CONSTRUCTION OF A SYMMETRIC SUBDIVISION SCHEME REPRODUCING POLYNOMIALS

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Abstract. In this work, we study on subdivision schemes reproducing polynomials and build a symmetric subdivision scheme reproducing polynomials of a certain predetermined degree, which is a slight variant of the family of Deslauries-Dubic interpolatory ones. Related to polynomial reproduction, a necessary and sufficient condition for a subdivision scheme to reproduce polynomials of degree L was recently established under the assumption of non-singularity of subdivision schemes. In case of stepwise polynomial reproduction, we give a characterization for a subdivision scheme to reproduce stepwise all polynomials of degree $\leq L$ without the assumption of non-singularity. This characterization shows that we can investigate the polynomial reproduction property only by checking the odd and even masks of the subdivision scheme.

The minimal-support condition being relaxed, we present explicitly a general formula for the mask of $(2n + 4)$-point symmetric subdivision scheme with two parameters that reproduces all polynomials of degree $\leq 2n+1$. The uniqueness of such a symmetric subdivision scheme is proved, provided the two parameters are given arbitrarily. By varying the values of the parameters, this scheme is shown to become various other well known subdivision schemes, ranging from interpolatory to approximating.

1. Introduction

In this work, we study on subdivision schemes reproducing polynomials and build a symmetric subdivision scheme reproducing polynomials of a certain predetermined degree. Subdivision schemes have been a popular way to generate curves and surfaces in Computer Aided Geometric Design (CAGD). (For more background on subdivision, we refer to the excellent works [1], [13], and [23].) Let $\mathbb{Z}$ be the integer set and $\mathbf{a} = \{a_i\}_{i \in \mathbb{Z}}$ a set of constants. A stationary binary subdivision scheme is a process which recursively defines a sequence of

control points $f^k = \{f_i^k\}_{i \in \mathbb{Z}}$ by a rule of the form

$$f_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{i-2j} f_j^k, \quad k \in \{0, 1, 2, \dots\},$$

which is denoted formally by $f^{k+1} = Sf^k = S^{k+1}f^0$. The set $\mathbf{a}$ is called the mask of the subdivision scheme. In this work, we consider masks of a finite set of non-zero coefficients $\mathbf{a}$. Then a point of f^{k+1} is defined by a finite affine combination of points in f^k with two different rules:

$$\begin{aligned} f_{2i}^{k+1} &= \sum_{j \in \mathbb{Z}} a_{2j} f_{i-j}^k, \\ f_{2i+1}^{k+1} &= \sum_{j \in \mathbb{Z}} a_{1+2j} f_{i-j}^k. \end{aligned}$$

A subdivision scheme is said to be *uniformly convergent* if for every initial data $f^0 = \{f_i\}_{i \in \mathbb{Z}}$, there is a continuous function $f \in C(\mathbb{R})$ such that for any interval $[a, b]$

$$\lim_{k \to \infty} \sup_{i \in \mathbb{Z} \cap 2^k[a,b]} |f_i^k - f(2^{-k}i)| = 0,$$

and such that $f \not\equiv 0$ for some initial data. In this work, we consider only the prime parametrization of a subdivision uniformly convergent scheme. (See [12] for the definitions of prime and dual parametrization of a subdivision scheme.) We denote the function f by $S^\infty f^0$, and call it a limit function of S or a function generated by S. It was shown by Cavaretta et al. [1], Dyn [8], and Ko et al. [20] that for a convergent binary scheme, the corresponding mask $\{a_i\}_{i \in \mathbb{Z}}$ necessarily satisfies

$$\sum_{i \in \mathbb{Z}} a_{2i} = \sum_{i \in \mathbb{Z}} a_{2i+1} = 1.$$

Among the criteria for a convergent subdivision scheme S, the polynomial reproduction property is desirable. Sampling an initial data from a given function f which is smooth enough, the quantity of approximation power is deeply related to the asymptotic rate of the sequence of approximations obtained at each step through the subdivision scheme to the original function f. The approximation power can be measured by the polynomial reproducing property (see [16], [21] for details): A subdivision scheme S reproduces polynomials of degree $\leq L$ (in the limit) if $S^\infty f^0 = p$ for any polynomial p of degree $\leq L$ and initial data $f^0 = p(i)$, $i \in \mathbb{Z}$. In the literature, another notion of polynomial reproduction is used: A subdivision scheme S with a mask $\{a_i\}_{i \in \mathbb{Z}}$ is said to reproduce polynomials of degree $\leq L$ in each subdivision step if S reproduces all polynomials p of degree $\leq L$ in the sense that for any $k \geq 0$, we have

$$p\left(\frac{i}{2^{k+1}}\right) = \sum_{j \in \mathbb{Z}} a_{i-2j} p\left(\frac{j}{2^k}\right), \quad i \in \mathbb{Z}.$$

This is called the stepwise polynomial reproduction, which implies the polynomial reproduction from the following theorem.

Theorem 1.1 (Dyn, Hormann, Sabin, and Shen [12]). *Let S be a convergent subdivision scheme. If S is stepwise polynomial reproducing up to degree L, then S reproduces all polynomials of degree $\leq L$.*

And the converse holds for non-singular (or stable) subdivision schemes ([8] and [12]). A convergent subdivision scheme S is called *non-singular* if $S^\infty f^0 = 0$ only when $f^0 = 0$. For more details on the polynomial reproduction property, we refer to [3], [12], [14], [21], [23]. Recently, Hormann and Sabin [16] derived the degree of polynomial reproduction for a family of schemes using algebraic considerations and Dyn et al. [12] generalized the method by establishing a necessary and sufficient condition for a subdivision scheme to reproduce polynomials of degree L under the assumptions of non-singularity and polynomial generation. A subdivision scheme S with mask $\{a_i\}_{i\in\mathbb{Z}}$ is said to generate polynomials of degree $\leq M$ if for any polynomial p of degree $\leq M$, there exists an initial data f^0 such that

$$S^\infty f^0 = p.$$

They showed that a non-singular subdivision scheme with mask $\{a_i\}_{i\in\mathbb{Z}}$ generating polynomials of degree $\leq M$ reproduces all polynomials up to degree $L(\leq M)$ if and only if for the corresponding Laurent polynomial $a(z)$ given as a symbol $a(z) = \sum_{i\in\mathbb{Z}} a_i z^i$,

$$a(z) - 2 \quad \text{is divisible by } (1-z)^{L+1}.$$

Conti and Hormann [3] extended the result for a non-singular subdivision scheme and derived a unified condition for polynomial reproduction that covers symmetric and non-symmetric schemes and applies to m-ary subdivision schemes (also see [15]).

