Abstract. All rings are commutative with $1 \neq 0$ and $n$ is a positive integer. Let $\phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ be a function where $\mathcal{J}(R)$ denotes the set of all ideals of $R$. We say that a proper ideal $I$ of $R$ is $\phi$-$n$-absorbing primary if whenever $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1a_2 \cdots a_{n+1} \in I \setminus \phi(I)$, either $a_1a_2 \cdots a_n \in I$ or the product of $a_{n+1}$ with $(n-1)$ elements of $a_1, \ldots, a_n$ is in $\sqrt{I}$. The aim of this paper is to investigate the concept of $\phi$-$n$-absorbing primary ideals.

1. Introduction

Throughout this paper $R$ will be a commutative ring with a nonzero identity. In [2], Anderson and Smith called a proper ideal $I$ of a commutative ring $R$ to be weakly prime if whenever $a, b \in R$ and $0 \neq ab \in I$, either $a \in I$ or $b \in I$. In [9], Bhatwadekar and Sharma defined a proper ideal $I$ of an integral domain $R$ to be almost prime (resp. $m$-almost prime) if for $a, b \in R$ with $ab \in I \setminus I^2$, (resp. $ab \in I \setminus I^m$, $m \geq 3$) either $a \in I$ or $b \in I$. This definition can obviously be made for any commutative ring $R$. Later, Anderson and Batanich [1] gave a generalization of prime ideals which covers all the above mentioned definitions. Let $\phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ be a function. A proper ideal $I$ of $R$ is said to be $\phi$-prime if for $a, b \in R$ with $ab \in I \setminus \phi(I)$, $a \in I$ or $b \in I$. Since $I \setminus \phi(I) = I \setminus (I \cap \phi(I))$, without loss of generality we may assume that $\phi(I) \subseteq I$. We henceforth make this assumption. Weakly primary ideals were first introduced and studied by Ebrahimi Atani and Farzalipour in [10]. A proper ideal $I$ of $R$ is called weakly primary if for $a, b \in R$ with $0 \neq ab \in I$, either $a \in I$ or $b \in I$. In [25], Yousefian Darani called a proper ideal $I$ of $R$ to be $\phi$-primary if for $a, b \in R$ with $ab \in I \setminus \phi(I)$, then either $a \in I$ or $b \in \sqrt{I}$. He defined the map $\phi_0 : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ as follows:

1. $\phi_0 : \phi(I) = \emptyset$ defines primary ideals.
2. $\phi_0 : \phi(I) = 0$ defines weakly primary ideals.
(3) $\phi_2 : \phi(I) = I^2$ defines almost primary ideals.
(4) $\phi_m (m \geq 2) : \phi(I) = I^m$ defines $m$-almost primary ideals.
(5) $\phi_\omega : \phi(I) = \cap_{n=1}^{\infty} I^n$ defines $\omega$-primary ideals.
(6) $\phi_1 : \phi(I) = I$ defines any ideals.

Given two functions $\psi_1, \psi_2 : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(J) \subseteq \psi_2(J)$ for each $J \in \mathcal{J}(R)$. Note in this case that

$\phi_0 \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{m+1} \leq \phi_m \leq \cdots \leq \phi_2 \leq \phi_1$.

Badawi in [4] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal $I$ of $R$ to be a 2-absorbing ideal of $R$ if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [3] generalized the concept of 2-absorbing ideals to $n$-absorbing ideals. According to their definition, a proper ideal $I$ of $R$ is called an $n$-absorbing (resp. strongly $n$-absorbing) ideal if whenever $a_1, \ldots, a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of $R$), then there are $n$ of the $a_i$'s (resp. $n$ of the $I_i$'s) whose product is in $I$. Thus a strongly 1-absorbing ideal is just a prime ideal. Clearly a strongly $n$-absorbing ideal of $R$ is also an $n$-absorbing ideal of $R$. Anderson and Badawi conjectured that these two concepts are equivalent, e.g., they proved that an ideal $I$ of a Prüfer domain $R$ is strongly $n$-absorbing if and only if $I$ is an $n$-absorbing ideal of $R$. [3, Corollary 6.9]. They also gave several results relating strongly $n$-absorbing ideals. The concept of 2-absorbing ideals has another generalization, called weakly 2-absorbing ideals, which has studied in [8]. A proper ideal $I$ of $R$ is a weakly 2-absorbing ideal of $R$ if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Generally, Mostafanasab et al. [15] called a proper ideal $I$ of $R$ to be a weakly $n$-absorbing (resp. strongly weakly $n$-absorbing) ideal if whenever $0 \neq a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$ (resp. $0 \neq I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of $R$), then there are $n$ of the $a_i$'s (resp. $n$ of the $I_i$'s) whose product is in $I$. Clearly a weakly $n$-absorbing ideal of $R$ is also a weakly $n$-absorbing ideal of $R$. Let $\phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ be a function. We say that a proper ideal $I$ of $R$ is a $\phi$-$n$-absorbing (resp. strongly $\phi$-$n$-absorbing) ideal of $R$ if $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for $a_1, a_2, \ldots, a_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ and $I_1 \cdots I_{n+1} \subseteq \phi(I)$ for ideals $I_1, \ldots, I_{n+1}$ of $R$) implies that there are $n$ of the $a_i$'s (resp. $n$ of the $I_i$'s) whose product is in $I$. Notice that $\phi$-$n$-absorbing ideals of a commutative ring $R$ have already been investigated by Ebrahimpour and Nekooei [11] as $(n, n+1)$-$\phi$-prime ideals.

