Some Properties for Certain Subclasses of Starlike Functions Defined by Convolution

R. M. EL-ASHWAH AND F. M. ABDULKAREM*
Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt
e-mail: r_elashwah@yahoo.com and fammari76@gmail.com

M. K. AOUF
Department of Mathematics, Faculty of Science, Damietta University, Mansoura University, Mansoura 35516, Egypt
e-mail: mkaouf127@yahoo.com

ABSTRACT. In this paper, we obtained some properties for subclasses of starlike functions defined by convolution such as partial sums, integral means, square root and integral transform for these classes.

1. Introduction

Let $S$ denote the class of functions of the form

\[(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\]

which are analytic and univalent in the open unit disc $U = \{ z : |z| < 1 \}$ and normalized by $f(0) = 0 = f'(0) - 1$. We denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of $S$ consisting of all functions which are, respectively, starlike and convex of order $\alpha$ $(0 \leq \alpha < 1)$. Thus,

\[(1.2) \quad S^*(\alpha) = \left\{ f \in S : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha \ (0 \leq \alpha < 1; z \in U) \right\} \]

* Corresponding Author.
Received October 20, 2013; accepted January 29, 2014.
2010 Mathematics Subject Classification: 30C45.
Key words and phrases: Analytic functions, starlike functions, convolution, integral means.
and

\[(1.3) \quad K(\alpha) = \left\{ f \in S : \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \ (0 \leq \alpha < 1; \ z \in U) \right\}.\]

The classes \(S^*(\alpha)\) and \(K(\alpha)\) were introduced by Robertson [18]. From (1.2) and (1.3) it follows that

\[(1.4) \quad f(z) \in K(\alpha) \iff zf'(z) \in S^*(\alpha).\]

We note that

\[S^*(0) = S^*\quad \text{and} \quad K(0) = K,
\]

which are, respectively, starlike and convex functions.

Let \(f \in S\) be given by (1.1) and \(g \in S\) given by

\[(1.5) \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (b_k > 0).\]

We define the Hadamard product (or convolution) of \(f\) and \(g\) as follows:

\[(1.6) \quad (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).\]

We denote by \(S_{A,B}(f, g, \alpha, \beta, \gamma)\) \((-1 \leq A < B \leq 1, \ 0 < B \leq 1)\) the subclass of \(S\), where \(f\) and \(g\) are given by (1.1) and (1.2), respectively and satisfies

\[(1.7) \quad \left| \frac{z(f*g)'(z)}{(f*g)(z)} - 1 \right| < \frac{B}{2(B - A)} \gamma \left( \frac{z(f*g)'(z)}{(f*g)(z)} - \alpha \right) - B \left( \frac{z(f*g)'(z)}{(f*g)(z)} - 1 \right) < \beta\]

\[(z \in U; 0 \leq \alpha < 1; 0 < \beta \leq 1)\]

where \((f * g)(z)\) is given by (1.6) and \(\frac{B}{2(B - A)} < \gamma \leq \left\{ \frac{B}{(B - A)\alpha} \right\} \), \(\alpha \neq 0\), \(\alpha = 0\).

We also let

\[(1.8) \quad T_{A,B}(f, g, \alpha, \beta, \gamma) = S_{A,B}(f, g, \alpha, \beta, \gamma) \cap T, \quad \text{where}\]

\[T = \left\{ f \in S : f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k ; z \in U \right\}.\]

We note that:

(i) \(T_{A,B}(f, S_A, \alpha, \beta, \gamma)\)
Some Properties for Certain Subclasses of Starlike Functions

for $S_3 = \frac{z}{(1 - z)^{2(1 - \delta)}}$, $0 \leq \delta < 1$ (see Magesh et al. [10 with $m = 0$]);

(ii) $T_{A,B} \left( f, z + \sum_{k=2}^{\infty} \left( \frac{1 + b}{k + b} \right)^{\nu} \frac{\lambda! (n + 1 - k)!}{(n - 1)! (k + \lambda - 1)!} \right) z^k, \alpha, \beta, \gamma$

$$= \left\{ f \in T : \left| \frac{z(fS_1)'(z)}{(fS_1)(z)} - \frac{1}{2} \frac{(B - A) \gamma(z(fS_1)'(z) - \alpha) - B \frac{(z(fS_1)'(z)}{(fS_1)(z)} - 1)}{2} \right| < \beta \ (z \in U) \right\},$$

for $n \geq 2, \lambda > -1, \mu \in \mathbb{C}, b \in \mathbb{C}\backslash \{ \mathbb{Z}_0 = 0, -1, -2, \ldots \}$ (see Owa et al. [16]);