In this paper, we characterize the stepwise polynomial reproduction property of a subdivision scheme S. We show that a subdivision scheme S with mask $\{a_i\}_{i\in\mathbb{Z}}$ reproduces all polynomials of degree $\leq L$ in each step if and only if for the corresponding Laurent polynomials $a_{odd}(z) := \sum_{k\in\mathbb{Z}} a_{2k+1} z^{2k+1}$ and $a_{even}(z) = \sum_{k\in\mathbb{Z}} a_{2k} z^{2k}$,

$$a_{odd}(z) - 1 \text{ and } a_{even}(z) - 1 \text{ are divisible by } (1-z)^{L+1}.$$

This characterization shows that even though the polynomial generation assumption is unavoidable, we can check the polynomial reproduction property only by estimating the factorizations of the Laurent polynomials $a_{even}(z) - 1$ and $a_{odd}(z) - 1$ related to the even and odd mask coefficients respectively, without checking the non-singularity or stability of the subdivision scheme beforehand (Theorem 2.8). Due to half the amount of the mask coefficients, the factorizations of $a_{even}(z) - 1$ and $a_{odd}(z) - 1$ are much easier than those of $a(z)$ and $a(z) - 2$. From (iii) of Lemma 2.7, we remark that the characterization can be obtained by combining Theorems 3.2 and 4.6 in [12].

Deslauriers and Dubuc [4] derived a $(2n+2)$-interpolatory subdivision scheme of minimal support reproducing all polynomials of degree $\leq 2n + 1$ (hereafter referred to as DD). By taking a convex combination of the two DD schemes, Dyn [9] reconstructed the Dyn, Gregory and Levin (DGL) 4-point [10] and the Weissman 6-point schemes with a tension parameter [24].

The minimal-support condition being relaxed, we obtain a $(2n + 4)$-point symmetric subdivision scheme reproducing all polynomials of degree $\leq 2n + 1$. We also prove that a symmetric subdivision scheme with mask $\{a_k\}_{k=-2n-3}^{2n+3}$ reproducing polynomials of degree $\leq 2n + 1$ in each step is unique, provided that a_{2n+3} and a_{2n+2} are given arbitrarily (see Theorem 2.3). The uniqueness of such a scheme verifies that the proposed subdivision scheme can be obtained by an affine combination with two parameters of the three subdivision schemes with Laurent polynomials $a_{n+1}^n(z)$, $a_{n+2}^n(z)$, and $a_{n+3}^n(z)$ given in [12]. These schemes were introduced by Dong and Shen [5], and they and Dyn et al. investigated the properties of these schemes in [6, 7] and [12].

At the cost of relaxing the minimal support, that is, with the appearance of two free parameters, these schemes generalize well-known many other subdivision schemes ranging from interpolatory to approximating such as the DGL 4-point, the Weissman 6-point schemes, the $(2n + 4)$-interpolatory symmetric scheme with a parameter, and the $(2n + 2)$, $(2n + 4)$ DD schemes as well as cubic and 6-th order B-splines. We can obtain also the mask of the subdivision scheme proposed Choi et al. [2], who did not get the explicit formulation for the masks. Despite the proposed scheme is involved in two parameters, no attempt has been made here to look into any practical impact in computer aided geometric design because this topic exceeds the scope of this paper.

This paper is organized as follows. In Section 2, we obtain a general rule for the construction of the mask of $(2n + 4)$-point symmetric subdivision schemes reproducing all polynomials of degree $\leq 2n + 1$. Then, we present an explicit formulation of the longer masks of a $(2n + 4)$-point symmetric subdivision schemes in terms of two parameters. Also we give a factorization of the Laurent polynomial, through which we can estimate proper properties such as smoothness, polynomial reproduction, and so on. In the section, we give a characterization for a subdivision scheme to reproduce stepwise polynomials of degree $\leq L$. With numerical performances, we provide numerical illustrations which show the convergence and smoothness of the proposed scheme for two special cases in Section 3.

2. Construction of the polynomial reproducing mask

The derivation in this section is based on the work done by de Villiers, Goosen and Herbst [22] and [19]. We denote by P_{2n+1} the space of all polynomials of degree $\leq 2n + 1$ for a nonnegative integer n. In our argument, the Lagrange fundamental polynomials $\{L_k(x)\}_{k=-n}^{n+1}$ corresponding to the nodes $\{k\}_{k=-n}^{n+1}$ play quite an important role. We define the Lagrange fundamental

polynomials $\{L_k(x)\}_{k=-n}^{n+1}$ by

$$(1) \qquad L_k(x) = \prod_{j \neq k, j=-n}^{n+1} \frac{x-j}{k-j}, \quad k = -n, \dots, n+1,$$

for which

$$(2) \qquad L_k(j) = \delta_{k,j}, \quad k, j = -n, \dots, n+1,$$

and

$$(3) \qquad \sum_{k=-n}^{n+1} p(k) L_k(x) = p(x), \quad p \in P_{2n+1}.$$

Then it is easy to see that for each $j = -n-1, \dots, n$,

$$(4) \qquad L_{-j}\left(\frac{1}{2}\right) = (-1)^j \frac{(n+1)}{2^{4n+1}(2j+1)} \binom{2n+1}{n} \binom{2n+1}{n+j+1},$$

$$(5) \qquad L_{-j}(n+2) = (-1)^{j+n+1} \frac{(2n+2)!}{(n-j)!(n+j+1)!(n+j+2)},$$

$$(6) \qquad L_{-j}(-n-1) = (-1)^{j+n} \frac{(2n+2)!}{(n-j)!(n+j+1)!(n-j+1)},$$

and

$$(7) \quad L_{-j}(n+2) + L_{-j}(-n-1) = (-1)^{j+n} \binom{2n+1}{n+j+1} \frac{(2n+2)(2j+1)}{(n+j+2)(n-j+1)}.$$

These quantities are crucial to find the explicit form of masks considered in the following process.

