_1$, we have $(c, 1)(d, 1) \notin \sqrt{I_1 \times I_2}$. Then we can conclude that $(1, i_2)(c, 1) \notin I_1 \times I_2$ or $(1, i_2)(d, 1) \in \sqrt{I_1 \times I_2}$. Hence either $c \in I_1$ or $d \in \sqrt{I_1}$ and so $I_1$ is a primary ideal. Therefore $I_1 \times I_2$ is a 2-absorbing primary ideal of $R_1$. By Theorem 2.17.

**Theorem 4.2.** A non-zero ideal $I$ of semiring $(\mathbb{Z}_0^+, \gcd, \text{lcm})$ is 2-absorbing primary ideal if and only if $I = \langle p^n \rangle$ for some positive integer $n > 1$ or $I = \langle p_1^n p_2^m \rangle$ for some pairwise distinct prime numbers $p_1, p_2$ and some positive integer $n, m > 1$.

**Proof.** Assume that $I$ is a 2-absorbing primary ideal. Then by Theorem 2.22, $I$ is a 2-absorbing ideal. By [10, Lemma 2.2], $I = \langle d \rangle$ such that $d \in \mathbb{Z}_0^+ \setminus \{0, 1\}$. Set $d = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where $p_1, p_2, \cdots, p_k$ are pairwise distinct prime. Now we consider $k > 2$. Then $p_1^{r_1} \otimes p_2^{r_2} \otimes (p_3^{r_3} \otimes \cdots \otimes p_k^{r_k}) = d \in I$ so neither $p_1^{r_1} \otimes p_2^{r_2} \in I$ nor
$p_2^{n_2} \otimes (p_3^{n_3} \otimes \cdots \otimes p_k^{n_k}) \in I$ nor $p_1^{n_1} \otimes (p_3^{n_3} \otimes \cdots \otimes p_k^{n_k}) \in I$, that is a contradiction. Hence $k \leq 2$. Therefore $I = < p^n >$ for some positive integer $n > 1$ or $I = < p_1^n p_2^n >$ for some pairwise distinct prime numbers $p_1, p_2$ and some positive integer $n, m > 1$.

Conversely, if $I = < p_1^n >$ for some positive integer $n > 1$, we are done. So we assume that $I = < p_1^n p_2^n >$ for some pairwise distinct prime numbers $p_1, p_2$ and some positive integer $n, m > 1$. Then $I = < p_1^n > \cap < p_2^n >$. Since $p_1, p_2$ are prime numbers, $< p_1^n >$ and $< p_2^n >$ are prime ideals, by [10, Theorem 2.7]. Hence $I$ is a 2-absorbing primary ideal, by Theorem 2.6.

Let $R = (\mathbb{Z}_0^+ \cup \{\infty\}, \max, \min)$ be a semiring with identity $\infty$, where $a \lor b = \max\{a, b\}$ and $a \land b = \min\{a, b\}$. Moreover, if $I$ is an ideal of $R$, then $I = \{0, 1, 2, \cdots, t\}$ for some $t \in \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^+$ or $I = R$, [13, Theorem 5]. In the next theorem we show that a characterization in semiring $R = (\mathbb{Z}_0^+ \cup \{\infty\}, \max, \min)$.

**Theorem 4.3.** Every ideal in $R = (\mathbb{Z}_0^+ \cup \{\infty\}, \max, \min)$ is a 2-absorbing primary ideal.

**Proof.** Let $I$ be a proper ideal of $R$. Then $I = \{0, 1, 2, \cdots, t\}$ for some $t \in \mathbb{Z}_0^+$ or $I = \mathbb{Z}_0^+$. Assume that $a \land b \land c \in I$ for some $a, b, c \in R$. Hence $a$ or $b$ or $c = \min\{a, b, c\} = a \land b \land c \langle I$. So we can conclude that $a \land b \in I$ or $b \land c \in \sqrt{I}$ or $a \land c \in \sqrt{I}$. Therefore $I$ is a 2-absorbing primary ideal of $R$.

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**References**