(iii) $T_{A,B} \left( f, z + \sum_{k=2}^{\infty} \frac{(\Gamma(n + 1))}{\Gamma(n + \eta - 1)} \frac{(\Gamma(2 + \eta - 1))}{\Gamma(n + \eta + 1)} z^k, \alpha, \beta, \gamma \right)$

$$= \left\{ f \in T : \left| \frac{z(3\alpha f(z))'}{(3\alpha f(z))} - 1 \right| < \beta \ (z \in U) \right\},$$

for $\eta - 1 < \mu < \eta < 2$ (see Murugusundramoorthy and Thilagavathi [13]);

(iv) $T_{A,B} \left( f, z + \sum_{k=2}^{\infty} \frac{(2 + \eta - 1)\Gamma(q_1 + B_1 (n - 1)) \Gamma(q_m + B_m (n - 1))}{\Gamma(q_1 + B_1 (n - 1)) \Gamma(q_m + B_m (n - 1))} z^k, \alpha, \beta, \gamma \right)$

$$= \left\{ f \in T : \left| \frac{z(W[p_1,q_1]f(z))'}{(W[p_1,q_1]f(z))} - 1 \right| < \beta \ (z \in U) \right\},$$

for $\ell \leq m + 1, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \ldots\}, \Omega = \{\prod_{i=0}^{\ell} \Gamma(p_i)\}^{-1} \{\prod_{i=0}^{m} \Gamma(q_i)\}$

(see Murugusundramoorthy and Magesh [14]).

Also we note that:

(i) $T_{A,B}(f, z, \alpha, \beta, \gamma) = S_{A,B} (\alpha, \beta, \gamma)$

$$= \left\{ f \in T : \left| \frac{z(f'(z))}{f(z)} - 1 \right| < \beta \ (z \in U) \right\};$$

(ii) $T_{A,B} \left( f, z \frac{z}{(1 - z)^{\gamma}}, \alpha, \beta, \gamma \right) = K_{A,B} (\alpha, \beta, \gamma)$

$$= \left\{ f \in T : \left| \frac{z(f''(z))}{f(z)} - 1 \right| < \beta \ (z \in U) \right\};$$

(iii) $T_{A,B} \left( f, z + \sum_{k=2}^{\infty} k^n z^k, \alpha, \beta, \gamma \right) = T_{A,B} (n, \alpha, \beta, \gamma)$

$$= \left\{ f \in T : \left| \frac{D_{A,B}^{n+1}f(z)}{D_{A,B}^nf(z)} - 1 \right| < \beta \ (z \in U) \right\},$$

where $D_{A,B}^n f(z)$ is the $n$th derivative of $f(z)$.
for \( n \in \mathbb{N}_0 \) and where \( D^n \) is the Salagean operator (see [20]);

(iv) \( T_{A,B} \left( f, z + \sum_{k=2}^{\infty} \left[ 1 + \lambda (k - 1) \right] z^k, \alpha, \beta, \gamma \right) = T_{A,B} (u, \alpha, \beta, \gamma) \)

\[
\begin{aligned}
\{ f \in T : \left| \frac{z(D^k f(z))'}{(D^k f(z))'} - 1 \right| < \beta \ (z \in U) \},
\end{aligned}
\]

for \( \lambda > 0, n \in \mathbb{N}_0 \) and where \( D^k \) is the Al-Oboudi operator (see [2]);

(v) \( T_{A,B} \left( f, z + \sum_{k=2}^{\infty} (k + \lambda - 1) z^k, \alpha, \beta, \gamma \right) = S_{A,B} (\lambda, \alpha, \beta, \gamma) \)

\[
\begin{aligned}
\{ f \in T : \left| \frac{z(D^k f(z))'}{(D^k f(z))'} - 1 \right| < \beta \ (z \in U) \},
\end{aligned}
\]

for \( \lambda > -1 \) and where \( D^k \) is the \( \lambda \)-th order Ruscheweyh derivative of \( f(z) \in S \) (see [1], [19]);