Recall from [6] that a proper ideal $I$ of $R$ is said to be a 2-absorbing primary ideal of $R$ if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. For more studies concerning 2-absorbing primary (submodules) ideals we refer to [16], [17]. Also, recall from [7] that a proper ideal $I$ of $R$ is said to be a weakly 2-absorbing primary ideal of $R$ if whenever $a, b, c \in R$ with $0 \neq abc \in I$ implies $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$. We call a proper ideal $I$ of $R$ to be a $\phi$-$n$-absorbing primary (resp. strongly $\phi$-$n$-absorbing primary) ideal of $R$ if $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for some elements $a_1, a_2, \ldots, a_{n+1} \in R$ (resp.
$I_1 \cdots I_{n+1} \subseteq I$ and $I_1 \cdots I_{n+1} \nsubseteq \phi(I)$ for ideals $I_1, \ldots, I_{n+1}$ of $R$ implies that either $a_1 a_2 \cdots a_n \in I$ or the product of $a_n+1$ with $(n-1)$ of $a_1, a_2, \ldots, a_n$ is in $\sqrt{T}$ (resp. either $I_1 I_2 \cdots I_n \subseteq I$ or the product of $I_{n+1}$ with $(n-1)$ of $I_1, I_2, \ldots, I_n$ is in $\sqrt{T}$). We can define the map $\phi_n : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ as follows: Let $I$ be a $\phi_n$-absorbing primary ideal of $R$. Then

1. $\phi_0(I) = \emptyset \Rightarrow I$ is an $n$-absorbing primary ideal.
2. $\phi(I) = I^2 \Rightarrow I$ is a weakly $n$-absorbing primary ideal.
3. $\phi(I) = I^2 \Rightarrow I$ is an almost $n$-absorbing primary ideal.
4. $\phi(I) = \bigcap_{m=1}^{\infty} I^m$ (m $\geq 2$) $\Rightarrow I$ is an $m$-almost $n$-absorbing primary ideal.
5. $\phi(I) = \bigcap_{m=1}^{\infty} I^m$ $\Rightarrow I$ is an $n$-absorbing primary ideal.
6. $\phi_1(I) = I \Rightarrow I$ is any ideal.

Some of our results use the $R(+)M$ construction. Let $R$ be a ring and $M$ be an $R$-module. Then $R(+)M = R \times M$ is a ring with identity $(1,0)$ under addition defined by $(r,m) + (s,n) = (r+s, m+n)$ and multiplication defined by $(r,m)(s,n) = (rs, rn + sm)$.

In [22], Quartararo et al. said that a commutative ring $R$ is a $u$-ring provided $R$ has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals. They show that every Bézout ring is a $u$-ring. Moreover, they proved that every Prüfer domain is a $u$-domain. Also, any ring which contains an infinite field as a subring is a $u$-ring. [24, Exercise 3.63].

Let $R$ be a ring and $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. In Section 2, we give some basic properties of $\phi$-n-absorbing primary ideals. For instance, we prove that if $\phi$ reverses the inclusion and for every $1 \leq i \leq k$, $I_i$ is a $\phi$-n-absorbing primary ideal of $R$ such that $\sqrt{I_i}$ is a $\phi$-n-absorbing ideal of $R$, respectively, then $I_1 \cap I_2 \cap \cdots \cap I_k$ and $I_1 I_2 \cdots I_k$ are two $\phi$-n-absorbing primary ideals of $R$ where $n = n_1 + n_2 + \cdots + n_k$. It is shown that a Noetherian domain $R$ is a Dedekind domain if and only if a nonzero $n$-absorbing primary ideal of $R$ is in the form of $I = M_1^{n_1} M_2^{n_2} \cdots M_n^{n_n}$ for some $1 \leq i \leq n$ and some distinct maximal ideals $M_1, M_2, \ldots, M_n$ of $R$ and some positive integers $t_1, t_2, \ldots, t_i$. Moreover, we prove that if $I$ is an ideal of a ring $R$ such that $\sqrt{I} = M_1 \cap M_2 \cap \cdots \cap M_n$, where $M_i$’s are maximal ideals of $R$, then $I$ is an $n$-absorbing primary ideal of $R$. We show that if $I$ is a $\phi$-n-absorbing primary ideal of $R$ that is not an $n$-absorbing primary ideal, then $I^{n+1} \subseteq \phi(I)$.

In Section 3, we investigate $\phi$-n-absorbing primary ideals of direct products of commutative rings. For example, it is shown that if $R$ is an indecomposable ring and $J$ is a finitely generated $\phi$-n-absorbing primary ideal of $R$, where $\phi$ $\leq \phi_{n+2}$, then $J$ is weakly $n$-absorbing primary. Let $n \geq 2$ be a natural number and $R = R_1 \times \cdots \times R_{n+1}$ be a decomposable ring with identity. Then we prove that $R$ is a von Neumann regular ring if and only if every proper ideal of $R$ is an $n$-almost $n$-absorbing primary ideal of $R$ if and only if every proper ideal of $R$ is an $\omega$-n-absorbing primary ideal of $R$.

In Section 4, we study the stability of $\phi$-n-absorbing primary ideals with respect to idealization. As a result of this section, we establish that if $I$ is a
proper ideal of $R$ and $M$ is an $R$-module such that $IM = M$, then $I(+M)$ is an $n$-almost $n$-absorbing primary ideal of $R(+M)$ if and only if $I$ is an $n$-almost $n$-absorbing primary ideal of $R$.

In Section 5, we prove that over a u-ring $R$ the two concepts of strongly $\phi$-$n$-absorbing primary ideals and of $\phi$-$n$-absorbing primary ideals are coincide. Moreover, if $M$ is a Prüfer domain and $I$ is an ideal of $R$, then $I$ is an $n$-absorbing primary ideal of $R$ if and only if $I[X]$ is an $n$-absorbing primary ideal of $R[X]$.

2. Properties of $\phi$-$n$-absorbing primary ideals

Let $n$ be a positive integer. Consider elements $a_1, \ldots, a_n$ and ideals $I_1, \ldots, I_n$ of a ring $R$. Throughout this paper we use the following notations:

- $a_1 \cdots \hat{a}_i \cdots a_n$: $i$-th term is excluded from $a_1 \cdots a_n$.
- $I_1 \cdots \hat{I}_i \cdots I_n$: $i$-th term is excluded from $I_1 \cdots I_n$.

It is obvious that any $n$-absorbing primary ideal of a ring $R$ is a $\phi$-$n$-absorbing primary ideal of $R$. Also it is evident that the zero ideal is a weakly $n$-absorbing primary ideal of $R$. Assume that $p_1, p_2, \ldots, p_{n+1}$ are distinct prime numbers. We know that the zero ideal $I = \{0\}$ is a weakly $n$-absorbing primary ideal of the ring $\mathbb{Z}_{p_1p_2 \cdots p_{n+1}}$. Notice that $p_1p_2 \cdots p_{n+1} = 0 \in I$, but neither $p_1p_2 \cdots p_n \in I$ nor $p_1 \cdots p_i \cdots p_{n+1} \in \sqrt{I} = \text{Nil}(\mathbb{Z}_{p_1p_2 \cdots p_{n+1}})$ for every $1 \leq i \leq n$. Hence $I$ is not an $n$-absorbing primary ideal of $\mathbb{Z}_{p_1p_2 \cdots p_{n+1}}$.

