AUTOMORPHISMS OF THE ZERO-DIVISOR GRAPH OVER 2 × 2 MATRICES

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Abstract. The zero-divisor graph of a noncommutative ring $R$, denoted by $\Gamma(R)$, is a graph whose vertices are nonzero zero-divisors of $R$, and there is a directed edge from a vertex $x$ to a distinct vertex $y$ if and only if $xy = 0$. Let $R = M_2(F_q)$ be the 2 × 2 matrix ring over a finite field $F_q$. In this article, we investigate the automorphism group of $\Gamma(R)$.

1. Introduction

Zero-divisor graphs have received a lot of attention (see [1, 2, 4, 6, 13]), because they are helpful for revealing the ring-theoretic properties via their graph-theoretic properties. In 1988, I. Beck first introduced the concept of zero-divisor graphs of commutative rings in [7], where all elements of a commutative ring $R$ are defined to be vertices and distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. In such a graph, the vertex 0 is adjacent to every other vertex, and non-zero-divisors are adjacent only to 0. In order to better illustrate the zero-divisor structure of a ring, D. F. Anderson and P. S. Livingston [5] redefined the notion of a zero-divisor graph by cutting off 0 and non-zero-divisors from the graph. Let $R$ be a commutative ring (with 1) and let $Z(R)$ be the set of zero-divisors of $R$. The zero-divisor graph $\Gamma(R)$ of $R$ (defined by D. F. Anderson and P. S. Livingston [5]) is a graph with vertices $Z(R)^* = Z(R) \setminus \{0\}$, the set of nonzero zero-divisors of $R$, and distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. S. P. Redmond [16] further extended the concept of a zero-divisor graph to a noncommutative ring $R$, also written as $\Gamma(R)$, by taking the vertex set to be $Z(R)^*$, and there is a directed edge from a vertex...
$x$ to a distinct vertex $y$ if and only if $xy = 0$. In [9] and [10] F. DeMeyer et al. introduced the definition of zero-divisor graphs over arbitrary semigroups.

Automorphisms of graphs play an important role both in graph theory and in algebra, and characterization of the full automorphisms of a graph is often a difficult work. Until now, little is known (as far as we know) about the automorphisms of zero-divisor graphs. Now we list some known results related to ours. In 1999, D. F. Anderson and P. S. Livingston [5] proved that if $n \geq 4$ is a non-prime integer, then Aut$(\Gamma(Z_n))$ is a direct product of some symmetric groups. In 2002, S. B. Mulay [14] established some group-theoretic properties of the group of graph-automorphisms of zero-divisor graphs over commutative rings. In 2002, F. DeMeyer and K. Schneider [11] studied the relationship between Aut$(\Gamma(R))$ and Aut$(\Gamma(R))$ when $R$ is a commutative ring. In 2008, J. Han (see [12, Theorem 3.9]) showed that Aut$(\Gamma(M_2(\mathbb{Z}_p)))$ is isomorphic to the symmetric group $S_{p+1}$ of degree $p+1$ when $p$ is a prime. In 2011, S. Park and J. Han [15] generalized [12, Theorem 3.9] to an arbitrary finite field $F_q$ with $q$ elements, and proved that Aut$(\Gamma(M_2(F_q))) \cong S_{q+1}$. When reading the proof of [15, Theorem 3.8], we find some major mistakes there (the automorphism $\sigma$ they constructed in the proof fails to be a bijection), which inspires us to determine the full automorphisms of $\Gamma(M_2(F_q))$ again. As an application of our main theorem, we show that [12, Theorem 3.9] and [15, Theorem 3.8] are both wrong.

This article is organized as follows. In Section 2, we give some preliminary results and introduce the compressed zero-divisor graph $\Gamma_E(M_2(F_q))$. In Section 3, we determine the automorphisms of $\Gamma_E(M_2(F_q))$. Section 4 is devoted to investigate the automorphism group of $\Gamma(M_2(F_q))$ via what has been obtained in Section 3.

2. Preliminaries and notations

Let $\Gamma$ be a directed graph with vertex set $V(\Gamma)$. We write $x \rightarrow y$ to mean that there is a directed edge from a vertex $x$ to a distinct vertex $y$, and write $x \leftrightarrow y$ to mean $x \rightarrow y$ and $y \rightarrow x$. A graph $H$ is called a subgraph of $\Gamma$ if $V(H) \subseteq V(\Gamma)$ and for any $x, y \in V(H)$, $x \rightarrow y$ in $\Gamma$ whenever $x \rightarrow y$ in $H$. Further, $H$ is called an induced subgraph of $\Gamma$ if for any $x, y \in V(H)$, $x \rightarrow y$ in $\Gamma$ if and only if $x \rightarrow y$ in $H$. An induced subgraph $K$ of $\Gamma$ is called a clique if $x \leftrightarrow y$ for any distinct vertices $x$ and $y$ in $K$. A set of vertices that induces a subgraph with no edges is called an independent set. For any vertex $x$ of $\Gamma$, $N_l(x) = \{y \in V(\Gamma) \mid y \rightarrow x\}$ and $N_r(x) = \{y \in V(\Gamma) \mid x \rightarrow y\}$ are respectively called the left neighborhood and the right neighborhood of $x$. For any set $X$, we denote by $|X|$ the cardinality of $X$. $|N_l(x)|$ is called the in-degree of $x$, and $|N_r(x)|$ is called the out-degree of $x$. A bijection $\sigma$ on $V(\Gamma)$ is said to be an automorphism of $\Gamma$ if for any two vertices $x, y \in V(\Gamma)$, $x \rightarrow y$ if and only if $\sigma(x) \rightarrow \sigma(y)$. Denote by Aut$(\Gamma)$ the automorphism group of $\Gamma$. 