(vi) \( T_{A,B} \left( f, z + \sum_{k=2}^{\infty} \left(1 + \ell + \lambda (k - 1)\right) \frac{m}{1 + \ell} z^k, \alpha, \beta, \gamma \right) = \)

\[
\begin{aligned}
\{ f \in T : \left| \frac{z(J^m(\ell, \lambda)(f(z))'}{(J^m(\ell, \lambda)(f(z))'} - 1 \right| < \beta \ (z \in U) \},
\end{aligned}
\]

for \( \lambda \geq 0, \ell > -1, m \in \mathbb{Z} = \{0, \pm 1, ...\} \) and where \( J^m(\ell, \lambda) \) is the Prapajap operator (see [17], [4], [6], with \( p = 1 \));

(vii) \( T_{A,B} \left( f, z + \sum_{k=2}^{\infty} \frac{(a_1)_{k-1} \cdots (a_q)_{k-1}}{\ell_1 \cdots \ell_{q-1} (k - 1)!} z^k, \alpha, \beta, \gamma \right) = \)

\[
\begin{aligned}
\{ f \in T : \left| \frac{z(J^m(\ell, \lambda)(f(z))'}{(J^m(\ell, \lambda)(f(z))'} - 1 \right| < \beta \ (z \in U) \},
\end{aligned}
\]

for \( \alpha_i > 0, i = 1, ..., q, \beta_j > 0, j = 1, ..., s, q \leq s + 1, q, s \in \mathbb{N}_0 \) and where \( H_{q,s} (\alpha_1) f(z) \) is the Dzoik-Srivastava operator (see [5]).

Now we recall the following lemma and definition which are very needed for our study.

**Lemma 1.** ([7]) Let the function \( f(z) \) be defined by (1.8). Then \( f(z) \) is in the class \( T_{A,B} (f, g, \alpha, \beta, \gamma) \) if and only if

\[
\sum_{k=2}^{\infty} \Psi(\alpha, \beta, \gamma, k, A, B) \left| a_k \right| \leq 2 (B - A) \beta \gamma (1 - \alpha),
\]

where

\[
\Psi(\alpha, \beta, \gamma, k, A, B) = [2 (B - A) \beta \gamma (k - \alpha) + (1 - \beta B) (k - 1)] b_k.
\]

**Definition 1.** ([12]) (Subordination) For analytic functions \( f \) and \( g \) with

(1.9) \[ \sum_{k=2}^{\infty} \Psi(\alpha, \beta, \gamma, k, A, B) \left| a_k \right| \leq 2 (B - A) \beta \gamma (1 - \alpha), \]
f(0) = g(0), f is said to be subordinate to g, denote by \( f \prec g \), if there exists an analytic function \( w \) such that \( w(0) = 0 \), \( |w(z)| < 1 \) and \( f(z) = g(w(z)) \), for all \( z \in U \).

2. Partial Sums

Unless otherwise mentioned, we assume in the reminder of this paper that \(-1 \leq A < B \leq 1\), \( 0 < B \leq 1 \), \( 0 \leq \alpha < 1 \), \( 0 \leq \beta < 1 \), \( z \in U \), \( g \) is given by (1.5) and \( \Psi(\alpha, \beta, \gamma, k, A, B) \) is given by (1.10).

Following the earlier works by Silverman [21] and Siliva [22] on partial sums of analytic functions, we consider in this section partial sums of functions in the class \( T_{A,B}(f, g, \alpha, \beta, \gamma) \) and obtain sharp lower bounds for the ratios of real part of \( f(z) \) to \( f_n(z) \), \( f(z) \) to \( f_n'(z) \) and \( f_n'(z) \) to \( f'(z) \), respectively.