**Remark 2.1.** Let $I$ be a proper ideal of a ring $R$ and $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function.

1. $I$ is $\phi$-primary if and only if $I$ is $\phi$-$1$-absorbing primary.
2. If $I$ is $\phi$-$n$-absorbing primary, then it is $\phi$-$i$-absorbing primary for all $i > n$.
3. If $I$ is $\phi$-primary, then it is $\phi$-$n$-absorbing primary for all $n > 1$.
4. If $I$ is $\phi$-$n$-absorbing primary for some $n \geq 1$, then there exists the least $n_0 \geq 1$ such that $I$ is $\phi$-$n_0$-absorbing primary. In this case, $I$ is $\phi$-$n$-absorbing primary for all $n \geq n_0$ and it is not $\phi$-$i$-absorbing primary for $n_0 > i > 0$.

**Remark 2.2.** If $I$ is a radical ideal of a ring $R$, then clearly $I$ is a $\phi$-$n$-absorbing primary (resp. strongly $\phi$-$n$-absorbing primary) ideal if and only if $I$ is a $\phi$-$n$-absorbing (resp. strongly $\phi$-$n$-absorbing) ideal.

**Theorem 2.3.** Let $R$ be a ring and let $\phi : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Then the following conditions are equivalent:

1. $I$ is $\phi$-$n$-absorbing primary,
2. For every elements $x_1, \ldots, x_n \in R$ with $x_1 \cdots x_n \notin \sqrt{I}$,

\[
(I :R x_1 \cdots x_n) \subseteq \bigcup_{i=0}^{n-1} (\sqrt{I} :R x_1 \cdots \hat{x}_i \cdots x_n) \cup (I :R x_1 \cdots x_{n-1}) \cup (\phi(I) :R x_1 \cdots x_n).
\]
Proof. (1)⇒(2) Suppose that \( x_1, \ldots, x_n \in R \) such that \( x_1 \cdots x_n \notin \sqrt{T} \). Let \( a \in (I :_R x_1 \cdots x_n) \). So \( ax_1 \cdots x_n \in I \). If \( ax_1 \cdots x_n \in \phi(I) \), then \( a \in (\phi(I) :_R x_1 \cdots x_n) \). Assume that \( ax_1 \cdots x_n \notin \phi(I) \). Since \( x_1 \cdots x_n \notin \sqrt{T} \), then either \( ax_1 \cdots x_{n-1} \in I \), i.e., \( a \in (I :_R x_1 \cdots x_{n-1}) \) or for some \( 1 \leq i \leq n-1 \) we have \( ax_1 \cdots x_i \cdots x_n \in \sqrt{T} \), i.e., \( a \in (\sqrt{T} :_R x_1 \cdots x_i \cdots x_n) \). Consequently

\[
(I :_R x_1 \cdots x_n) \subseteq \left[ \bigcup_{n-1} \left( \sqrt{T} :_R x_1 \cdots x_i \cdots x_n \right) \right]
\cup (I :_R x_1 \cdots x_{n-1} \cup (\phi(I) :_R x_1 \cdots x_n).
\]

(2)⇒(1) Let \( a_1a_2 \cdots a_{n+1} \in I \setminus \phi(I) \) for some \( a_1, a_2, \ldots, a_{n+1} \in R \) such that \( a_1a_2 \cdots a_n \notin I \). Then \( a_1 \in (I :_R a_2 \cdots a_{n+1}) \). If \( a_2 \cdots a_{n+1} \in \sqrt{T} \), then we are done. Hence we may assume that \( a_2 \cdots a_{n+1} \notin \sqrt{T} \) and so by part (2),

\[
(I :_R a_2 \cdots a_{n+1}) \subseteq \left[ \bigcup_{n-2} (\sqrt{T} :_R a_2 \cdots a_i \cdots a_{n+1}) \right]
\cup (I :_R a_2 \cdots a_n) \cup (\phi(I) :_R a_2 \cdots a_{n+1}).
\]

Since \( a_1a_2 \cdots a_{n+1} \notin \phi(I) \) and \( a_1a_2 \cdots a_n \notin I \), the only possibility is that \( a_1 \in \bigcup_{n-2} (\sqrt{T} :_R a_2 \cdots a_i \cdots a_{n+1}) \). Then \( a_1a_2 \cdots a_i \cdots a_{n+1} \in \sqrt{T} \) for some \( 2 \leq i \leq n \). Consequently \( I \) is \( \phi \)-absorbing primary.

Let \( R \) be an integral domain with quotient field \( K \). Badawi and Houston [5] defined a proper ideal \( I \) of \( R \) to be strongly primary if, whenever \( ab \in I \) with \( a, b \in K \), we have \( a \in I \) or \( b \in \sqrt{T} \). In [25], a proper ideal \( I \) of \( R \) is called strongly \( \phi \)-primary if whenever \( ab \in I \setminus \phi(I) \) with \( a, b \in K \), we have either \( a \in I \) or \( b \in \sqrt{T} \). We say that a proper ideal \( I \) of \( R \) is quotient \( \phi \)-absorbing primary if whenever \( x_1x_2 \cdots x_{n+1} \in I \setminus \phi(I) \) with \( x_1, x_2, \ldots, x_{n+1} \in K \), we have either \( x_1x_2 \cdots x_n \in I \) or \( x_1 \cdots x_i \cdots x_{n+1} \in \sqrt{T} \) for some \( 1 \leq i \leq n \). Consequently, \( I \) is quotient \( \phi \)-absorbing primary.

\begin{proposition}
Let \( V \) be a valuation domain with the quotient field \( K \), and let \( \phi : \mathfrak{V} \rightarrow \mathfrak{V} \cup \{0\} \) be a function. Then every \( \phi \)-absorbing primary ideal of \( V \) is quotient \( \phi \)-absorbing primary.
\end{proposition}

\begin{proof}
Assume that \( I \) is a \( \phi \)-absorbing primary ideal of \( V \). Let \( x_1x_2 \cdots x_{n+1} \in I \) for some \( x_1, x_2, \ldots, x_{n+1} \in K \) such that \( x_1x_2 \cdots x_n \notin I \). If \( x_{n+1} \notin \sqrt{T} \), then \( x_{n+1} \in V \), since \( V \) is valuation. So \( x_1 \cdots x_n x_{n+1} x_{n+1} = x_1 \cdots x_n \in I \), a contradiction. Hence \( x_{n+1} \in V \). If \( x_i \in V \) for every \( 1 \leq i \leq n \), then there is nothing to prove. If \( x_i \notin V \) for some \( 1 \leq i \leq n \), then \( x_1 \cdots x_i \cdots x_{n+1} \in I \subseteq \sqrt{T} \).