We consider the problem of finding symmetric masks $\mathbf{a} = \{a_j\}_{j=-2n-3}^{2n+3}$ reproducing polynomials of degree $\leq 2n+1$ in each step, that is

$$(8) \qquad \sum_{k \in \mathbb{Z}} a_{j-2k} p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad p \in P_{2n+1}.$$

Throughout this section, we let $v = a_{2n+2}$ and $w = a_{2n+3}$, for convenience's sake. Setting $j = 0$ in (8), and using (2) and (3), we obtain from (8)

$$(9) \qquad \sum_{k=-n-1}^{n+1} a_{-2k} L_{-j}(k) = \delta_{j,0}, \quad j = -n-1, \dots, n.$$

We split the summation on the left-hand side of the equation (9) as

$$\sum_{k=-n-1}^{n+1} a_{-2k} L_{-j}(k) = \sum_{k=-n}^{n+1} a_{-2k} L_{-j}(k) + a_{2n+2} L_{-j}(-n-1)$$

$$= a_{2j} + a_{2n+2} L_{-j}(-n-1).$$

Thus substituting (6) into the above equation gives the explicit form of a_{2j} for $j = -n - 1, \ldots, n$,

$$a_{2j} = \delta_{j,0} - vL_{-j}(-n-1)$$

(10)
$$= \delta_{j,0} + (-1)^{j+n+1}v\frac{(2n+2)!}{(n-j)!(n+j+1)!(n-j+1)}$$

$$= \delta_{j,0} + (-1)^{j+n+1}\binom{2n+2}{n+j+1}v.$$

Also setting $j = 1$ in (8), we get

(11)
$$\sum_{k=-n-1}^{n+2} a_{1-2k}L_{-j}(k) = L_{-j}\left(\frac{1}{2}\right), \quad j = -n-1, \ldots, n.$$

We split the summation on the left-hand side of the equation (11) as

$$\sum_{k=-n-1}^{n+2} a_{1-2k}L_{-j}(k) = \sum_{k=-n}^{n+1} a_{1-2k}L_{-j}(k)$$
$$+ a_{2n+3}[L_{-j}(n+2) + L_{-j}(-n-1)].$$

By applying the relation (2), we get

$$\sum_{k=-n-1}^{n+2} a_{1-2k}L_{-j}(k) = a_{1+2j} + w[L_{-j}(n+2) + L_{-j}(-n-1)].$$

Using the identities (4)-(7), we have the explicit form for a_{2j+1}

$$a_{2j+1} = L_{-j}\left(\frac{1}{2}\right) - w[L_{-j}(n+2) + L_{-j}(-n-1)]$$

(12)
$$= \frac{n+1}{2^{4n+1}}\binom{2n+1}{n}\frac{(-1)^j}{2j+1}\binom{2n+1}{n+j+1}$$
$$+ (-1)^{j+n+1}w\binom{2n+1}{n+j+1}\frac{(2n+2)(2j+1)}{(n+j+2)(n-j+1)}$$

for $j = -n-1, \ldots, n$.

It is easy to see that the mask $\{a_j\}_{j=-2n-3}^{2n+3}$ with a_{2j} as given in (10) and a_{2j+1} as given in (12) satisfies the conditions of symmetry and the proposed scheme satisfies the polynomial reproducing property up to degree $2n + 1$, because this property is the starting point of the construction of the mask as formulated in (8).

Note that by applying $p(x) = 1$ to the relation (8), we have the identity

$$\sum_{j\in\mathbb{Z}} a_{2j} = \sum_{j\in\mathbb{Z}} a_{2j+1} = 1.$$

And we take the value of the parameter v as $v = 0$, the proposed scheme becomes the $(2n + 4)$-point interpolatory symmetric subdivision scheme, and

when $v = w = 0$, it becomes the well-known $(2n + 2)$-DD scheme [22] of which the mask, denoted by $\{a_{2i+1}^{DD,2n+2}\}$, is given as

$$(13) \quad a_{2i+1}^{DD,2n+2} = \frac{n+1}{2^{4n+1}}\binom{2n+1}{n}\frac{(-1)^i}{2i+1}\binom{2n+1}{n+i+1}, \quad i = -n-1, \ldots, n.$$

In general, it is not an interpolatory scheme if $v \neq 0$ since, in this case, we have

$$a_{2j} \neq \delta_{j,0}.$$

In summary, we have the following theorem:

Theorem 2.1 ([19]). *For each integer $n \geq 0$, define the symmetric subdivision scheme with a mask $\{a_j\}_{j=-2n-3}^{2n+3}$ given as $a_{2n+2} = a_{-2n-2} = v$, $a_{2n+3} = a_{-2n-3} = w$, and*

$$(14) \qquad a_{2j} = \delta_{j,0} + (-1)^{j+n+1}\binom{2n+2}{n+j+1}v, \quad \text{for} \quad j = -n \ldots, n,$$

and for $j = -n - 1, -n, \ldots, n$,

$$(15) \qquad a_{2j+1} = \frac{n+1}{2^{4n+1}}\binom{2n+1}{n}\frac{(-1)^j}{2j+1}\binom{2n+1}{n+j+1}$$
$$+ (-1)^{j+n+1}\binom{2n+1}{n+j+1}\frac{(2n+2)(2j+1)}{(n+j+2)(n-j+1)}w.$$

Then the subdivision scheme has the properties:

  (i) *the scheme reproduces all polynomials of degree $\leq 2n + 1$.*
  (ii) *In case when $v = 0$, it becomes a $(2n+4)$-point interpolatory symmetric subdivision scheme (ISSS) with one parameter.*
  (iii) *In case when $v = 0$ and $w = w_n$, where*

$$(16) \qquad\qquad w_n = (-1)^{n+1}\frac{(n+2)}{2^{4n+5}(2n+3)}\binom{2n+3}{n+1},$$

  it becomes the $(2n + 4)$-point DD scheme so that it reproduces all polynomials of degree $\leq 2n + 3$.
  (iv) *In case when $v = w = 0$, it becomes the $(2n + 2)$-point DD scheme.*

Proof. We have only to show (iii). For the two parameters, we take the specific values as $v = 0$ and w given in (16). In this case, it is easy to see that $a_{2n+3} = a_{2n+3}^{DD,2n+4}$ and for $i = 0, 1, \ldots, n$,

$$a_{2i+1} = a_{2i+1}^{DD,2n+2} + (-1)^{i+n+1}w\binom{2n+1}{n+i+1}\frac{(2n+2)(2i+1)}{(n+i+2)(n-i+1)}$$
$$= \frac{n+1}{2^{4n+1}}\binom{2n+1}{n}\frac{(-1)^i}{2i+1}\binom{2n+1}{n+i+1}$$
$$+ \frac{n+2}{2^{4n+5}}\binom{2n+3}{n+1}\frac{1}{2n+3}(-1)^i\binom{2n+1}{n+i+1}\frac{(2n+2)(2i+1)}{(n+i+2)(n-i+1)}$$

$$= \frac{n+2}{2^{4n+5}} \binom{2n+3}{n+1} \frac{(-1)^i}{2i+1} \binom{2n+3}{n+i+2}$$
$$= a_{2i+1}^{DD,2n+4}.$$

Then, the symmetry of the mask verifies that the $(2n+4)$-point (interpolatory symmetric) subdivision scheme becomes the $(2n+4)$-point DD scheme. And the $(2n+4)$-point DD scheme reproduces all polynomials of degree $2n+3$, which completes the proof. $\qquad\square$

Remark 2.2. Choi et al. [2] presented a new class of subdivision schemes unifying not only the 4-point DD scheme but also the quadratic and cubic B-spline schemes. They proved the convergence, smoothness and approximation order. But they did not get the explicit formulation for the masks. Instead, they proposed the forms of the masks $\{b_j\}_{j=-L}^{L}$ of the subdivision schemes for $L = 1, 2, \ldots, 10$. With the mask given in Theorem 2.1 for $w = 0$, we can obtain the mask of the subdivision scheme which they proposed (L is even).