**Theorem 1.** Let the function \( f(z) \) defined by (1.1) be in the class \( S_{A,B}(f, g, \alpha, \beta, \gamma) \). Define the partial sums \( f_1(z) \) and \( f_n(z) \), by

\[
f_1(z) = z \quad \text{and} \quad f_n(z) = z + \sum_{k=2}^{n} a_k z^k \quad (n \in \mathbb{N}/\{1\}).
\]

Suppose also that

\[
\sum_{k=2}^{\infty} d_k |a_k| \leq 1,
\]

where

\[
d_k = \frac{\Psi(\alpha, \beta, \gamma, 2, A, B)}{2(B-A) \beta \gamma (1-\alpha)}.
\]

Then \( f \in T_{A,B}(f, g, \alpha, \beta, \gamma) \). Furthermore,

\[
\Re \left\{ \frac{f(z)}{f_n(z)} \right\} > 1 - \frac{1}{d_{n+1}}, \quad z \in U, \ n \in \mathbb{N},
\]

and

\[
\Re \left\{ \frac{f_n(z)}{f(z)} \right\} > \frac{d_{n+1}}{1+d_{n+1}}.
\]

The result is sharp for the extremal function is given by

\[
f(z) = z + \frac{z^{n+1}}{d_{n+1}},
\]

**Proof.** For \( d_k \) given by (2.2) it is easily to show that \( d_{k+1} > d_k > 1 \). Therefore we have

\[
\sum_{k=2}^{n} |a_k| + d_{n+1} \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} d_k |a_k| \leq 1,
\]
by using (2.2). By setting

$$k_1(z) = d_{n+1} \left\{ \frac{f(z)}{f_n(z)} - \left( 1 - \frac{1}{d_{n+1}} \right) \right\}$$

(2.7)

$$= 1 + \frac{d_{n+1} \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}},$$

and using (2.6), we have

(2.8)

$$\left| \frac{k_1(z) - 1}{k_1(z) + 1} \right| \leq \frac{d_{n+1} \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^{n} |a_k| - d_{n+1} \sum_{k=n+1}^{\infty} |a_k|} \leq 1 \ (z \in U)$$

which yields the assertion (2.3) of Theorem 1. For $z = re^{i\pi n}$ that

$$\frac{f(z)}{f_n(z)} = 1 + \frac{z^n}{d_{n+1}} \rightarrow 1 - \frac{1}{d_{n+1}}$$

as $r \rightarrow 1^-$. Similarly, if we take

$$k_2(z) = (1 + d_{n+1}) \left\{ \frac{f_n(z)}{f(z)} - \frac{d_{n+1}}{1 + d_{n+1}} \right\}$$

(2.9)

$$= 1 - \frac{(1 + d_{n+1}) \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}},$$

and using (2.6), we have

(2.10)

$$\left| \frac{k_2(z) - 1}{k_2(z) + 1} \right| \leq \frac{(1 + d_{n+1}) \sum_{k=n+1}^{\infty} |a_k|}{2 - 2 \sum_{k=2}^{n} |a_k| - (1 - d_{n+1}) \sum_{k=n+1}^{\infty} |a_k|}$$

which leads us immediately to the assertion (2.4). This completes the proof of Theorem 1.

**Theorem 2.** Let the function $f(z)$ defined by (1.1) be in the class $S_{A,B}(f, g, \alpha, \beta, \gamma)$ satisfies the condition (1.9), then

(2.11)

$$\Re \left\{ \frac{f'(z)}{f'_n(z)} \right\} \geq 1 - \frac{n + 1}{d_{n+1}}.$$  

The result is sharp for the extremal function is given by (2.5).

**Proof.** Let

$$k(z) = d_{n+1} \left\{ \frac{f'(z)}{f'_n(z)} - \left( 1 - \frac{n + 1}{d_{n+1}} \right) \right\}$$

$$= 1 + \frac{d_{n+1} \sum_{k=n+1}^{\infty} ka_k z^{k-1} + \sum_{k=2}^{n} ka_k z^{k-1}}{1 + \sum_{k=2}^{n} ka_k z^{k-1}}$$

$$= 1 + \frac{d_{n+1} \sum_{k=n+1}^{\infty} ka_k z^{k-1}}{1 + \sum_{k=2}^{n} ka_k z^{k-1}},$$
using (2.6), we have

\[ \frac{|k(z) - 1|}{|k(z) + 1|} \leq \frac{\frac{d_{n+1}}{n+1} \sum_{k=n+1}^{\infty} k |a_k|}{2 - 2 \sum_{k=2}^{\infty} k |a_k| - \frac{d_{n+1}}{n+1} \sum_{k=n+1}^{\infty} k |a_k|} \leq 1, \]

if

\[ \sum_{k=2}^{n} k |a_k| + \frac{d_{n+1}}{n+1} \sum_{k=n+1}^{\infty} k |a_k| \leq 1 \]

Since the left hand of (2.13) is bounded above by \( \sum_{k=2}^{\infty} d_k |a_k| \) if

\[ \sum_{k=2}^{n} (d_k - k) |a_k| + \sum_{k=n+1}^{\infty} (d_k - \frac{d_{n+1}}{n+1} k |a_k|) \geq 0. \]

This completes the proof of Theorem 2.