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Hereafter, \( R \) will always denote \( M_2(F_q) \), the \( 2 \times 2 \) matrix ring over a finite field \( F_q \) with \( q \geq 2 \) elements. By \( M_{1 \times 2}(F_q) \) we mean the set of all \( 1 \times 2 \) matrices over \( F_q \), and by \( \alpha^t \) we mean the transpose of \( \alpha \in M_{1 \times 2}(F_q) \). Let \( M_{1 \times 2}^1(F_q) \) be a subset of \( M_{1 \times 2}(F_q) \) consisting of the vectors whose first nonzero component is 1, i.e., \( M_{1 \times 2}^1(F_q) = \{(0 \ 1), (1 \ a) \ | \ a \in F_q \} \). By \( Z(R) \) we denote the set of all zero-divisors of \( R \), and by \( U(R) \) we denote the set of all units of \( R \). Then \( R = Z(R) \cup U(R) \). For every matrix \( A \in R \), let \( \det(A) \) be the determinant of \( A \), and let \( r(A) \) be the rank of \( A \). In \( R \), the matrix unit which has 1 in the 
(i, j) position and 0 elsewhere is denoted by \( E_{ij} \), the zero matrix is denoted by \( 0 \), and the identity matrix is denoted by \( I \). For any subset \( X \) of \( F_q \) (resp., \( R; M_{1 \times 2}(F_q) \)), let \( X^* = X - \{0\} \). If \( X \) is either an element or a subset of \( R \), then the left annihilator of \( X \) is \( \text{ann}_l X = \{A \in R \ | \ AX = 0\} \) and the right annihilator of \( X \), denoted by \( \text{ann}_r X \), is similarly defined. As usual, \( \Gamma(R) \) denotes the zero-divisor graph over \( R \), i.e., \( \Gamma(R) \) is a graph with vertex set \( Z(R)^* \), and \( A \rightarrow B \) if and only if \( A \neq B \) and \( AB = 0 \). By \( S_{F_q} \) we mean the symmetric group over \( F_q \), and \( S_{F_q} \) is similarly defined.

**Lemma 2.1.** The following three conditions are equivalent:

(i) \( A \in Z(R)^* \);

(ii) \( r(A) = 1 \);

(iii) \( A = \alpha \beta \) for \( \alpha, \beta \in M_{1 \times 2}(F_q)^* \).

**Proof.** Note that \( A \in Z(R) \) if and only if \( \det(A) = 0 \). Thus \( A \in Z(R)^* \) if and only if \( r(A) = 1 \), since \( A \) is a \( 2 \times 2 \) matrix. This yields (i) \( \Leftrightarrow \) (ii). Clearly, we have (ii) \( \Leftrightarrow \) (iii). \( \square \)

From the lemma above, each vertex \( A \) in \( \Gamma(R) \) can be written as \( A = \alpha \beta \) for some \( \alpha, \beta \in M_{1 \times 2}(F_q)^* \). The set of nonzero scalar multiples of \( A \in R \) is denoted by \( [A] \), i.e., \( [A] = \{aA \ | \ a \in F_q^* \} \). Then the multiplication \( [A] [B] = [AB] \) is well-defined, and \( [A] = [B] \) if and only if \( B = aA \) for some \( a \in F_q^* \). If \( A \in V(\Gamma(R)) \), then there exist unique \( \alpha, \beta \in M_{1 \times 2}(F_q) \) such that \( [A] = [\alpha \beta] \). We call \( \alpha \beta \) the standard representation of \( [A] \).

**Lemma 2.2.** If \( \alpha \in M_{1 \times 2}(F_q) \), then there exists a unique \( \beta \in M_{1 \times 2}(F_q) \) such that \( \alpha \beta^t = 0 \). Meanwhile \( \beta \alpha^t = 0 \), and

(i) if \( \alpha = (0 \ 1) \), then \( \beta = (1 \ 0) \),

(ii) if \( \alpha = (1 \ 0) \), then \( \beta = (0 \ 1) \),

(iii) if \( \alpha = (1 \ a) \) for some \( a \neq 0 \), then \( \beta = (1 - a^{-1}) \).

**Proof.** It is easy to get these results by direct calculations. \( \square \)

**Lemma 2.3.** If \( A \in Z(R)^* \), then \( A^2 = 0 \) if and only if \( A \in [E_{12}] \cup [E_{21}] \cup_{a \in F_q^*} \{[1 - a^{-1} \ a] \} \).

**Proof.** Obviously, we get the ‘if’ part. For the ‘only if’ part, assume that \( \alpha \beta \) is the standard representation of \( [A] \), where \( \alpha, \beta \in M_{1 \times 2}(F_q) \). Then it follows from \( A^2 = 0 \) if and only if \( [A]^2 = 0 \) that \( \beta \alpha^t = 0 \). By Lemma 2.2, we have
whose vertices are the equivalence classes induced by $\sim$ ann $\mathbb{R}$ compressed zero-divisor graph of $\mathbb{R}$

Applying Lemma 2.2 on Eq. (1), we get

$$D = \{ 1 \} \cup \{ 0 \}$$

Definition 2.5. A definition to noncommutative rings.

Lemma 2.4. Let $A, B \in Z(\mathbb{R})^*$. Then ann$_1 A \cap$ ann$_1 B \neq \{ 0 \}$ and ann$_r A \cap$ ann$_r B \neq \{ 0 \}$ if and only if $[A] = [B]$. In particular, ann$_1 A =$ ann$_1 B$ if and only if $[A] = [B]$.

Proof. Clearly, “$\Rightarrow$” holds. In order to prove “$\Leftarrow$”, assume that $C \neq 0$ and $C \in$ ann$_1 A \cap$ ann$_1 B$. Then $[C] \in$ ann$_1 A \cap$ ann$_1 B$. That means $[C][A] = 0$ and $[C][B] = 0$. Suppose that $\alpha_1^t, \beta_1^t, \gamma_2^t, \delta_2^t$ are the standard representations of $[A], [B], [C]$, respectively. It follows from $[C][A] = 0$ and $[C][B] = 0$ that

$$\gamma_2 \alpha_1^t = \gamma_2 \beta_1^t = 0.$$

Applying Lemma 2.2 on Eq. (1), we get $\alpha_1 = \beta_1$. Similarly, if $D \neq 0$ and $D \in$ ann$_r A \cap$ ann$_r B$, then $\alpha_2 = \beta_2$. Thus $[A] = [\alpha_1 \alpha_2] = [\beta_1 \beta_2] = [B]$. □

S. B. Mulay [14] introduced the compressed zero-divisor graph of a commutative ring while studying automorphisms. The relation on a commutative ring $\mathbb{R}$ gives $s \sim t$ if and only if ann$_R(s) =$ ann$_R(t)$ is an equivalence relation.