Theorem 2.3. *There exists a unique symmetric subdivision scheme with mask $\{a_j\}_{j=-2n-3}^{2n+3}$ reproducing stepwise all polynomials of degree $\leq 2n + 1$, provided that a_{2n+3} and a_{2n+2} are given arbitrarily.*

Proof. We claim that if a symmetric subdivision scheme with mask $\{a_j\}_{j=-2n-1}^{2n+1}$ is stepwise polynomial reproducing up to degree $2n + 1$, that is,

$$(17) \qquad \sum_{k\in\mathbb{Z}} a_{j-2k} p(k) = 0, \quad j \in \mathbb{Z}, \quad p \in P_{2n+1},$$

then $a_j = 0$ for all $j = -2n + 1, \ldots, 2n + 1$. Indeed, it is easily seen that substituting $p(x) = x^\ell, \ell = 1, \ldots, 2n + 1$, into equation (17) and applying the non-singularity of the Vandermonde matrix shows the claim, which prove the theorem. $\qquad\square$

We notice that we can see in the proof that such a subdivision scheme is also unique, provided that any two mask coefficients a_{2i} and a_{2j+1} with different parities of $\{a_k\}_{k=-2n-3}^{2n+3}$ are arbitrarily given.

To investigate useful properties of a subdivision scheme S with mask $\{a_i\}_{i\in\mathbb{Z}}$, we use the so-called Laurent polynomial (or symbol)

$$a(z) = \sum_{i\in\mathbb{Z}} a_i z^i.$$

Remark 2.4. Theorem 2.3 shows that by suitable choices of v and w, the symmetric subdivision scheme constructed in Theorem 2.1 induces the schemes mentioned in Theorem 2.1 and Remark 2.2. Also, we can see that the proposed subdivision scheme is an affine combination of the three subdivision schemes with Laurent polynomials $a_{n+1}^n(z)$, $a_{n+2}^n(z)$, and $a_{n+3}^n(z)$ given in [7] and [12] (see also [5] and [6]).

With the Laurent polynomial, Dyn et al. [12] established the following theorem for the polynomial reproduction property of a subdivision scheme under the assumptions of polynomial generation and non-singularity.

Theorem 2.5. *Let S be a non-singular subdivision scheme with mask $\{a_i\}_{i\in\mathbb{Z}}$ that generates polynomials of degree $\leq M$. Then S reproduces polynomials up to degree L $(\leq M)$ if and only if for its Laurent polynomial $a(z)$, $a(z) - 2$ is divisible by $(1-z)^{L+1}$.*

Proof. See Theorem 4.6 in [12] (and Theorem 1.1 also). $\square$

For an integer $n \geq 0$, let S be the proposed scheme with mask $\{a_j\}_{j=-2n-3}^{2n+3}$ given as $a_{2n+2} = a_{-2n-2} = v$, $a_{2n+3} = a_{-2n-3} = w$, and

$$(18) \qquad a_{2j} = \delta_{j,0} + (-1)^{j+n+1}\binom{2n+2}{n+j+1}v \quad \text{for} \quad j = -n\ldots,n,$$

and for $j = -n-1, -n, \ldots, n$,

$$a_{2j+1} = a_{2j+1}^{DD,2n+2} + (-1)^{j+n+1}\binom{2n+1}{n+j+1}\frac{(2n+2)(2j+1)}{(n+j+2)(n-j+1)}w$$

$$(19) \qquad = a_{2j+1}^{DD,2n+2} + (-1)^{j+n+1}\binom{2n+3}{n+j+2}\frac{(2j+1)}{(2n+3)}w,$$

where $a_{2j+1}^{DD,2n+2}$ are given as in (13). The Laurent polynomial $a(z)$ is decomposed into three parts,

$$a(z) = a^{DD,2n+2}(z) + va_{even}(z) + wa_{odd}(z),$$

where $a^{DD,2n+2}(z)$ is the Laurent polynomials corresponding to the $(2n+2)$-DD scheme,

$$a^{DD,2n+2}(z) = \sum_{j=-n-1}^{n} a_{2j+1}^{DD,2n+2}z^{2i+1} + 1,$$

and $a_{even}(z)$ and $a_{odd}(z)$ are given as

$$a_{even}(z) = \sum_{j=-n-1}^{n+1} (-1)^{j+n+1}\binom{2n+2}{n+j+1}z^{2j},$$

$$a_{odd}(z) = \sum_{j=-n-2}^{n+1} (-1)^{j+n+1}\binom{2n+3}{n+j+2}\frac{(2j+1)}{(2n+3)}z^{2j+1}.$$

We simplify $a_{even}(z)$ as

$$a_{even}(z) = z^{-2n-2}\sum_{j=-n-1}^{n+1} (-1)^{j+n+1}\binom{2n+2}{n+j+1}z^{2n+2j+2}$$

$$= z^{-2n-2}\sum_{j=0}^{2n+2} (-1)^j\binom{2n+2}{j}z^{2j}$$

$$= z^{-2n-2}(1-z^2)^{2n+2}.$$

To simplify $a_{odd}(z)$, we split the summation into two parts as

$$a_{odd}(z) = \sum_{j=-n-2}^{n+1} (-1)^{j+n+1} \binom{2n+3}{n+j+2} \frac{(2j+1)}{(2n+3)} z^{2j+1}$$

$$= z^{-2n-3} \sum_{j=0}^{2n+3} (-1)^{j+1} \binom{2n+3}{j} \frac{(2j-2n-3)}{(2n+3)} z^{2j}$$

$$= z^{-2n-3} \Big((1-z^2)^{2n+3} - \frac{2}{2n+3} \sum_{j=0}^{2n+3} (-1)^j j \binom{2n+3}{j} z^{2j} \Big).$$

On the other hand, we can simplify the last summation as follows.

$$\sum_{j=0}^{2n+3} (-1)^j j \binom{2n+3}{j} z^{2j} = z \sum_{j=0}^{2n+3} (-1)^j j \binom{2n+3}{j} z^{2j-1}$$

$$= \frac{z}{2} \frac{d}{dz} \Big(\sum_{j=0}^{2n+3} (-1)^j \binom{2n+3}{j} z^{2j} \Big)$$

$$= \frac{z}{2} \frac{d}{dz} \big((1-z^2)^{2n+3} \big)$$

$$= -(2n+3) z^2 (1-z^2)^{2n+2}.$$

That is, we have obtained the factorization

$$-\frac{2}{2n+3} \sum_{j=0}^{2n+3} (-1)^j j \binom{2n+3}{j} z^{2j} = 2z^2 (1-z^2)^{2n+2}.$$