Consequently, \( I \) is quotient \( \phi \)-absorbing primary.
\end{proof}

\begin{proposition}
Let \( R \) be a von Neumann regular ring and let \( \phi : \mathfrak{R} \rightarrow \mathfrak{R} \cup \{0\} \) be a function. Then \( I \) is a \( \phi \)-absorbing primary ideal of \( R \) if and only if \( e_1e_2 \cdots e_{n+1} \in I \setminus \phi(I) \) for some idempotent elements \( e_1, e_2, \ldots, e_{n+1} \in R \) implies that either \( e_1e_2 \cdots e_n \in I \) or \( e_1 \cdots e_i \cdots e_{n+1} \in \sqrt{T} \) for some \( 1 \leq i \leq n \).
\end{proposition}

\begin{proof}
Notice the fact that any finitely generated ideal of a von Neumann regular ring \( R \) is generated by an idempotent element.
\end{proof}
Theorem 2.6. Let $R$ be a ring and let $\phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ be a function. If $I$ is a $\phi$-$n$-absorbing primary ideal of $R$ such that $\sqrt{\phi(I)} = \phi(\sqrt{I})$, then $\sqrt{I}$ is a $\phi$-$n$-absorbing ideal of $R$.

Proof. Let $x_1x_2 \cdots x_{n+1} \in \sqrt{I} \cap \phi(\sqrt{I})$ for some $x_1, x_2, \ldots, x_{n+1} \in R$ such that $x_1 \cdots x_i \cdots x_{n+1} \notin \sqrt{I}$ for every $1 \leq i \leq n$. Then there is a natural number $m$ such that $x_1^m x_2^m \cdots x_{n+1}^m \in I$. If $x_1^m x_2^m \cdots x_{n+1}^m \in \phi(I)$, then $x_1x_2 \cdots x_{n+1} \in \sqrt{\phi(I)} = \phi(\sqrt{I})$, which is a contradiction. Since $I$ is $\phi$-$n$-absorbing primary, our hypothesis implies that $x_1^m x_2^m \cdots x_{n+1}^m \in I$. Hence $x_1x_2 \cdots x_n \in \sqrt{I}$. Therefore $\sqrt{I}$ is a $\phi$-$n$-absorbing ideal of $R$. □

Corollary 2.7. Let $I$ be a $\phi$-$n$-absorbing primary ideal of $R$. Then $\sqrt{I} = P_1 \cap P_2 \cap \cdots \cap P_n$ where $1 \leq i \leq n$ and $P_i$'s are the only distinct prime ideals of $R$ that are minimal over $I$.

Proof. In Theorem 2.6, suppose that $\phi = \phi_0$. Now apply [3, Theorem 2.5]. □

Theorem 2.8. Let $R$ be a ring, and let $\phi : \mathcal{J}(R) \to \mathcal{J}(R) \cup \{\emptyset\}$ be a function that reverses the inclusion. Suppose that for every $1 \leq i \leq k$, $I_i$ is a $\phi$-$n_i$-absorbing primary ideal of $R$ such that $\sqrt{I_i} = P_i$ is a $\phi$-$n_i$-absorbing ideal of $R$, respectively. Set $n := n_1 + n_2 + \cdots + n_k$. The following conditions hold:

1. $I_1 \cap I_2 \cap \cdots \cap I_k$ is a $\phi$-$n$-absorbing primary ideal of $R$.
2. $I_1I_2 \cdots I_k$ is a $\phi$-$n$-absorbing primary ideal of $R$.

Proof. (1) Set $L = I_1 \cap I_2 \cap \cdots \cap I_k$. Then $\sqrt{L} = P_1 \cap P_2 \cap \cdots \cap P_k$. Suppose that $a_1a_2 \cdots a_{n+1} \in L \cap \phi(L)$ for some $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1 \cdots a_i \cdots a_{n+1} \notin \sqrt{L}$ for every $1 \leq i \leq n$. By, $\sqrt{L} = P_1 \cap P_2 \cap \cdots \cap P_k$ is $\phi$-$n$-absorbing, then $a_1a_2 \cdots a_n \in P_1 \cap P_2 \cap \cdots \cap P_k$. We claim that $a_1a_2 \cdots a_n \notin L$. For every $1 \leq i \leq k$, $P_i$ is $\phi$-$n_i$-absorbing and $a_1a_2 \cdots a_n \in P_i \setminus \phi(P_i)$, then there exist elements $1 \leq \beta_1^i, \beta_2^i, \ldots, \beta_{n_i}^i \leq n$ such that $a_{\beta_1^i}a_{\beta_2^i} \cdots a_{\beta_{n_i}^i} \in P_i$. If $\beta_m^i = \beta_s$ for two pairs $i, r$ and $m, s$, then

$$a_{\beta_1^i}a_{\beta_2^i} \cdots a_{\beta_k^i} \cdots a_{\beta_1^r}a_{\beta_2^r} \cdots a_{\beta_k^r} \cdots a_{\beta_1^n} \cdots a_{\beta_1^n} \in \sqrt{L}.$$ 

Therefore $a_1 \cdots a_{n+1} \notin \sqrt{L}$, a contradiction. So $\beta_j^i$'s are distinct. Hence

$$\{a_{\beta_1^i}, a_{\beta_2^i}, \ldots, a_{\beta_{n_i}^i}, a_{\beta_1^r}, a_{\beta_2^r}, \ldots, a_{\beta_{n_i}^r}, \ldots, a_{\beta_1^n}, \ldots, a_{\beta_1^n}\} = \{a_1, a_2, \ldots, a_n\}.$$ 