The compressed zero-divisor graph $\Gamma_E(\mathbb{R})$ is the (undirected) graph whose vertices are the equivalence classes induced by $\sim$ other than $\{0\}$ and $[1]$, such that distinct vertices $[s]$ and $[t]$ are adjacent in $\Gamma_E(\mathbb{R})$ if and only if $st = 0$. This graph was later studied more extensively in [3, 8, 17]. Now, we extend this definition to noncommutative rings.

Definition 2.5. The relation on a noncommutative ring $\mathbb{R}$ gives $s \sim t$ if and only if ann$_R(s) =$ ann$_R(t)$ is an equivalence relation. The compressed zero-divisor graph of $\mathbb{R}$, denoted by $\Gamma_E(\mathbb{R})$, is the directed graph whose vertices are the equivalence classes induced by $\sim$ other than $\{0\}$ and $[1]$, such that $[s] \rightarrow [t]$ in $\Gamma_E(\mathbb{R})$ if and only if $[s] = [t]$ and $st = 0$.

Let $\Gamma_E(\mathbb{R})$ be the compressed zero-divisor graph of $\mathbb{R} = M_2(F_q)$, i.e., $\Gamma_E(\mathbb{R})$ is a graph with vertex set $\{ [A] : A \in Z(\mathbb{R})^* \}$, and there is a directed edge from a vertex $[A]$ to a distinct $[B]$ if and only if $AB = 0$. For any vertex $[A]$ in $\Gamma_E(\mathbb{R})$, suppose that $A = \alpha^t \beta = (a_1 a_2)(b_1 b_2)$ and the first nonzero component of $\alpha$ (resp., $\beta$) is $a_i$ (resp., $b_j$), and call $[A]$ a vertex of type $(i, j)$. Then all vertices in $\Gamma_E(\mathbb{R})$ can be categorized into four types as follows.

- type (1,1): $[A] = [(1 0)^t(1 b)] = [(1 b a b)]$, $a, b \in F$,
- type (1,2): $[A] = [(1 a)^t(0 1)] = [(a 0 b^t)]$, $a \in F$,
- type (2,1): $[A] = [(0 1)^t(1 b)] = [(0 b 1)]$, $b \in F$,
- type (2,2): $[A] = [(0 1)^t(0 1)] = [E_{22}]$.

It is obvious to see that in $\Gamma_E(\mathbb{R})$, there are $q^2, q, q, 1$ vertices in types (1,1), (1,2), (2,1), (2,2) respectively. It follows that there are $(q + 1)^2$ vertices in $\Gamma_E(\mathbb{R})$, and $(q - 1)(q + 1)^2$ vertices in $\Gamma(\mathbb{R})$ since each vertex $[A] \in V(\Gamma_E(\mathbb{R}))$ contains $q - 1$ different vertices in $\Gamma(\mathbb{R})$. 

Lemma 3.1. \( \sigma \in E \) we prove that \( \Pi \) is a subgroup of \( \text{Aut}(\Gamma_{FE}) \) over any given set, particular, \( \sigma(P) = P \).

Proof. Clearly, we get \( \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( P^{-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \). Since \( \sigma_P([E_{11}]) = [E_{11}] \), we get
\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]
Thus \( \left( \begin{array}{cc} a a_1 & ab_1 \\ c a_1 & cb_1 \end{array} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). This yields \( a \neq 0, a_1 \neq 0, \text{ and } b_1 = c = 0 \). Hence
\[
P = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} a_1 & 0 \\ c_1 & d_1 \end{pmatrix}.
\]
From \( \sigma_P([E_{12}]) = [E_{12}] \), we have
\[
\left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \left( \begin{array}{cc} a_1 & 0 \\ c_1 & d_1 \end{array} \right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]
It follows that \( \left( \begin{array}{cc} ac_1 & ad_1 \\ 0 & 0 \end{array} \right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), and so \( ac_1 = 0 \). Thus \( c_1 = 0 \), and therefore
\[
P = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}.
\]
Since \( \sigma_P([\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}]) = [\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}] \), it follows that
\[
\left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right) \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \left( \begin{array}{cc} a^{-1} & 0 \\ 0 & d^{-1} \end{array} \right) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.
\]
Thus \( \left( \begin{array}{cc} 0 & 0 \\ 1 & ad^{-1} \end{array} \right) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \). This gives \( ad^{-1} = 1 \). Thus \( a = d \), and \( P \) is a nonzero scalar matrix.

Lemma 3.2. \( \Pi \cong U(R)/F_q^*I \).

Proof. Set
\[
\psi : U(R) \rightarrow \Pi, \quad P \mapsto \sigma_P.
\]
Then \( \psi \) is well-defined and \( \psi \) is a surjection. It is easy to prove that \( \sigma_P \sigma_Q = \sigma_{PQ} \), and so \( \psi \) is a surjective homomorphism. By Lemma 3.1, we get \( \text{Ker} \psi = F_q^*I \). Thus \( \Pi \cong U(R)/F_q^*I \). \( \square \)
Next, let us introduce another automorphism $\tau_f$ of $\Gamma_E(R)$.

**Definition 3.3.** Let $f \in S_{F_q}$ that fixes 0, set $f^* \in S_{F_q}$ as follows, and call $f^*$ the companion permutation of $f$.

$$f^*(a) = \begin{cases} 0, & a = 0, \\ -f(-a^{-1})^{-1}, & a \neq 0. \end{cases}$$

**Lemma 3.4.** If $f, g \in S_{F_q}$ both fix 0, then $(f^*)^* = f$, $(f^{-1})^* = (f^*)^{-1}$, and $(fg)^* = f^*g^*$.

**Proof.** Apparently, $(f^*)^*(0) = f(0)$.

In the case $a \neq 0$, it follows from $f^*(-a^{-1})f(a) = -1$ and $(f^*)^*(a)f^*(-a^{-1}) = -1$ that $(f^*)^*(a) = f(a)$. Thus $(f^*)^* = f$.

Clearly, we have $(f^{-1})^*f^*(0) = 0$. For any $a \neq 0$, set $b = f(-a^{-1})$.

Then $(f^{-1})^*f^*(a) = (f^{-1})^*(-f(-a^{-1})^{-1}) = (f^{-1})^*(-b^{-1}) = -f^{-1}(b)^{-1} = -(-a)^{-1} = a$. Thus $(f^{-1})^*f^* = 1$, and so $(f^{-1})^* = (f^*)^{-1}$.