Thus, $a_{odd}(z)$ is simplified as

$$a_{odd}(z) = z^{-2n-3}(1-z^2)^{2n+2}(1+z^2).$$

Combining these simplifications, we express $a(z)$ as

(20) $a(z) = a^{DD,2n+2}(z) + z^{-2n-3}(1-z^2)^{2n+2}(w+vz+wz^2).$

Since the $(2n+2)$-point DD scheme reproduces polynomials of degree $\leq 2n+1$, Theorem 2.5 implies that the Laurent polynomial $a^{DD,2n+2}(z) - 2$ is divided by $(1-z)^{2n+2}$. From (20), we show that $a(z) - 2$ is $(1-z)^{2n+2}$, which accords with Theorem 2.5.

As given in Theorem 2.5, Dyn et al. [12] found a necessary and sufficient condition under the assumptions of non-singularity and polynomial generation. In the following lemmas, however, we shall show that even though the polynomial generation assumption is unavoidable, the equivalence relation holds without the assumption of non-singularity (and of continuity as well).

Lemma 2.6. *Let S_b be a subdivision scheme with mask $\{b_i\}_{i \in \mathbb{Z}}$ and $b(z)$ its Laurent polynomial. Then the following holds.*

(i) $b(z)$ is divisible by $(1 - z)^{L+1}$ if and only if

$$b(z)p(z) = 0, \quad p \in P_L,$$

where $p(z) = \sum_{i \in \mathbb{Z}} p(i) z^i$;

(ii) or, equivalently, $b(z)$ is divisible by $(1 + z)^{L+1}$ if and only if

$$b(z)p(-z) = 0, \quad p \in P_L.$$

Proof. The equivalence of (i) and (ii) is straightforward. And the claim (i) comes from Lemma 4.2 in [12]. $\square$

Now, we are to characterize a subdivision scheme having the stepwise polynomial reproduction property. Assume that S_b is a subdivision scheme with a finite set of mask $\{b_i\}_{i \in \mathbb{Z}}$ reproducing polynomials of degree $\leq L$ in each step. Let $b_{odd}(z)$, $b_{even}(z)$ be the Laurents polynomials defined by

$$(21) \qquad b_{odd}(z) = \sum_{k \in \mathbb{Z}} b_{2k+1} z^{2k+1} \quad \text{and} \quad b_{even}(z) = \sum_{k \in \mathbb{Z}} b_{2k} z^{2k}.$$

Then we can see that for any polynomial $p(x) \in P_L$,

$$p\Big(\frac{2j+1}{2^{\ell+1}}\Big) = \sum_{k \in \mathbb{Z}} b_{2j-2k+1} p\Big(\frac{k}{2^\ell}\Big), \quad j \in \mathbb{Z} \quad \text{and} \quad \ell = 0, 1, \dots.$$

That is, the (interpolatory) subdivision S_{odd} with mask $\{\tilde{b}_k\}_{k \in \mathbb{Z}}$ given as

$$\tilde{b}_{2k+1} = b_{2k+1} \quad \text{and} \quad \tilde{b}_{2k} = \delta_{k,0}, \quad k \in \mathbb{Z}$$

also reproduces stepwise all polynomials of degree $\leq L$. In this case, Theorem 2.5 implies that the corresponding Laurent polynomial $\tilde{b}(z) - 2$ as well as $b(z) - 2$ is divisible by $(1 - z)^{L+1}$. On the other hand, we have the relation between $b_{odd}(z)$ and $\tilde{b}(z)$ as

$$\tilde{b}(z) = b_{odd}(z) + 1.$$

Thus, we obtain that $b_{even}(z) - 1 = b(z) - 2 - \big(b_{odd}(z) - 1\big)$ is divisible by $(1 - z)^{L+1}$. Consequently, we obtain that if a subdivision scheme S_b has the stepwise polynomial reproduction property of degree $\leq L$, then $b_{odd}(z) - 1$ and $b_{even}(z) - 1$ are divisible by $(1 - z)^{L+1}$. Furthermore, we shall show that the opposite statement holds also. Before we characterize such subdivision schemes, we need the following lemma:

Lemma 2.7. *For an integer $L \geq 0$, let S_b be a subdivision scheme with a finite set of mask $\{b_i\}_{i \in \mathbb{Z}}$. And let $b_{odd}(z)$, $b_{even}(z)$ be the Laurents polynomials defined as in* (21) *Then the following statements are all equivalent:*

(i) $b_{odd}(z) - 1$ and $b_{even}(z) - 1$ are divisible by $(1 - z)^{L+1}$;

(ii) $b_{odd}(z) + 1$ and $b_{even}(z) - 1$ are divisible by $(1 + z)^{L+1}$;

(iii) $b(z)$ and $b(z) - 2$ are divisible by $(1 + z)^{L+1}$ and $(1 - z)^{L+1}$, respectively.

Proof. The proof is straightforward. By writing $b(z)$ and $b(z) - 2$ as

$$b(z) = b_{odd}(z) + 1 + b_{even}(z) - 1 \quad \text{and} \quad b(z) - 2 = b_{odd}(z) - 1 + b_{even}(z) - 1$$

and replacing z with $-z$, we can show these equivalences. $\qquad\square$

Now, we are ready to characterize the stepwise polynomial reproduction property of a subdivision scheme. We can see that the following characterization holds without the non-singularity assumption of a subdivision scheme.

Theorem 2.8. *For an integer $L \geq 0$, let S_b be a subdivision scheme with a finite set of mask $\{b_i\}_{i \in \mathbb{Z}}$. And let $b_{odd}(z)$, $b_{even}(z)$ be the Laurents polynomials defined by*

$$b_{odd}(z) = \sum_{k \in \mathbb{Z}} b_{2k+1} z^{2k+1} \quad \text{and} \quad b_{even}(z) = \sum_{k \in \mathbb{Z}} b_{2k} z^{2k}.$$

Then the mask $\{b_i\}_{i \in \mathbb{Z}}$ satisfies the polynomial reproduction property of degree L,

$$(22) \qquad\qquad \sum_{k \in \mathbb{Z}} b_{j-2k} p(k) = p\left(\frac{j}{2}\right), \quad j \in \mathbb{Z}, \quad p \in P_L,$$

if and only if the identities

$$(23) \qquad\qquad\qquad \sum_{k \in \mathbb{Z}} b_{2k+1} = \sum_{k \in \mathbb{Z}} b_{2k} = 1$$

hold, and $b_{odd}(z) - 1$ and $b_{even}(z) - 1$ are divisible by $(1 - z)^{L+1}$.