**Theorem 3.** Let the function \( f(z) \) defined by (1.1) be in the class \( S_{A,B}(f, g, \alpha, \beta, \gamma) \) satisfies the condition (1.9), then

\[ \Re \left\{ \frac{f_n'(z)}{f'(z)} \right\} \geq 1 - \frac{d_{n+1}}{n+1 + d_{n+1}}. \]

**Proof.** Let

\[
k(z) = \left[ (k + 1) + d_{n+1} \right] \left\{ \frac{f_n'(z)}{f'(z)} - \frac{d_{n+1}}{n + 1 + d_{n+1}} \right\}
\[
= 1 - \frac{\left( 1 + \frac{d_{n+1}}{n+1} \right) \sum_{k=n+1}^{\infty} k a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} k a_k z^{k-1}},
\]

and using (2.14), we have

\[
\frac{|k(z) - 1|}{|k(z) + 1|} \leq \frac{(1 + \frac{d_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k |a_k|}{2 - 2 \sum_{k=2}^{\infty} k |a_k| - (1 + \frac{d_{n+1}}{n+1}) \sum_{k=n+1}^{\infty} k |a_k|} \leq 1,
\]

This completes the proof of Theorem 3.

3. **Integral Means**

In 1925, Littlewood [9] proved the following subordination theorem.

**Lemma 2.** If the functions \( f \) and \( g \) are analytic in \( U \) with \( g \prec f \), then for \( \delta > 0 \), and \( 0 < r < 1 \),

\[ \int_{0}^{2\pi} |g(re^{i\phi})|^\delta d\phi \leq \int_{0}^{2\pi} |f(re^{i\phi})|^\delta d\phi. \]
Using Lemma 1 and Lemma 2, we prove the following result.

**Theorem 4.** Let \( f \in T_{A,B}(f, g, \alpha, \beta, \gamma), \) \( \delta > 0, \) \( 0 \leq \alpha < 1, \) \( 0 \leq \gamma < 1, \) \( n \geq 0 \) and \( f_2(z) \) is given by

\[
f_2(z) = z - \frac{2(B-A)\beta\gamma(1-\alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} z^2,
\]

where \( \Psi(\alpha, \beta, \gamma, 2, A, B) \) is defined by (1.10). Then for \( z = re^{i\phi}, \) \( 0 < r < 1, \) we have

\[
\int_0^{2\pi} |f(z)|^\delta d\phi \leq \int_0^{2\pi} |f_2(z)|^\delta d\phi.
\]

**Proof.** For \( f(z) \) is given by (1.8), (3.2) is equivalent proving that

\[
\int_0^{2\pi} \left| 1 - \sum_{k=2}^{\infty} |a_k| z^{k-1} \right|^\delta d\phi \leq \int_0^{2\pi} \left| 1 - \frac{2(B-A)\beta\gamma(1-\alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} z \right| d\phi.
\]

By Lemma 2, it suffices to show that

\[
1 - \sum_{k=2}^{\infty} |a_k| z^{k-1} < 1 - \frac{2(B-A)\beta\gamma(1-\alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} z.
\]

Setting

\[
(3.3) \quad 1 - \sum_{k=2}^{\infty} |a_k| z^{k-1} = 1 - \frac{2(B-A)\beta\gamma(1-\alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} w(z).
\]

From (3.3) and (1.9), we obtain

\[
|w(z)| = \left| \sum_{k=2}^{\infty} \frac{\Psi(\alpha, \beta, \gamma, k, A, B)}{2(B-A)\beta\gamma(1-\alpha)} a_k z^{k-1} \right| \\
\leq |z| \sum_{k=2}^{\infty} \frac{\Psi(\alpha, \beta, \gamma, k, A, B)}{2(B-A)\beta\gamma(1-\alpha)} |a_k| \leq |z|
\]

This completes the proof of Theorem 4.