If $a_{\beta_1^i}a_{\beta_2^i} \cdots a_{\beta_{n_i}^i} \in I_i$ for every $1 \leq i \leq k$, then

$$a_1a_2 \cdots a_n = a_{\beta_1^i}a_{\beta_2^i} \cdots a_{\beta_{n_i}^i} a_{\beta_1^r}a_{\beta_2^r} \cdots a_{\beta_{n_i}^r} \cdots a_{\beta_1^n}a_{\beta_2^n} \cdots a_{\beta_{n_i}^n} \in L,$$

thus we are done. Therefore we may assume that $a_{\beta_1^i}a_{\beta_2^i} \cdots a_{\beta_{n_i}^i} \notin I_i$. Since $I_1$ is $\phi$-$n_1$-absorbing primary and

$$a_{\beta_1^1}a_{\beta_2^1} \cdots a_{\beta_{n_1}^1} a_{\beta_1^2}a_{\beta_2^2} \cdots a_{\beta_{n_2}^2} \cdots a_{\beta_1^k}a_{\beta_2^k} \cdots a_{\beta_{n_k}^k} a_{n+1} = a_1 \cdots a_{n+1} \notin I_1 \setminus \phi(I_1),$$

we are done.
then we have $a_{\beta_1} \cdots a_{\beta_{t-1}} a_{\beta_t} a_{\beta_{t+1}} \cdots a_{\beta_{n_k}} a_{\beta_{n_k+1}} \in P_1$ for some $1 \leq t \leq n_1$. On the other hand

$$a_{\beta_1} \cdots a_{\beta_{t-1}} a_{\beta_t} a_{\beta_{t+1}} \cdots a_{\beta_{n_k}} a_{\beta_{n_k+1}} \in P_2 \cap \cdots \cap P_k.$$ Consequently $a_{\beta_1} \cdots a_{\beta_t} a_{\beta_{t+1}} \cdots a_{\beta_{n_k}} a_{\beta_{n_k+1}} \in \sqrt{L}$, which is a contradiction. Similarly we deduce that $a_{\beta_1} a_{\beta_2} \cdots a_{\beta_{n_k}} \in I_i$ for every $2 \leq i \leq k$. Then $a_1 a_2 \cdots a_n \in L$.

(2) The proof is similar to that of part (1). \qed

Corollary 2.9. Let $R$ be a ring with $1 \neq 0$ and let $P_1, P_2, \ldots, P_n$ be prime ideals of $R$. Suppose that for every $1 \leq i \leq n$, $P_i$ is a $1$-primary ideal of $R$ where $t_i$ is a positive integer. Then $P_1^{1} \cap P_2^{t_2} \cap \cdots \cap P_n^{t_n}$ and $P_1^{t_1} P_2^{t_2} \cdots P_n^{t_n}$ are $n$-absorbing primary ideals of $R$. In particular, $P_1 \cap P_2 \cap \cdots \cap P_n$ and $P_1 P_2 \cdots P_n$ are $n$-absorbing primary ideals of $R$.

Example 2.10. Let $R = \mathbb{Z}[X_2, X_3, \ldots, X_n] + 3X_1 \mathbb{Z}[X_2, X_3, \ldots, X_n, X_1]$. Set $P_i := X_{i+1} R$ for $1 \leq i \leq n-1$ and $P_n := 3X_1 \mathbb{Z}[X_2, X_3, \ldots, X_n, X_1]$. Note that for every $1 \leq i \leq n$, $P_i$ is a prime ideal of $R$. Let $I = P_1 P_2 \cdots P_{n-2} \cap P_n$. Then $3X_1^2 X_2 \cdots X_n, 3 \not\in I$ and $3X_1^2 X_2 \cdots X_n = 3X_1^2 X_2 \cdots X_n \not\in I$. On the other hand $X_2 \cdots X_n, 3 = 3X_2 \cdots X_n \not\in \sqrt{I} \subseteq P_n$ and $3X_1^2 X_2, \cdots, X_1^i X_2 \cdots X_n, 3 \not\in \sqrt{I} \subseteq P_{i-1}$ for every $2 \leq i \leq n$. Hence $I$ is not an $n$-absorbing primary.

In [6, Example 2.7], the authors offered an example to show that if $I \subseteq J$ such that $I$ is a 2-absorbing primary ideal of $R$ and $\sqrt{J} = \sqrt{I}$, then $J$ need not be a 2-absorbing ideal of $R$. They considered the ideal $J = \langle XYZ, Z^2, ZX^3 \rangle$ of the ring $R = \mathbb{Z}[X, Y, Z]$ and showed that $\sqrt{J} = \langle XY \rangle$. But $X \in \sqrt{J}$, which is a contradiction. Therefore their example is incorrect. In the following example we show that if $I \subseteq J$ such that $I$ is an $n$-absorbing primary ideal of $R$ and $\sqrt{J} = \sqrt{I}$, then $J$ need not be an $n$-absorbing ideal of $R$.

Example 2.11. Let $R = K[X_1, X_2, \ldots, X_{n+2}]$ where $K$ is a field. Consider the ideal $J = \langle X_1 X_2 \cdots X_{n+1}, X_1^2 X_2 \cdots X_n, X_1^3 X_{n+2} \rangle$ of $R$. Then

$$\sqrt{J} = \langle X_1 X_2 \cdots X_n, X_1^2 X_{n+2} \rangle = \langle X_1 \rangle \cap \langle X_2, X_{n+2} \rangle \cap \langle X_3, X_{n+2} \rangle \cap \cdots \cap \langle X_n, X_{n+2} \rangle.$$

Set $P_1 = \langle X_1 \rangle$ and $P_i = \langle X_i, X_{n+2} \rangle$ for every $2 \leq i \leq n$. Note that $P_i$‘s are prime ideals of $R$. Let $I = P_1^2 P_2 \cdots P_n$. Then $I \subseteq J$ and $\sqrt{J} = \sqrt{I} = \cap_{i=1}^n P_i$.

By Corollary 2.9, $I$ is an $n$-absorbing primary ideal of $R$ but $J$ is not an $n$-absorbing primary ideal of $R$ because $X_1 X_2 \cdots X_{n+1} \not\in J$, but $X_1 X_2 \cdots X_n \not\in J$ and $X_2 \cdots X_{n+1} \not\in \sqrt{J} \subseteq \langle X_1 \rangle$ and $X_1 \cdots X_{n+1} \not\in \sqrt{J} \subseteq \langle X_1, X_{n+2} \rangle$ for every $2 \leq i \leq n$. 