Proof. Assume that the mask $\{b_i\}_{i \in \mathbb{Z}}$ satisfies the polynomial reproduction property (22). Substituting $p(x) = 1$ into (22), we obtain the identities (23). To prove that $b_{odd}(z) - 1$ and $b_{even}(z) - 1$ are divisible by $(1 - z)^{L+1}$, it is sufficient to show that for each $i = 0, \ldots, L$, the Laurent polynomials $b_{odd}(z)$ and $b_{even}(z)$ satisfy

$$b_{odd}^{(i)}(1) = \delta_{i,0} \quad \text{and} \quad b_{even}^{(i)}(1) = \delta_{i,0},$$

where $b^{(i)}(z)$ denotes the i-th derivative $\frac{d^i}{dz^i} b(z)$ and δ is the Kronecker delta function. From (23), we have directly that

$$b_{odd}(1) = 1 \quad \text{and} \quad b_{even}(1) = 1,$$

and it remains to show that

$$b_{odd}^{(i)}(1) = 0 \quad \text{and} \quad b_{even}^{(i)}(1) = 0, \quad i = 1, \ldots, L.$$

For our argument, we define polynomials $p_\ell(x)$ of degree ℓ by

$$p_\ell(x) = (-2x + 1)(-2x)(-2x - 1) \cdots (-2x - \ell + 2), \quad \ell = 1, \ldots, L.$$

Then for each $\ell = 1, \ldots, L$, we can see that

$$b_{odd}^{(\ell)}(1) = \sum_{k \in \mathbb{Z}} a_{1-2k} p_\ell(k).$$

Since p_ℓ is a polynomial of degree $\leq L$, we have from (22) that

$$\sum_{k\in\mathbb{Z}} a_{1-2k} p_\ell(k) = p_\ell\left(\frac{1}{2}\right).$$

On the other hand, the definition of p_ℓ verifies that $p_\ell(\frac{1}{2}) = 0$, which shows $b^{(\ell)}(1) = 0$. To show that $b^{(\ell)}_{even}(1) = 0$ for each $\ell = 1, \ldots, L$, we apply the same argument to polynomials $q_\ell(x)$ of degree ℓ defined by

$$q_\ell(x) = 2x(2x-1)\cdots(2x-\ell+1), \quad \ell = 1, \ldots, L,$$

and we obtain from (22) that for each $i = 0, \ldots, L$, $b^{(i)}_{even}(1) = \delta_{i,0}$, which shows that $b_{odd}(z) - 1$ and $b_{even}(z) - 1$ are divisible by $(1-z)^{L+1}$.

Now we assume that the mask $\{b_i\}_{i\in\mathbb{Z}}$ satisfies the identities (23) and $b_{odd}(z) - 1$ and $b_{even}(z) - 1$ are divisible by $(1-z)^{L+1}$. Then from Lemma 2.7, the Laurent polynomial $b(z)$ given as

$$b(z) = \sum_{i\in\mathbb{Z}} b_i z^i = b_{even}(z) + b_{odd}(z)$$

is divisible by $(1+z)^{L+1}$ and $b(z) - 2$ is divisible by $(1-z)^{L+1}$. Let $p \in P_L$ be a polynomial of degree $\leq L$ and let $f^1 = \{f^1_j\}_{j\in\mathbb{Z}}$ be the refined data from $f^0 = \{f^0_j\}_{j\in\mathbb{Z}}$ with $f^0_j = p(j)$ given by

$$f^1_j = \sum_{k\in\mathbb{Z}} b_{j-2k} p(k), \quad j \in \mathbb{Z}.$$

Also, let $q = \{q_i\}_{i\in\mathbb{Z}}$ with $q_i = p(\frac{i}{2})$. Since the refinement relation induces the identity

$$f^1(z) = b(z) f^0(z^2),$$

the corresponding Laurent polynomials satisfy the identities

$$f^1(z) = b(z)f^0(z^2) = \frac{1}{2}(b(z)q(z) + b(z)q(-z)) = \frac{1}{2}b(z)q(z).$$

Here we used the identity $q(z) + q(-z) = 2f^0(z^2)$ and we applied the condition (ii) of Lemma 2.6 to obtain the equation $b(z)q(-z) = 0$. On the other hand, since $b(z) - 2$ is divisible by $(1-z)^{L+1}$, we obtain from (i) of Lemma 2.6 that

$$(b(z) - 2)q(z) = 0 \quad \text{or} \quad b(z)q(z) = 2q(z),$$

for $q(z)$ is stemmed from the polynomial $q(x) = p(\frac{x}{2})$ of degree $\leq L$. Consequently, we obtain the equation

$$f^1(z) = q(z).$$

Identifying the coefficients, we conclude that we have

$$f^1(j) = q(j), \quad j \in \mathbb{Z},$$

that is, the mask satisfies the polynomial reproduction property of degree L,

$$\sum_{k \in \mathbb{Z}} b_{j-2k} p(k) = p\Big(\frac{j}{2}\Big), \quad j \in \mathbb{Z}, \quad p \in P_L.$$

This completes the proof. $\square$

In the following example, we list the masks of the proposed scheme for $n = 0, 1, 2$.

Examples ([19, Example 1]).

- For $n = 0$, we have the mask of non-interpolatory scheme:

$$\Big[w, \ v, \ \frac{1}{2} - w, \ 1 - 2v, \ \frac{1}{2} - w, \ v, \ w\Big].$$

In case when $v = 0$, it becomes the DGL 4-point scheme:

$$\Big[w, \ 0, \ \frac{1}{2} - w, \ 1, \ \frac{1}{2} - w, \ 0, \ w\Big].$$

When we set $w = 0$, we get the same mask as Choi et al. [2] proposed:

$$\Big[v, \frac{1}{2}, 1 - 2v, \frac{1}{2}, v\Big].$$

If we set $v = 1/8$, this scheme becomes the cubic B-spline subdivision scheme:

$$\Big[\frac{1}{8}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{8}\Big].$$

Also, when we set $v = 3/16, w = 1/32$, the proposed scheme becomes the 6-th order B-spline subdivision scheme:

$$\Big[\frac{1}{32}, \frac{6}{32}, \frac{15}{32}, \frac{20}{32}, \frac{15}{32}, \frac{6}{32}, \frac{1}{32}\Big].$$

- For $n = 1$, we get the scheme:

$$\Big[w, v, -\frac{1}{16} - 3w, -4v, \frac{9}{16} + 2w, 1 + 6v, \frac{9}{16} + 2w, -4v, -\frac{1}{16} - 3w, v, w\Big].$$