**4. Square Root Transformation**

**Definition 2.** Let \( f \in S \) and \( h(z) = \sqrt{f(z^2)}, \) then \( h \in S \) and \( h(z) = z + \sum_{k=2}^{\infty} c_{2k-1} z^{2k-1} \) for \( |z| < 1, \) the function \( m \) is called a square root transformation of \( f. \)

**Theorem 5.** Let the function \( f(z) \) defined by (1.8) be in the class \( T_{A,B}(f, g, \alpha, \beta, \gamma), \)
\( 2(B-A)\beta\gamma(1-\alpha) \leq \Psi(\alpha, \beta, \gamma, 2, A, B) \) and \( h \) be the square root transformation
of $f$, then

$$r \sqrt{1 - \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)}} \leq |h(z)|$$

and

(4.1) $$|h(z)| \leq r \sqrt{1 + \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)}} r^2,$$

where

(4.2) $$f(z) = z - \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} z^2 \ (|z| = \pm r).$$

Proof. In the view of Lemma 1, we have

(4.3) $$r^2 - \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} r^4 \leq |f(z^2)| \leq r^2 + \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} r^4$$

Using (4.3) in the definition (2) we find

$$|h(z)| = \sqrt{|f(z^2)|} \leq \sqrt{r^2 + \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)}} r^4$$

(4.4)

$$= r \sqrt{1 + \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)}} r^2.$$ 

Since, $2(B - A) \beta \gamma (1 - \alpha) \leq \Psi(\alpha, \beta, \gamma, 2, A, B)$ and $r = |z| < 1$, we have

(4.5) $$1 + \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} r^2 \geq 1 - \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)} r^2$$

and hence,

$$|h(z)| = \sqrt{|f(z^2)|} \geq \sqrt{r^2 - \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)}} r^4$$

(4.6)

$$= r \sqrt{1 - \frac{2(B - A) \beta \gamma (1 - \alpha)}{\Psi(\alpha, \beta, \gamma, 2, A, B)}} r^2.$$ 

This completes the proof of Theorem 5.
5. Integral Transform of the Class $T_{A,B}(f,g,\alpha,\beta,\gamma)$

For $f \in S$ we define the integral transform

$$V_\mu(f(z)) = \int_0^1 \mu(t)\frac{f(tz)}{t} dt,$$

where $\mu(t)$ is a real valued, non-negative weight function normalized so that $\int_0^1 \mu(t)dt = 1$. Since special cases of $\mu(t)$ are particularly interesting such as $\mu(t) = (1 + c)t^c$, $c > -1$, for which $V_\mu$ is known as the Bernardi operator [3], and

$$\mu(t) = \frac{(c + 1)^\eta}{\Gamma(\eta)} \left( \log \frac{1}{t} \right)^{\eta-1}, \quad c > -1, \quad \eta \geq 0,$$

which gives the Komatu operator [8], see also [15].

Now we show that the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$ is closed under $V_\mu(f(z))$.

**Theorem 6.** Let the function $f(z)$ defined by (1.8) be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$, then $V_\mu(f(z)) \in T_{A,B}(f,g,\alpha,\beta,\gamma)$.

**Proof.** From (5.1), we have

$$V_\mu(f(z)) = \frac{(c + 1)^\eta}{\Gamma(\eta)} \int_0^1 (-1)^{\eta-1} t^{\eta-1} \left( z - \sum_{k=2}^\infty a_k z^k \right) dt$$

$$= \frac{(-1)^{\eta-1}(c + 1)^\eta}{\Gamma(\eta)} \lim_{r \to 0^+} \left\{ \int_r^1 t^c \left( \log \frac{1}{t} \right)^{\eta-1} \left( z - \sum_{k=2}^\infty a_k z^k \right) dt \right\}$$

$$= z - \sum_{k=2}^\infty \left( \frac{c + 1}{c + k} \right)^\eta a_k z^k.$$

We need to prove that

$$\sum_{k=2}^\infty \frac{\Psi(\alpha,\beta,\gamma,k,A,B)}{2(B-A) \beta_7 (1-\alpha)} \left( \frac{c + 1}{c + k} \right)^\eta a_k \leq 1.$$

On the other hand by Lemma 1, $f(z) \in T_{A,B}(f,g,\alpha,\beta,\gamma)$ if and only if

$$\sum_{k=2}^\infty \frac{\Psi(\alpha,\beta,\gamma,k,A,B)}{2(B-A) \beta_7 (1-\alpha)} a_k \leq 1.$$

Since $\frac{c + 1}{c + k} < 1$, therefore (5.3) holds and the proof of Theorem 6 is completed.