Let $a \in F^*_q$. Set $b = g(-a^{-1})$. Then $(fg)^*(0) = f^*g^*(0)$ and $(fg)^*(a) = -(fg(-a^{-1}))^{-1} = -f(b)^{-1} = f^*(-b^{-1}) = f^*(-g(-a^{-1})^{-1}) = f^*g^*(a)$ yielding $(fg)^* = f^*g^*$. □

Let $f \in S_{F_q}$ that fixes 0, and let $\Sigma = \{\tau_f \in S_{F_q}, f(0) = 0\}$, where the map $\tau_f : V(\Gamma_E(R)) \mapsto V(\Gamma_E(R))$ is defined as the following.

$$\tau_f : [A] = [(1 a)^t(1 b)] \mapsto [(1 f^*(a))^t(1 f(b))], a, b \in F_q,$$

$$[A] = [(1 a)^t(0 1)] \mapsto [(1 f^*(a))^t(0 1)], a \in F_q,$$

$$[A] = [(0 1)^t(1 b)] \mapsto [(0 1)^t(1 f(b))], b \in F_q,$$

$$[A] = [(0 1)^t(0 1)] \mapsto [(0 1)^t(0 1)].$$

**Lemma 3.5.** (i) $\tau_{fg} = \tau_f\tau_g$, $\tau_{f^{-1}} = \tau_f^{-1}$ for every $\tau_f, \tau_g \in \Sigma$.

(ii) $\Sigma$ is a group, and $\Sigma \cong S_{F_q}$.

**Proof.** (i) For any $a, b \in F_q$, we have

$$\tau_{fg}([(1 a)^t(1 b)]) = [(1 (fg)^*(a))^t(1 fg(b))]$$

$$= [(1 f^*g^*(a))^t(1 fg(b))]$$

$$= \tau_f([(1 g^*(a))^t(1 g(b))])$$

$$= \tau_f\tau_g([(1 a)^t(1 b)]).$$

Thus

$$\tau_{fg}([A]) = \tau_f\tau_g([A])$$

for each $[A]$ of type (1,1). Similarly, Eq. (2) holds for any $[A]$ of types (1,2), (2,1), (2,2). Hence $\tau_{fg} = \tau_f\tau_g$, and so $\tau_{f^{-1}}\tau_f = \tau_{f^{-1}}f = \tau_1 = 1$. 
(ii) By (i) we know Σ is a group. Apparently, $S_{F_q}$ is a group and $S_{F_q} \cong \{ f \in S_{F_q} \mid f(0) = 0 \}$. Each $f \in S_{F_q}$ that fixes 0 induces a $\tau_f$ in Σ. Let

$$\varphi : \{ f \in S_{F_q} \mid f(0) = 0 \} \rightarrow \Sigma, \ f \mapsto \tau_f.$$ 

We show that $\varphi$ is an isomorphism, thus $\Sigma \cong \{ \tau_f \mid f \in S_{F_q}, f(0) = 0 \} \cong S_{F_q}$.

Indeed, by (i) we get that the surjection $\varphi$ satisfies $\varphi(fg) = \varphi(f)\varphi(g)$. In what follows, we prove that $\varphi$ is injective, and so $\varphi$ is an isomorphism. In fact, if there exist $f, g \in S_{F_q}$ fix 0 such that $\tau_f = \tau_g$, then for every $b \in F_q$ we have $\tau_f([(0 1)^t(1 b)]) = \tau_g([(0 1)^t(1 b)])$. Thus $f(b) = g(b)$, and so $f = g$. Hence $\varphi$ is injective.

Now we show that $\tau_f$ is an automorphism of $\Gamma_E(R)$.

**Lemma 3.6.** $\tau_f$ is an automorphism of $\Gamma_E(R)$, and $\Sigma$ is a subgroup of $\text{Aut}(\Gamma_E(R))$.

**Proof.** Clearly, $\tau_f$ preserves the type of each vertex in $\Gamma_E(R)$. Since $f$ and $f^*$ are permutations over $F_q$, it follows that $\tau_f$ is bijective. If we can prove that $[A] \rightarrow [B]$ if and only if $\tau_f([A]) \rightarrow \tau_f([B])$ for each $[A], [B] \in V(\Gamma_E(R))$, then $\tau_f$ is an automorphism of $\Gamma_E(R)$. In fact, it follows from $\tau_f([A]) \rightarrow \tau_f([B])$ that $[A] \rightarrow [B]$ since $\tau_{f^{-1}} \tau_f = 1$. Conversely, if $[A] \rightarrow [B]$, then $[A] \neq [B]$ and $[\alpha_1^t\alpha_2][\beta_1^t\beta_2] = 0$, where $\alpha_1^t\alpha_2$ and $\beta_1^t\beta_2$ are respectively the standard representations of $[A]$ and $[B]$. Thus $\alpha_2^t\beta_1 = 0$. Lemma 2.2 yields $\alpha_2, \beta_1$ satisfying one of the three cases as follows.

(i) If $\alpha_2 = (0 1)$, then $\beta_1 = (1 0)$, and so

$$\tau_f([A])\tau_f([B]) = [(\ast \ast)^t(0 1)][(0 1)^t(\ast \ast)] = 0.$$ 

(ii) If $\alpha_2 = (1 0)$, then $\beta_1 = (0 1)$. Now

$$\tau_f([A])\tau_f([B]) = [(\ast \ast)^t(1 0)][(0 1)^t(\ast \ast)] = 0.$$ 

(iii) If $\alpha_2 = (1 a)$ for some $a \neq 0$, then $\beta_1 = (1 - a^{-1})$. Thus

$$\tau_f([A])\tau_f([B]) = [(\ast \ast)^t(1 f(a))][(1 f^*(-a^{-1}))^t(\ast \ast)] = 0.$$ 

By (i)-(iii), we always have $\tau_f([A])\tau_f([B]) = 0$. Note that, $[A] \neq [B]$ and $\tau_f$ is bijective, so we get $\tau_f([A]) \neq \tau_f([B])$. Hence $\tau_f([A]) \rightarrow \tau_f([B])$. □

**Lemma 3.7.** Let $P, Q \in U(R)$, and let $f, g \in S_{F_q}$ fix 0. Then

(i) $\tau_f = 1$ if and only if $f = 1$,

(ii) if $f$ fixes 1 and $\sigma \neq \tau_f$, then $\sigma P = \tau_f = 1$,

(iii) if $f, g \in S_{F_q}$ fix 1 and $\sigma P \tau_f = \sigma Q \tau_g$, then $f = g$ and $P = aQ$ for some $a \in F^*$.