In case when $v = 0$, it becomes the 6-point Weissman scheme:

$$\Big[w, 0, -\frac{1}{16} - 3w, 0, \frac{9}{16} + 2w, 1, \frac{9}{16} + 2w, 0, -\frac{1}{16} - 3w, 0, w\Big].$$

In the case of $v = w = 0$, it becomes the 4-point DD scheme:

$$\Big[-\frac{1}{16}, \ 0, \ \frac{9}{16}, \ 1, \ \frac{9}{16}, \ 0, \ -\frac{1}{16}\Big].$$

- For $n = 2$, we obtain the scheme:

$$\Big[\quad \ldots, 1 - 20v, \frac{75}{128} - 5w, \ 15v, \ -\frac{25}{256} + 9w, \ -6v, \ \frac{3}{256} - 5w, \ v, \ w\Big],$$

where only the masks $\{a_i\}_{i=0}^8$ are given and the others are obtained from the symmetry of the mask. In case when $v = 0$, it becomes the 8-point ISSS:

$$\left[\ \ldots, 1, \frac{75}{128} - 5w, \ 0, \ -\frac{25}{256} + 9w, \ 0, \ \frac{3}{256} - 5w, \ 0, \ w \right].$$

In the case of $v = w = 0$, it becomes the 6-point DD scheme:

$$\left[\frac{3}{256}, \ 0, \ -\frac{25}{256}, \ 0, \ \frac{75}{128}, \ 1, \ \frac{75}{128}, \ 0, \ -\frac{25}{256}, \ 0, \ \frac{3}{256} \right].$$

3. Analysis of the proposed schemes

The proposed subdivision schemes in this work have two free parameters. In this section, we illustrate the performance of the subdivision scheme with a mask given as in (10) and (12). In this section, we analyse the smoothness of the proposed scheme for the two special cases when $n = 0$ and $n = 1$. By choosing appropriate values for two parameters, the proposed scheme provides up to C^4-smoothness and satisfies the sum rule of order 6. In the univariate case, a mask $\{a_i\}_{i\in\mathbb{Z}}$ is said to satisfy the sum rules of order k if

$$(24) \qquad \sum_{j\in\mathbb{Z}}(2j)^m a_{2j} = \sum_{j\in\mathbb{Z}}(2j+1)^m a_{2j+1}, \quad m = 0, 1, \ldots, k-1.$$

We also present some numerical examples by setting the free parameters to various values.

In a convergent subdivision scheme, the corresponding mask $\{a_i\}_{i\in\mathbb{Z}}$ necessarily satisfies the condition

$$(25) \qquad \sum_{i\in\mathbb{Z}} a_{2i} = \sum_{i\in\mathbb{Z}} a_{2i+1} = 1.$$

Dyn [8] provided a sufficient and necessary condition for a uniformly convergent subdivision scheme: For a subdivision scheme S, S is uniformly convergent if and only if there is an integer $L \geq 1$, such that

$$\left\| \left(\tfrac{1}{2}S_1\right)^L \right\|_\infty < 1,$$

where S_1 is the subdivision scheme associated with the mask q, where $a(z) = (\frac{1+z}{2})q(z)$ and satisfying

$$df^k = S_1 df^{k-1}, \quad k = 1, 2, \ldots,$$

for the control points $f^k = S^k f^0$ and $df^k = \{(df^k)_i = 2^k(f_{i+1}^k - f_i^k)\}_{i\in\mathbb{Z}}$, and the norm $\|S\|_\infty$ of a subdivision scheme S with a mask $\{a_i\}_{i\in\mathbb{Z}}$ is defined by

$$\|S\|_\infty = \max\left\{ \sum_{i\in\mathbb{Z}} |a_{2i}|, \sum_{i\in\mathbb{Z}} |a_{2i+1}| \right\}.$$

By the linearity, the smoothness of the limit function $S^\infty f^0$ for a given sequence f^0 of control points is equivalent to that of $\varphi = S^\infty \delta, \delta = \{\delta_{n,0}\}_{n\in\mathbb{Z}}$. The

function φ is called the basic limit function of a subdivision scheme. Note that when $v = 0$, this scheme becomes interpolatory.

We say that a subdivision scheme is C^m if for the data $\delta_{n,0}$ the basic limit function has continuous derivatives up to order m. The following theorem is available to check the smoothness of the subdivision scheme.

Theorem 3.1 ([8], [11]). *Consider a subdivision scheme S with symbol*

$$a(z) = \left(\frac{1+z}{2z}\right)^m a_m(z).$$

If the subdivision scheme S_m corresponding to $a_m(z)$ converges uniformly, then S is C^m.

Based on Theorem 3.1, we investigate the smoothness range of the two free parameters v and w for the proposed 4-point and 6-point subdivision schemes.

3.1. The proposed 4-point subdivision schemes [19]

For the proposed 4-point ($n = 0$) subdivision scheme with a mask

$$a(z) = wz^{-3} + vz^{-2} + (\tfrac{1}{2} - w)z^{-1} + (1 - 2v) + (\tfrac{1}{2} - w)z + vz^2 + wz^3,$$

we have the mask of subdivision scheme S_1

$$a_1(z) = 2[wz^{-2} + (v - w)z^{-1} + (\tfrac{1}{2} - v) + (\tfrac{1}{2} - v)z + (v - w)z^2 + wz^3],$$

where $a_1(z) = \frac{2z}{1+z}a(z)$. It is easy to verify that $a(z)$ and $a_1(z)$ satisfy the necessary condition (25) for the convergence of S and S_1. If

$$\left\|\frac{1}{2}S_1\right\|_\infty = |w| + |\tfrac{1}{2} - v| + |v - w| < 1,$$

then this scheme converges to a continuous limit function. We have the mask of scheme S_2 by using the relation $a_2(z) = \frac{2z}{1+z}a_1(z)$:

$$a_2(z) = 4[wz^{-1} + (v - 2w) + (\tfrac{1}{2} - 2v + 2w)z + (v - 2w)z^2 + wz^3].$$

If

$$\left\|\frac{1}{2}S_2\right\|_\infty = \max\{4|w| + 2|\tfrac{1}{2} - 2v + 2w|, 4|v - 2w|\} < 1,$$

then this scheme is $C^1(\mathbb{R})$.