**Theorem 7.** Let the function $f(z)$ defined by (1.8) be in the class $T_{A,B}(f,g,\alpha,\beta,\gamma)$. Then $V_\mu(f(z))$ is starlike of order $\xi$ ($0 \leq \xi < 1$) in the disc $|z| < r_1$, where

$$r_1 = \inf_{k \geq 2} \left[ \left( \frac{c + k}{c + 1} \right)^\eta \frac{(1 - \xi)\Psi(\alpha,\beta,\gamma,k,A,B)}{(k - \xi)[2(B-A) \beta_7 (1-\alpha)]} \right]^{1/\eta}.$$


Proof. It is sufficient to show that
\begin{equation}
\left| \frac{z[V_\mu (f(z))]' - 1}{V_\mu (f(z))} \right| \leq 1 - \xi \text{ for } |z| < r_1,
\end{equation}
where \( r_1 \) is given by (5.4). Indeed we find, again from the definition (1.8) that
\begin{equation}
\left| \frac{z[V_\mu (f(z))]' - 1}{V_\mu (f(z))} \right| \leq \sum_{k=2}^{\infty} \left( \frac{c+1}{c+k} \right)^\eta \left( \frac{k-\xi}{1-\xi} \right) a_k |z|^{k-1}.
\end{equation}
Thus
\begin{equation}
\left| \frac{z[V_\mu (f(z))]' - 1}{V_\mu (f(z))} \right| \leq 1 - \xi,
\end{equation}
if
\begin{equation}
\sum_{k=2}^{\infty} \left( \frac{c+1}{c+k} \right)^\eta \left( \frac{k-\xi}{1-\xi} \right) a_k |z|^{k-1} \leq 1.
\end{equation}
But, by Lemma 1, (5.6) will be true if
\begin{equation}
\left( \frac{c+1}{c+k} \right)^\eta \left( \frac{k-\xi}{1-\xi} \right) |z|^{k-1} \leq \frac{\Psi (\alpha, \beta, \gamma, k, A, B)}{2 (B - A) \beta \gamma (1 - \alpha)},
\end{equation}
that is, if
\begin{equation}
r_1 = |z| \leq \left[ \left( \frac{c+k}{c+1} \right)^\eta \frac{(1-\xi)\Psi (\alpha, \beta, \gamma, k, A, B)}{(k-\xi)2 (B - A) \beta \gamma (1 - \alpha)} \right]^{1/\eta}.
\end{equation}
Theorem 7 follows easily from (5.7).

**Theorem 8.** Let the function \( f(z) \) defined by (1.8) be in the class \( T_{A,B}(f, g, \alpha, \beta, \gamma) \). Then \( V_\mu (f(z)) \) is convex of order \( \xi \) \((0 \leq \xi < 1)\) in the disc \(|z| < r_2\), where
\begin{equation}
r_2 = \inf_{k \geq 2} \left[ \left( \frac{c+k}{c+1} \right)^\eta \frac{(1-\xi)\Psi (\alpha, \beta, \gamma, k, A, B)}{k(\xi)2 (B - A) \beta \gamma (1 - \alpha)} \right]^{1/\eta}.
\end{equation}

**Remark.** (i) Putting \( g = \frac{z}{(1 - z)^{2(1-\delta)}} \) \((0 \leq \delta < 1)\), in the above results, we obtain the corresponding results obtained by Magesh et al. [10, with \( m = 0 \)];
(ii) Specializing the function \( g(z) \), we obtain different results corresponding to the classes mentioned in the introduction.

**Acknowledgment.** The authors thank the referees for their valuable suggestions which led to the improvement of this paper.
References