**Proof.** (i) Clearly, if $f = 1$, then $f^* = 1$, and so $\tau_f = 1$. Conversely, by the definition of $\tau_f$, we conclude $f = 1$ if $\tau_f = 1$.

(ii) Note that $\sigma P ([E_{11}]) = \tau_f([E_{11}]) = [E_{11}],$ $\sigma P ([E_{12}]) = \tau_f([E_{12}]) = [E_{12}]$, and $\sigma P ([(0 0)]) = \tau_f([(0 0)])[[(0 1)]] = [f(1 1)]$, it follows from Lemma 3.1 that $\tau_f = \sigma P = 1$. 

Lemma 3.8. By Eq. (3) and (ii), we have
\[
\sigma_{Q^{-1}P} = \tau_{gf^{-1}} \quad (4)
\]
By (i), Eq. (4), and Lemma 3.1, we get \(Q^{-1}P\) is a nonzero scalar matrix and \(gf^{-1} = 1\). Thus \(f = g\), and \(P = aQ\) for some \(a \in F^*\).

In the following of this section, we develop some lemmas to show that any automorphism of \(\Gamma_E(R)\) can be expressed as \(\sigma P \tau_f\), where \(\sigma P\) and \(\tau_f\) are the automorphisms we constructed as above.

**Lemma 3.8.** The in-degree and out-degree of each vertex \([A]\) in \(\Gamma_E(R)\) are both equal to \(q + 1\) if \(A^2 \neq 0\), and are both equal to \(q\) if \(A^2 = 0\).

**Proof.** For any \([B], [A] \in \pi E(R)\) satisfies \([B][A] = 0\), assume that \(\alpha_1, \alpha_2, \beta_1, \beta_2\) are respectively the standard representations of classes \([A], [B]\). Thus \(\alpha_1, \alpha_2, \beta_1, \beta_2 \in M_{1,2}(F_q) \) and \([B][A] = 0\) imply \(\beta_2 \alpha_1 = 0\). By Lemma 2.2, \(\beta_2\) is uniquely determined by \(\alpha_1\). Thus the cardinality of \(\{[B]: [B][A] = 0, B \in \pi E(R)\}\) is depending on the choice of \(\beta_1 \in M_{1,2}(F_q)\). Hence there are \(q + 1\) nonzero classes in \(\{[B]: [B][A] = 0, B \in \pi E(R)\}\) since \(\pi E_{1,2}(F_q) = q + 1\) and different \(\beta_1\) implies different \([\beta_1^2, \beta_2]\). Then \(N_1([A]) = \{[B]: [B][A] = 0, B \in \pi E(R)\} - \{[A]\}\), and so the in-degree of \([A]\) is \(|N_1([A])| = q\) if \(A^2 = 0\), the in-degree of \([A]\) is \(|N_1([A])| = q + 1\) if \(A^2 \neq 0\). By a similar argument, we can get the out-degree of each vertex \([A]\). □

**Lemma 3.9.** If \(\sigma\) is an automorphism of \(\Gamma_E(R)\) and \(\sigma([A]) = [B]\), then \(A^2 = 0\) if and only if \(B^2 = 0\).

**Proof.** It immediately follows from Lemma 3.8 since \(\sigma\) preserves the in-degree and the out-degree of each vertex. □

**Lemma 3.10.** If \(\sigma\) is an automorphism of \(\Gamma_E(R)\) that fixes \([E_{11}], i = 1,2\), then it preserves the type of each vertex.

**Proof.** Note that, there is exactly one vertex \([E_{22}]\) in type (2,2), and \(\sigma\) already fixes \([E_{22}]\).

From \([E_{11}] \rightarrow ([0 \ 1 \ 0 \ b])\) we know \(\sigma([E_{11}]) \rightarrow \sigma([0 \ 1 \ 0 \ b])\). Assume that \(\sigma([0 \ 1 \ 0 \ b]) = ([0 \ 0 \ 1 \ b])\). Then \(\sigma([E_{11}]) \sigma([0 \ 1 \ 0 \ b]) = 0\) implies \(E_{11}(b_1 0 \ b_4) = 0\). Thus \(b_1 = b_2 = 0\), and hence \(\sigma([0 \ 1 \ 0 \ b]) = ([0 \ 0 \ 1 \ b])\). Since \(\sigma\) is a bijection and \(\sigma([E_{22}]) = [E_{22}]\), we get \(b_3 \neq 0\). That means
\[
\sigma([0 \ 0 \ 1 \ b]) = ([0 \ 0 \ b_3 \ b_4]) = ([0 \ 0 \ 1 \ b^{-1}_3 b_4]).
\]
This yields \(\sigma\) preserves vertices of type (2,1).

After a similar argument on \([0 \ 0 \ 1 \ a] \rightarrow [E_{11}]\), we get that \(\sigma\) preserves vertices of type (1,2).
Since \( \sigma \) is bijective and it preserves vertices of types (1,2), (2,1), (2,2), it follows that \( \sigma \) preserves vertices of type (1,1).

\[ \square \]

**Lemma 3.11.** If \( \sigma \) is an automorphism of \( \Gamma_E(R) \) that fixes \([E_{ii}], \ i = 1, 2, \) then there exists an \( f \in S_{F_q} \) fixes 0 such that \( \sigma = \tau_f \).

**Proof.** Since \( \sigma \) is an automorphism of \( \Gamma_E(R) \) and fixes \([E_{ii}], \ i = 1, 2, \) it follows from Lemma 3.10 that \( \sigma \) preserves the type of (2,1). For every \( a \in F_q \), \( \sigma([\begin{smallmatrix} 0 & 0 \\ 0 & a \end{smallmatrix}]) = [\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}] \) for some \( b \in F_q \). Set \( f(a) = b \). Then \( f \in S_{F_q} \). By Lemma 3.9, we have \( \sigma([E_{21}]) = [E_{21}] \), and so \( f(0) = 0 \). Thus \( \tau_f \) is well defined. Now, we show that \( \sigma = \tau_f \).

(i) Clearly, \( \sigma([A]) = \tau_f([A]) \) holds for \([A] \) of type (2,2), since \( \sigma([E_{22}]) = [E_{22}] = \tau_f([E_{22}]) \).