Theorem 2.12. Let $R$ be a ring, and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Suppose that $I$ is an ideal of $R$ such that $\sqrt{\phi(I)} \subseteq \phi(I)$. If $\sqrt{I}$ is a $\phi$-$(n-1)$-absorbing ideal of $R$, then $I$ is a $\phi$-$n$-absorbing primary ideal of $R$.

Proof. Let $\sqrt{I}$ be $\phi$-$(n-1)$-absorbing. Assume that $a_1a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for some $a_1, a_2, \ldots, a_{n+1} \in R$ and $a_1a_2 \cdots a_n \notin I$. Hence

$$(a_1a_{n+1})(a_2a_{n+1}) \cdots (a_na_{n+1}) = (a_1a_2 \cdots a_n)a_{n+1}^n \in I \subseteq \sqrt{I}.$$  

Notice that, if $(a_1a_2 \cdots a_n)a_{n+1}^n \in \phi(\sqrt{I})$, then $a_1a_2 \cdots a_na_{n+1} \in \sqrt{\phi(\sqrt{I})} \subseteq \phi(I)$ which is a contradiction. Therefore

$$(a_1a_{n+1})(a_2a_{n+1}) \cdots (a_na_{n+1}) \in \sqrt{I} \setminus \phi(\sqrt{I}).$$

Then for some $1 \leq i \leq n$,

$$(a_1a_{n+1}) \cdots (a_{i-1}a_{n+1}) \cdots (a_na_{n+1}) = (a_1 \cdots \hat{a_i} \cdots a_n)a_{n+1}^{n-1} \in \sqrt{I},$$

and so $a_1 \cdots \hat{a_i} \cdots a_na_{n+1} \in \sqrt{I}$. Consequently $I$ is $\phi$-$n$-absorbing primary. $\square$

The following example gives an ideal $J$ of a ring $R$ where $\sqrt{J}$ is an $n$-absorbing ideal of $R$, but $J$ is not an $n$-absorbing primary ideal of $R$.

Example 2.13. Let $R = K[X_1, X_2, \ldots, X_{n+2}]$ where $K$ is a field and let $J = \langle X_1X_2 \cdots X_{n+1}, X_2^2X_2 \cdots X_{n+1}, X_3^2X_{n+2} \rangle$. Then

$$\sqrt{J} = \langle X_1 \rangle \cap \langle X_2, X_{n+2} \rangle \cap \langle X_1, X_{n+2} \rangle \cap \cdots \cap \langle X_n, X_{n+2} \rangle.$$  

By [3, Theorem 2.1(c)], $\sqrt{J}$ is an $n$-absorbing ideal of $R$, but $J$ is not an $n$-absorbing primary ideal of $R$ as it is shown in Example 2.11.

We know that if $I$ is an ideal of a ring $R$ such that $\sqrt{I}$ is a maximal ideal of $R$, then $I$ is a primary ideal of $R$.

Theorem 2.14. Let $I$ be an ideal of a ring $R$. If $\sqrt{I} = M_1 \cap M_2 \cap \cdots \cap M_n$ where $M_i$'s are maximal ideals of $R$, then $I$ is an $n$-absorbing primary ideal of $R$.

Proof. Let $a_1a_2 \cdots a_{n+1} \in I$ for some $a_1, a_2, \ldots, a_{n+1} \in R$ such that $a_1 \cdots \hat{a_i} \cdots a_{n+1} \notin \sqrt{I}$ for every $1 \leq i \leq n$. If for some $1 \leq i \leq n$, $a_1 \cdots \hat{a_i} \cdots a_{n+1} \in M_j$ (for every $1 \leq j \leq n$), then $a_1 \cdots \hat{a_i} \cdots a_{n+1} \in \sqrt{I}$ and we are done. Without loss of generality we may assume that for every $1 \leq i \leq n$, $a_1 \cdots \hat{a_i} \cdots a_{n+1} \notin M_i$, respectively. Since $M_i$'s are maximal, then $M_i + R(a_1 \cdots \hat{a_i} \cdots a_{n+1}) = R$ for every $1 \leq i \leq n$. Therefore for every $1 \leq i \leq n$ there are $m_i \in M_i$ and $r_i \in R$ such that $m_i + r_i(a_1 \cdots \hat{a_i} \cdots a_{n+1}) = 1$. So

$$m_1m_2 \cdots m_n + \sum_{t=1}^{n} \sum_{\alpha_1=1}^{n-t+1} \sum_{\alpha_2=2}^{t} r_{\alpha_1}r_{\alpha_2} \cdots r_{\alpha_t}(m_1 \cdots \hat{m_{\alpha_1}} \cdots \hat{m_{\alpha_2}} \cdots \hat{m_{\alpha_t}} \cdots m_n)$$
\[ \prod_{i=1}^{t}(a_1 \cdots a_i, \cdots a_n] = 1. \]

Since \( m_1m_2 \cdots m_n \in \sqrt{I} \), hence \( (m_1m_2 \cdots m_n)^t \in I \) for some \( t \geq 1 \). Thus
\[
(m_1m_2 \cdots m_n)^t + s \left( \sum_{t=1}^{n} \sum_{a_1=1}^{n-t+1} \sum_{a_1 < a_2 < \ldots < a_t \leq n} [r_{a_1}, r_{a_2}, \ldots, r_{a_t}, (m_1 \cdots m_{a_1}, \cdots m_{a_2}, \ldots m_{a_t}, \ldots m_n)] \right)
\]
\[
\prod_{i=1}^{t}(a_1 \cdots a_i, \cdots a_n] = 1
\]
for some \( s \in R \). Multiply \( a_1a_2 \cdots a_n \) on both sides to get
\[
a_1a_2 \cdots a_n = a_1a_2 \cdots a_n((m_1m_2 \cdots m_n)^t +
\sum_{t=1}^{n} \sum_{a_1=1}^{n-t+1} \sum_{a_1 < a_2 < \ldots < a_t \leq n} [r_{a_1}, r_{a_2}, \ldots, r_{a_t}, (m_1 \cdots m_{a_1}, \cdots m_{a_2}, \ldots m_{a_t}, \ldots m_n)]
\]
\[
\prod_{i=1}^{t}(a_1 \cdots a_i, \cdots a_n] \in I.
\]

Hence \( I \) is an \( n \)-absorbing primary ideal. \( \square \)

Let \( R \) be an integral domain with \( 1 \neq 0 \) and let \( K \) be the quotient field of \( R \). A nonzero ideal \( I \) of \( R \) is said to be invertible if \( \overline{I}^{-1} = \overline{R} \), where \( \overline{I}^{-1} = \{ x \in K \mid xI \subseteq R \} \). An integral domain \( R \) is said to be a Dedekind domain if every nonzero proper ideal of \( R \) is invertible.