For C^2 continuity, $a_2(z)$ should satisfy the necessary condition (25). This implies

$$w = \frac{v}{2} - \frac{1}{16}.$$

From the relation $a_3(z) = \frac{2z}{1+z}a_2(z)$, we have the mask of scheme S_3:

$$a_3(z) = 8[(\tfrac{v}{2} - \tfrac{1}{16}) + (-\tfrac{v}{2} + \tfrac{3}{16})z + (-\tfrac{v}{2} + \tfrac{3}{16})z^2 + (\tfrac{v}{2} - \tfrac{1}{16})z^3],$$

and

$$\left\|\frac{1}{2}S_3\right\|_\infty = 2|v - \tfrac{1}{8}| + 2|\tfrac{3}{8} - v| < 1,$$

which implies that $0 < v < \frac{1}{2}$. Hence for the case $w = \frac{v}{2} - \frac{1}{16}$ and $0 < v < \frac{1}{2}$, this scheme is $C^2(\mathbb{R})$.

For C^3 continuity, $a_3(z)$ should satisfy the necessary condition (25), which is always true. The mask of S_4 is

$$a_4(z) = 16[(\tfrac{v}{2} - \tfrac{1}{16})z + (-v + \tfrac{1}{4})z^2 + (\tfrac{v}{2} - \tfrac{1}{16})z^3],$$

and

$$\left\| \tfrac{1}{2}S_4 \right\|_\infty = \max\{16|\tfrac{v}{2} - \tfrac{1}{16}|, 8| - v + \tfrac{1}{4}|\} < 1,$$

which implies that $\frac{1}{8} < v < \frac{3}{8}$. This scheme is $C^3(\mathbb{R})$ in case $w = \frac{v}{2} - \frac{1}{16}$ and $\frac{1}{8} < v < \frac{3}{8}$. From the fact that $a_4(z)$ should satisfy the necessary condition (25) for C^4 continuity, we get

$$v = \frac{3}{16}, \quad w = \frac{1}{32},$$

and we have the mask of scheme S_5

$$a_5(z) = z^2 + z^3,$$

and

$$\left\| \tfrac{1}{2}S_5 \right\|_\infty = \frac{1}{2}.$$

Hence this scheme is $C^4(\mathbb{R})$.

It is well-known ([8]) that for a given finite mask $\mathbf{a}$, in the univariate case, the basic limit function φ satisfies the refinement equation

$$\varphi = \sum_{j \in \mathbb{Z}} a_j \varphi(2 \cdot -j).$$

We can verify the accuracy of the function φ in terms of the sum rules associated with the mask. We say that φ has accuracy k if the linear space

$$S(\varphi) = \{\varphi *' b : b \in l(\mathbb{Z})\},$$

contains the polynomial space P_{k-1}. Here the semi-convolution $\varphi *' b$ is the sum defined by

$$\varphi *' b = \sum_{i \in \mathbb{Z}} \varphi(\cdot - i)b(i).$$

If the mask $\mathbf{a}$ satisfies the sum rules of order k in (24),

$$\sum_{j \in \mathbb{Z}}(2j)^m a_{2j} = \sum_{j \in \mathbb{Z}}(2j+1)^m a_{2j+1}, \quad m = 0, 1, \ldots, k - 1,$$

then it was proved in [17] (also see [18]) that φ has accuracy k.

For the proposed 4-point $(n = 0)$ subdivision scheme with a mask

$$a(z) = wz^{-3} + vz^{-2} + (\tfrac{1}{2} - w)z^{-1} + (1 - 2v) + (\tfrac{1}{2} - w)z + vz^2 + wz^3,$$

we can easily get the fact that in case $w = v/2 - 1/16$, the mask of the proposed scheme has the sum rules of order 4. And for $v = 3/16$ and $w = 1/32$, it has sum rules of order 6.

TABLE 1. By computing $||(\frac{1}{2}S_m)^{10}||_\infty < 1, m = 1, 2, 3, 4, 5$, for the proposed 4-point ($n = 0$) subdivision scheme S with mask $[w, v, 1/2 - w, 1 - 2v, 1/2 - w, v, w]$, we obtain the ranges of v and w with MAPLE, Digits:=30.

Smoothness	Range of v	Range of w
C^2	$0 < v < 1/2$	$w = 1/2(v - 1/8)$
C^3	$1/8 < v < 3/8$	$w = 1/2(v - 1/8)$
C^4	$3/16$	$1/32$

A summary of ranges of two parameters for smoothness of the proposed 4-point ($n = 0$) scheme can be seen in Table 1. The segment $w = 1/2(v - 1/8)$ represents the ranges of C^2 and C^3 smoothness for $0 < v < 1/2$ and $1/8 < v < 3/8$, respectively. When $v = 3/16$ and $w = 1/32$, the scheme becomes the 6-th order B-spline scheme which induces C^4 smoothness, as known well.

3.2. The proposed 6-point subdivision schemes [19]

For $n = 1$, we have a mask of the proposed 6-point ($n = 1$) subdivision scheme

$$
\begin{aligned}
a(z) = {} & wz^{-5} + vz^{-4} - (\tfrac{1}{16} + 3w)z^{-3} - 4vz^{-2} + (\tfrac{9}{16} + 2w)z^{-1} \\
& + (1 + 6v) + (\tfrac{9}{16} + 2w)z - 4vz^2 - (\tfrac{1}{16} + 3w)z^3 + vz^4 + wz^5.
\end{aligned}
$$

In much the same way to the proposed 4-point scheme, we summarize the range of two parameters v, w for the smoothness $C^m, m = 0, 1, \ldots, 5$ in Table 2.

TABLE 2. By computing $||(\frac{1}{2}S_m)^{10}||_\infty < 1, m = 1, 2, \ldots, 6$, for the proposed 6-point subdivision scheme S with mask $[w, v, -(1/16 + 3w), -4v, 9/16 + 2w, 1 + 6v, 9/16 + 2w, -4v, -(1/16 + 3w), v, w]$, we obtain the ranges of v and w with MAPLE, Digits:=30.

Smoothness	Range of v	Range of w
C^4	$-0.0654296875000000 < v < -0.0290527343750000$	$w = v/2 + 3/256$
C^5	$-0.0468750000000000 < v < -0.0382050771680549$	$w = v/2 + 3/256$

The proposed subdivision scheme in this work has two free parameters. At the cost of two parameters, we can see that the proposed scheme can provide up to C^4-smoothness of the basic limit function and the accuracy 6 in Table 1. Here, the accuracy 6 means that the mask has the sum rules of order 6 but this is different from the accuracy of a subdivision scheme which is determined by the polynomial reproducing property. Applying some suitable operator Q

(see [21]) before applying S, $S^\infty Q$ reproduces polynomials up to degree $k-1$, i.e.,

$$S^\infty Qf = f, \quad \forall f \in P_{k-1}.$$

The main idea of relaxing the minimal support condition is an interesting one; it leads to a generalization of the DD masks. With specific choices of the two free parameters, this generalization then includes various other well known subdivision masks from interpolatory scheme through approximating scheme including B-spline scheme.

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