(ii) \( \sigma([A]) = \tau_f([A]) \) holds for \([A] \) of type (2,1).

For any \( a \in F_q \), we get
\[
\sigma([\begin{smallmatrix} 0 & 0 \\ 1 & a \end{smallmatrix}]) = [\begin{smallmatrix} 0 & 0 \\ 1 & f(a) \end{smallmatrix}] = \tau_f([\begin{smallmatrix} 0 & 0 \\ 1 & a \end{smallmatrix}]).
\]

(iii) \( \sigma([A]) = \tau_f([A]) \) holds for \([A] \) of type (1,2).

Suppose that \([A] = [\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}], a \in F_q \). By Lemma 3.10 we can assume \( \sigma([\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}]) = [\begin{smallmatrix} 1 & a \end{smallmatrix}]) \). Apparently, by Lemma 3.9 we get \( \sigma([E_{12}]) = [E_{12}] = [1 f^*(0)](0 1) = \tau_f([E_{12}]) \). In the case \( a \neq 0 \), we have \( [\begin{smallmatrix} 0 & 0 \\ 1 & -a^{-1} \end{smallmatrix}] \) and so \( \sigma([\begin{smallmatrix} 0 & 0 \\ 1 & -a^{-1} \end{smallmatrix}]) \) \( \to \) \( [\begin{smallmatrix} 0 & 0 \\ 1 & a \end{smallmatrix}] \). By (ii) we know
\[
\sigma([\begin{smallmatrix} 0 & 0 \\ 1 & -a^{-1} \end{smallmatrix}]) = \tau_f([0 1](1 - a^{-1})) = [\begin{smallmatrix} 0 & 0 \\ 1 & f(-a^{-1}) \end{smallmatrix}],
\]
and so \( [\begin{smallmatrix} 0 & 0 \\ 1 & f(-a^{-1}) \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 0 \\ 1 & a \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 0 \\ 1 & f(a) \end{smallmatrix}] \).

(iv) \( \sigma([A]) = \tau_f([A]) \) holds for \([A] \) of type (1,1).

For each \([A] = [\begin{smallmatrix} 1 & a \\ b & 0 \end{smallmatrix}], a, b \in F_q \), assume \( \sigma([A]) = [\begin{smallmatrix} 1 & * \\ 1 & * \end{smallmatrix}] \). Then
\[
\sigma([\begin{smallmatrix} 1 & a \\ b & 0 \end{smallmatrix}]) = [\begin{smallmatrix} 1 & f^*(a) \\ b & 0 \end{smallmatrix}] = \tau_f([\begin{smallmatrix} 1 & a \\ b & 0 \end{smallmatrix}]).
\]
Indeed, if \( a = 0 \), then applying \( \sigma \) on \([E_{22}] \) \( \to \) \( [\begin{smallmatrix} 0 & 0 \\ b & 0 \end{smallmatrix}] \) we know
\[
\sigma([\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}]) = [\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}],
\]
In the case \( a \neq 0 \), from \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) we have \( \sigma([\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \). Thus \( \sigma([\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \). Note that, by (ii) we get \( \sigma([\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \) \( \to \) \( [\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}] \). Thus
\[
\sigma([\begin{smallmatrix} 1 & a \\ b & 0 \end{smallmatrix}]) = [\begin{smallmatrix} 1 & f^*(a) \\ b & 0 \end{smallmatrix}],
\]
If \( b = 0 \), then applying \( \sigma \) on \([1 \ a^t(1 \ 0)] \rightarrow [(0 \ 1)^t(0 \ 1)]\) and by Eqs. (5)-(6), we have
\[
\sigma([(1 \ a)^t(1 \ 0)]) = [(1 \ f^*(a))^t(1 \ 0)] = \tau_f([(1 \ a)^t(1 \ 0)]).
\]
In the case \( b \neq 0 \), applying \( \sigma \) on \([1 \ a^t(1 \ b)] \rightarrow [(1 - b^{-1})^t(0 \ 1)] \) and by Eqs. (5)-(6), we get that
\[
\sigma([(1 \ a)^t(1 \ b)]) = [(1 \ f^*(a))^t(1 \ f(b))] = \tau_f([(1 \ a)^t(1 \ b)]).
\]
From (i)-(iv), we conclude that \( \sigma([A]) = \tau_f([A]) \) for any \([A]\) of all types. Thus \( \sigma = \tau_f \).

**Theorem 3.12.** If \( \sigma \) is an automorphism of \( \Gamma_E(R) \), then \( \sigma = \sigma_P \tau_f \), where \( \sigma_P \in \Pi \) and \( \tau_f \in \Sigma \).

**Proof.** The proof is divided into two steps as follows.

**Step 1.** There exists \( P \in U(R) \) such that \( \sigma P \sigma \) fixes \([E_i], i = 1, 2\).

Since the rank of every matrix in \( Z(R)^* \) is 1, there exist \( P_1Q_1 \in U(R) \) such that \( \sigma([E_{11}]) = [P_1E_{11}Q_1] \). Write \( \sigma_1 = \sigma_{P_1} \sigma \). Then
\[
\sigma_1([E_{11}]) = [P_1^{-1}P_1E_{11}Q_1P_1] = [E_{11}Q_1P_1] = [aE_{11} + bE_{12}]
\]
for some \( a, b \in F_q \). Since \( E_{11}^2 \neq 0 \), by Lemma 3.9 we know \((aE_{11} + bE_{12})^2 \neq 0\). Thus \( a \neq 0 \). Set \( P_2 = (1 \ 0 \ -a^{-1}b) \) and set \( \sigma_2 = \sigma_{P_2} \sigma_1 \). We have
\[
\sigma_2([E_{11}]) = [P_2 \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix} P_2^{-1}] = [E_{11}].
\]
Since \([E_{22}] \in N_l([E_{11}]) \cap N_r([E_{11}])\), we get
\[
\sigma_2([E_{22}]) = [N_l(\sigma_2([E_{11}]) \cap N_r(\sigma_2([E_{11}])))
\]
\[
= [N_l([E_{11}]) \cap N_r([E_{11}])
\]
\[
= [\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \cap \begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}]
\]
\[
= \{[E_{22}]\}.
\]
From what is above, if we set \( P = P_2P_1^{-1} \), then \( P \in U(R) \) and \( \sigma_2 = \sigma_P \sigma \) preserves \([E_i], i = 1, 2\).