**Theorem 2.15.** Let \( R \) be a Noetherian integral domain with \( 1 \neq 0 \) that is not a field. The following conditions are equivalent:

1. \( R \) is a Dedekind domain;
2. A nonzero proper ideal \( I \) of \( R \) is an \( n \)-absorbing primary ideal of \( R \) if and only if \( I = M_1^i M_2^i \cdots M_n^i \) for some \( 1 \leq i \leq n \) and some distinct maximal ideals \( M_1, M_2, \ldots, M_n \) of \( R \) and some positive integers \( t_1, t_2, \ldots, t_i \);
3. If \( I \) is a nonzero \( n \)-absorbing primary ideal of \( R \), then \( I = M_1^i M_2^i \cdots M_n^i \) for some \( 1 \leq i \leq n \) and some distinct maximal ideals \( M_1, M_2, \ldots, M_n \) of \( R \) and some positive integers \( t_1, t_2, \ldots, t_i \);
4. A nonzero proper ideal \( I \) of \( R \) is an \( n \)-absorbing primary ideal of \( R \) if and only if \( I = P_1^i P_2^i \cdots P_i^i \) for some \( 1 \leq i \leq n \) and some distinct prime ideals \( P_1, P_2, \ldots, P_i \) of \( R \) and some positive integers \( t_1, t_2, \ldots, t_i \);
5. If \( I \) is a nonzero \( n \)-absorbing primary ideal of \( R \), then \( I = P_1^i P_2^i \cdots P_i^i \) for some \( 1 \leq i \leq n \) and some distinct prime ideals \( P_1, P_2, \ldots, P_i \) of \( R \) and some positive integers \( t_1, t_2, \ldots, t_i \).
Proof. (1)⇒(2) Assume that \( R \) is a Dedekind domain that is not a field. Then every nonzero prime ideal of \( R \) is maximal. Let \( I \) be a nonzero \( n \)-absorbing primary ideal of \( R \). Since \( R \) is a Dedekind domain, there are distinct maximal ideals \( M_1, M_2, \ldots, M_n \) of \( R \) (\( k \geq 1 \)) such that \( I = M_1^{t_1}M_2^{t_2} \cdots M_n^{t_n} \) in which \( t_j \)'s are positive integers. Therefore \( \sqrt{I} = M_1 \cap M_2 \cap \cdots \cap M_n \). Since \( I \) is \( n \)-absorbing primary and every prime ideal of \( R \) is maximal, then \( \sqrt{I} \) is the intersection of at most \( n \) maximal ideals of \( R \), by Corollary 2.7. So \( i \leq n \).

Conversely, suppose that \( I = M_1^{t_1}M_2^{t_2} \cdots M_n^{t_n} \) for some \( 1 \leq i \leq n \) and some distinct maximal ideals \( M_1, M_2, \ldots, M_n \) of \( R \) and some positive integers \( t_1, t_2, \ldots, t_n \). Then \( I \) is \( n \)-absorbing primary, by Corollary 2.9.

(1)⇒(4) The proof is similar to that of (1)⇒(2).

(2)⇒(3), (3)⇒(5) and (4)⇒(5) are evident.

(5)⇒(1) Let \( M \) be an arbitrary maximal ideal of \( R \) and \( I \) be an ideal of \( R \) such that \( M^2 \subset I \subset M \). Hence \( \sqrt{I} = M \) and so \( I \) is \( M \)-primary. Then \( I \) is \( n \)-absorbing primary, and thus by part (5) we have that \( I = P_1^{t_1}P_2^{t_2} \cdots P_i^{t_i} \) for some \( 1 \leq i \leq n \) and some distinct prime ideals \( P_1, P_2, \ldots, P_i \) of \( R \) and some positive integers \( t_1, t_2, \ldots, t_i \). Then \( \sqrt{I} = P_1 \cap P_2 \cap \cdots \cap P_i = M \) which shows that \( I \) is a power of \( M \), a contradiction. Therefore, there are no ideals properly between \( M^2 \) and \( M \). Consequently \( R \) is a Dedekind domain, by [13, Theorem 39.2, p. 470].

Since every principal ideal domain is a Dedekind domain, we have the following result as a consequence of Theorem 2.15.

**Corollary 2.16.** Let \( R \) be a principal ideal domain and \( I \) be a nonzero proper ideal of \( R \). Then \( I \) is an \( n \)-absorbing primary ideal of \( R \) if and only if \( I = R(P_1^{t_1}P_2^{t_2} \cdots P_i^{t_i}) \), where \( P_j \)'s are prime elements of \( R \), \( 1 \leq i \leq n \) and \( t_j \)'s are some integers.

The following example shows that an \( n \)-absorbing primary ideal of a ring \( R \) need not be of the form \( P_1^{t_1}P_2^{t_2} \cdots P_i^{t_i} \), where \( P_j \)'s are prime ideals of \( R \), \( 1 \leq i \leq n \) and \( t_j \)'s are some integers.

**Example 2.17.** Let \( R = K[X_1, X_2, \ldots, X_n] \) where \( K \) is a field and let \( I = (X_1, X_2, \ldots, X_{n-1}, X_n^2) \). Since \( I \) is \( (X_1, X_2, \ldots, X_n) \)-primary, then \( I \) is an \( n \)-absorbing primary ideal of \( R \). But \( I \) is not in the form of \( P_1^{t_1}P_2^{t_2} \cdots P_i^{t_i} \), where \( P_j \)'s are prime ideals of \( R \), \( 1 \leq i \leq n \) and \( t_j \)'s are some integers.

**Theorem 2.18.** Let \( R \) be a ring, \( a \in R \) a nonunit and \( m \geq 2 \) a positive integer. If \( (0 : R a) \subseteq (a) \), then \( (a) \) is \( \phi \)-absorbing primary for some \( \phi \) with \( \phi \leq \phi_m \) if and only if \( (a) \) is \( n \)-absorbing primary.