**Step 2.** There exists \( f \in S_{F_q} \) that fixes 0 such that \( \sigma_P \sigma = \tau_f \).

Thanks to Lemma 3.11, we immediately get this step.

By Steps 1-2, if \( \sigma \) is an automorphism of \( \Gamma_E(R) \), then there exist \( P \in U(R) \) and \( f \in S_{F_q} \) that fixes 0 such that \( \sigma_P \sigma = \tau_f \). Rewrite \( P^{-1} \) as \( P \). Then \( \sigma = \sigma_P \tau_f \).

In order to calculate the number of automorphisms of \( \Gamma_E(R) \) more easily, we develop the following theorem.

**Theorem 3.13.** If \( \sigma \) is an automorphism of \( \Gamma_E(R) \), then \( \sigma = \sigma_P \tau_f \), where \( P \in U(R) \) and \( f \in S_{F_q} \) that fixes 0 and 1.
Similarly, we get
Lemma 4.1. Let \( \sigma_2 = \sigma_Q \sigma_1 \) preserves \([E_{ii}]/(i = 1, 2)\) and \( [(1 - 1 - 1)] \).

By Lemmas 3.9-3.10, we know \( \sigma_1 \) preserves the type of each vertex and sends a square-zero class to a square-zero one. Thus \( \sigma_1([E_{12}]) = [E_{12}] \) and \( \sigma_1([E_{21}]) = [E_{21}] \).

Proof. By Lemma 2.4, we have \( \sigma \) so we get \( \sigma \).

Corollary 3.14. \(|\text{Aut}(\Gamma(E(R)))| = (q + 1)!.\)

Proof. Since \([\Pi] = |U(R)/F_q^2| = (q^2 - 1)(q^2 - q) = q(q^2 - 1), \|f \in S_{F_q}| f(0) = 0, f(1) = 1| = (q - 2)!, \) and by Lemma 3.7(iii) and Theorem 3.13 we have \(|\text{Aut}(\Gamma(E(R)))| = q(q^2 - 1) \cdot (q - 2)! = (q + 1)!\).

4. Automorphisms of \( \Gamma(R) \)

In this section, we discuss the automorphism group of \( \Gamma(R) \). Let \( \sigma \) be an automorphism of \( \Gamma(R) \), and let \( \text{Aut}(A) = \{\sigma(B) | B \in [A]\} \).

Lemma 4.1. Let \( \sigma \) be an automorphism of \( \Gamma(R) \). Then \( \text{Aut}(A) = |\text{Aut}(A)| \), \( A \in Z(R)^* \).

Proof. For every \( A \in Z(R)^* \), by Lemma 3.8 we have \(|N_1([A])| \geq q \) in \( \Gamma(R) \), and so \(|N_1([A])| \geq q(q - 1) \geq 2 \) in \( \Gamma(R) \). Thus \( |N_1([aA])| \neq 0, a \in F_q^* \).

That means \( N_1([aA]) \cap N_1([bA]) \neq \emptyset \), and so \( \text{ann}(\sigma(A)) \cap \text{ann}(\sigma(aA)) \neq \emptyset \).

Similarly, we get
\[ \text{ann}(\sigma(A)) \cap \text{ann}(\sigma(aA)) \neq \emptyset. \]

By Lemma 2.4, we have \( |\text{Aut}(A)| = |\text{Aut}(aA)|. \) Thus \( |\text{Aut}(aA)| \) for any \( a \in F_q^* \), and so \( |\text{Aut}(A)| \leq |\text{Aut}(A)| = q - 1 \).

We can assume that \( V(\Gamma(R)) = \bigcup_{i=1}^{q+1} [A_i] \), since there are \((q + 1)^2 \) vertices in \( \Gamma(E(R)) \). Let \( \sigma \) be a bijection on \( V(\Gamma(R)) \) satisfying \( \sigma(A_i) = [A_i] \) and \( \sigma(A_i) = A \) if \( A \notin [A_i] \), and let \( S_{[A_i]} \) be the set consisting of all such bijections. Clearly,
Proof. Let $K \in \sigma(A)$. Note that, if $A \neq 0$, then $[A]$ induces an independent set, and each of the vertices in $[A]$ have the same left neighborhood and the same right neighborhood. This gives $S$ is an automorphism of $\Gamma(R)$, and $S_{[A]}$ is a subgroup of $\text{Aut}(\Gamma(R))$. If $A = 0$, then $[A]$ induces a clique, and $N(A) = [A] = N(B) - [A]$, $N(A) - [A] = N(B) - [A]$ for every $A, B \in [A]$. Again $\sigma$ is an automorphism of $\Gamma(R)$ and $S_{[A]}$ is a subgroup of $\text{Aut}(\Gamma(R))$. Thus $K(\Gamma(R)) = \prod_{i=1}^{(q+1)/2} S_{[A]}$ is a subgroup of $\text{Aut}(\Gamma(R))$ since $K(\Gamma(R))$ is generated by $\cup_{j=1}^{(q+1)/2} S_{[A]}$. \(\Box\)

**Lemma 4.2.** $K(\Gamma(R))$ is a subgroup of $\text{Aut}(\Gamma(R))$.

**Proof.** Let $\sigma_i \in S_{[A_i]}$. Then $\sigma_i([A_i]) = [A_i]$ and $\sigma_i$ fixes any other vertex in $\Gamma(R)$. Note that, if $A_i = 0$, then $[A_i]$ induces an independent set, and each of the vertices in $[A_i]$ have the same left neighborhood and the same right neighborhood. Thus, $\phi$ is a bijection, and by Eq. (7) we know $\sigma_i([A_i]) = [A_i]$, and so $\sigma_i([A_i]) \in \text{Aut}(\Gamma(R))$. Then $K(\Gamma(R)) = \prod_{i=1}^{(q+1)/2} S_{[A_i]}$ is a subgroup of $\text{Aut}(\Gamma(R))$ since $K(\Gamma(R))$ is generated by $\cup_{j=1}^{(q+1)/2} S_{[A]}$. \(\Box\)

**Theorem 4.3.** $\text{Aut}(\Gamma(R))/K(\Gamma(R)) \cong \text{Aut}(\Gamma_E(R))$.

**Proof.** If $\sigma$ is an automorphism of $\Gamma(R)$, then by Lemma 4.1 we have $\sigma(A) = [\sigma(A)]$. And for each $A, B \in Z(R^*)$,

$$A \to B \iff \sigma(A) \to \sigma(B).$$

Thus

$$A \neq B, AB = 0 \iff \sigma(A) \neq \sigma(B), \sigma(A)\sigma(B) = 0,$$

and so

$$(7) \quad A \neq B, AB = 0 \iff \sigma(A) \neq \sigma(B), [\sigma(A)][\sigma(B)] = 0.$$

Set $[\sigma] : [A] \mapsto [\sigma(A)]$. Then $[\sigma]$ is an automorphism of $\Gamma_E(R)$. Indeed, $[\sigma]$ is a bijection, and by Eq. (7) we know $[A] \to [B]$ in $\Gamma_E(R)$ if and only if $[\sigma(A)] \to [\sigma(B)]$ in $\Gamma_E(R)$.

Set

$$\phi : \text{Aut}(\Gamma(R)) \mapsto \text{Aut}(\Gamma_E(R)), \sigma \mapsto [\sigma].$$

Then we prove that $\phi$ is a surjective homomorphism, and show that $\text{Ker} \phi = K(\Gamma(R))$. Thus $\text{Aut}(\Gamma(R))/K(\Gamma(R)) \cong \text{Aut}(\Gamma_E(R))$.

(i) $\phi$ is surjective.

For any $\delta \in \text{Aut}(\Gamma_E(R))$, let $\sigma$ be a bijection on $V(\Gamma(R))$ such that $[\sigma(A)] = [\delta([A])]$. Then $\sigma$ is well defined since $[\sigma(A)] = [\delta([A])] = q - 1$. Now we prove that $\sigma \in \text{Aut}(\Gamma(R))$. (iia) If $A \to B$ in $\Gamma(R)$, then $A, B \in Z(R^*)$ are distinct and satisfy $AB = 0$. If $[A] = [B]$, then from $AB = 0$ we get $A^2 = 0$, and so $\sigma(A) = \delta([A])$ is a square-zero class by Lemma 3.9. This gives $\sigma(A)\sigma(B) \in \sigma([A]) = 0$, and so $\sigma(A) \to \sigma(B)$ in $\Gamma(R)$. If $[A] \neq [B]$, then $\delta([A]) \to \delta([B])$ in $\Gamma_E(R)$ since $[A][B] = 0$. Thus $\sigma(A)\sigma(B) \in \sigma|A|\sigma|B| = \delta([A])\delta([B]) = 0$, and so $\sigma(A) \to \sigma(B)$ in $\Gamma(R)$. (iib) Conversely, if $\sigma(A) \to \sigma(B)$ in $\Gamma(R)$, then $A \neq B$ and

$$(8) \quad \sigma(A)\sigma(B) = 0.$$
Note that $\sigma(A) \in \sigma[A] = \delta([A])$, and so $[\sigma(A)] = \sigma[A] = \delta([A])$. Similarly, we have $[\sigma(B)] = \sigma[B] = \delta([B])$. Thus, by Eq. (8) we get
\begin{equation}
\delta([A])\delta([B]) = 0.
\end{equation}

If $\delta([A]) \neq \delta([B])$, then $\delta([A]) \rightarrow \delta([B])$ in $\Gamma_E(R)$. It follows that $[A] \rightarrow [B]$ in $\Gamma_E(R)$. Hence $AB = 0$, and so $A \rightarrow B$ in $\Gamma(R)$. If $\delta([A]) = \delta([B])$, then $[A] = [B]$. And by Eq. (9) we have $(\delta([A]))^2 = 0$. By Lemma 3.9 we get $A^2 = 0$. And note that $[A] = [B]$, so we have $AB = 0$. Thus $A \rightarrow B$ in $\Gamma(R)$. Hence $\sigma \in \text{Aut}(\Gamma(R))$, and so $\phi$ is a surjection.

(ii) $\phi(\sigma_1\sigma_2) = \phi(\sigma_1)\phi(\sigma_2)$.

$[\sigma_1\sigma_2][A] = ([\sigma_1][\sigma_2][A]) = \sigma_1([\sigma_2][A]) = [\sigma_1][\sigma_2][A], A \in Z(R)^*$ yields $[\sigma_1\sigma_2] = [\sigma_1][\sigma_2]$, and so $\phi(\sigma_1\sigma_2) = \phi(\sigma_1)\phi(\sigma_2)$.

(iii) $\text{Ker}\phi = K(\Gamma(R))$.

Write $V(\Gamma(R)) = \cup_{i=1}^{q+1} [A]$. If $[\sigma]$ is the identity automorphism of $\Gamma_E(R)$, then $\sigma[A_i] = [\sigma][A_i] = [A_i]$ for any $A_i \in V(\Gamma_E(R))$. Thus $\sigma \in K(\Gamma(R))$.

From (i)-(iii), we have $\text{Aut}(\Gamma(R))/K(\Gamma(R)) \cong \text{Aut}(\Gamma_E(R))$. \hfill $\Box$

**Corollary 4.4.** $|\text{Aut}(\Gamma(R))| = ((q - 1)!)(q+1)^2 \cdot (q + 1)!$

**Proof.** It immediately follows from $|K(\Gamma(R))| = ((q - 1)!)(q+1)^2$, Corollary 3.14 and Theorem 4.3. \hfill $\Box$

**Remark 4.5.** [15, Theorem 3.8] (resp., [12, Theorem 3.9]) said that the automorphism group of $\Gamma(R)$ (resp., $\Gamma(M_2(\mathbb{Z}_q))$ where $q$ is a prime) is isomorphic to the symmetric group $S_{q+1}$ of degree $q + 1$, which means $|\text{Aut}(\Gamma(R))| = (q + 1)!$ (resp., $\text{Aut}(\Gamma(M_2(\mathbb{Z}_q))) = (q + 1)!$). But our result $(q - 1)!)(q+1)^2 \cdot (q + 1)!$ is much greater than $(q + 1)!$ in general. Hence [12, Theorem 3.9] and [15, Theorem 3.8] are both incorrect. In fact, the automorphism $\sigma$ that constructed in [12, Theorem 3.9] (resp., [15, Theorem 3.8]) fails to be a bijection.

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