**Proof.** We may assume that \( (a) \) is \( \phi_m \)-absorbing primary. Let \( x_1x_2 \cdots x_{n+1} \in (a) \) for some \( x_1, x_2, \ldots, x_{n+1} \in R \). If \( x_1x_2 \cdots x_{n+1} \notin (a^m) \), then either \( x_1x_2 \cdots x_n \in (a) \) or \( x_1x_2 \cdots x_{n+1} \in \sqrt{(a)} \) for some \( 1 \leq i \leq n \). Therefore, assume that \( x_1x_2 \cdots x_{n+1} \in (a^m) \). Hence \( x_1x_2 \cdots x_n(x_{n+1} + a) \in (a) \). If \( x_1x_2 \cdots x_n(x_{n+1} + a) \notin (a^m) \), then either \( x_1x_2 \cdots x_n \in (a) \) or \( x_1x_2 \cdots x_n(x_{n+1} + a) \in \sqrt{(a)} \).
Corollary 2.19. Let $R$ be an integral domain, $a \in R$ a nonunit element and $m \geq 2$ a positive integer. Then $(a)$ is $\phi$-absorbing primary for some $\phi$ with $\phi \leq m$ if and only if $(a)$ is $n$-absorbing primary.

Theorem 2.20. Let $V$ be a valuation domain and $n$ be a natural number. Suppose that $I$ is an ideal of $V$ such that $I^{n+1}$ is not principal. Then $I$ is a $\phi_{n+1}$-primary ideal of $V$ and only if it is $n$-absorbing primary.

Proof. $(\Rightarrow)$ Assume that $I$ is $\phi_{n+1}$-primary ideal that is not $n$-absorbing primary. Therefore there are $a_1, \ldots, a_{n+1} \in R$ such that $a_1 \cdots a_{n+1} \in I$, but neither $a_1 \cdots a_n \in I$ nor $a_1 \cdots a_i a_{n+1} \in \sqrt{I}$ for every $1 \leq i \leq n$. Hence $(a_i) \nsubseteq I$ for every $1 \leq i \leq n+1$. Since $V$ is a valuation domain, thus $I \subset (a_i)$ for every $1 \leq i \leq n+1$, and so $I^{n+1} \subseteq (a_1 \cdots a_{n+1})$. Since $I^{n+1}$ is not principal, then $a_1 \cdots a_{n+1} \notin I^{n+1}$. Therefore $I$ $\phi_{n+1}$-primary implies that either $a_1 \cdots a_n \in I$ or $a_1 \cdots a_i a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$, which is a contradiction. Consequently $I$ is $n$-absorbing primary.

$(\Leftarrow)$ is trivial. 

Let $J$ be an ideal of $R$ and $\phi : \mathfrak{Z}(R) \to \mathfrak{Z}(R) \cup \{\emptyset\}$ be a function. Define $\phi_J : \mathfrak{Z}(R/J) \to \mathfrak{Z}(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ for every ideal $I \in \mathfrak{Z}(R)$ with $J \subseteq I$ (and $\phi_J(I/J) = \emptyset$ if $\phi(I) = \emptyset$).

Theorem 2.21. Let $J \subseteq I$ be proper ideals of a ring $R$, and let $\phi : \mathfrak{Z}(R) \to \mathfrak{Z}(R) \cup \{\emptyset\}$ be a function.

1. If $I$ is a $\phi$-primary ideal of $R$, then $I/J$ is a $\phi_J$-primary ideal of $R/J$.
2. If $J \subseteq \phi(I)$ and $I/J$ is a $\phi_J$-primary ideal of $R/J$, then $I$ is a $\phi$-primary ideal of $R$.
3. If $\phi(I) \subset J$ and $I$ is a $\phi$-primary ideal of $R$, then $I/J$ is a weakly $n$-primary ideal of $R/J$.
4. If $\phi(J) \subseteq \phi(I)$, $J$ is a $\phi$-primary ideal of $R$ and $I/J$ is a weakly $n$-primary ideal of $R/J$, then $I$ is a $\phi_J$-primary ideal of $R$.

Proof. (1) Let $a_1, a_2, \ldots, a_{n+1} \in R$ be such that $(a_1 + J)(a_2 + J) \cdots (a_{n+1} + J) \in (I/J)\cap (\phi(I)/J) = (I/J)\cap (\phi(I) + J)/J$. Then $a_1 a_2 \cdots a_{n+1} \in I\phi(I)$ and $I$ $\phi$-primary gives either $a_1 \cdots a_n \in I$ or $a_1 \cdots a_i a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. Therefore either $(a_1 + J) \cdots (a_n + J) + (a_{n+1} + J) \in I/J$ or $(a_1 + J) \cdots (a_i + J) \cdots (a_{n+1} + J) \in \sqrt{I}/J = \sqrt{I}/J$ for some $1 \leq i \leq n$. This shows that $I/J$ is $\phi_J$-primary.
(2) Suppose that $a_1 a_2 \cdots a_{n+1} \in I \backslash \phi(I)$ for some $a_1, a_2, \ldots, a_{n+1} \in R$. Then $(a_1 + J)(a_2 + J) \cdots (a_{n+1} + J) \in (I/J) \backslash \phi(I)/J = (I/J) \backslash \phi(J/I)$. Since $I/J$ is assumed to be a weakly $\phi$-primary ideal of $R$, we get either $(a_1 + J) \cdots (a_{n+1} + J) \in I/J$ or $(a_1 + J) \cdots (a_{i+1} + J) \cdots (a_{n+1} + J) \in \sqrt{I/J} = \sqrt{I}/J$ for some $1 \leq i \leq n$. Consequently, either $a_1 \cdots a_i \in I$ or $a_1 \cdots \hat{a}_i \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$, that is, $I$ is a weakly $\phi$-primary ideal of $R$.

(3) is a direct consequence of part (1).

(4) Let $a_1 \cdots a_{n+1} \in I \backslash \phi(I)$ where $a_1, a_2, \ldots, a_{n+1} \in R$. Note that $a_1 \cdots a_{n+1} \notin \phi(J)$ because $\phi(J) \subseteq \phi(I)$. If $a_1 \cdots a_{n+1} \notin J$, then either $a_1 \cdots a_n \in J$ or $a_1 \cdots \hat{a}_i \cdots a_{n+1} \in \sqrt{J} \subseteq \sqrt{I}$ for some $1 \leq i \leq n$, since $J$ is a weakly $\phi$-primary ideal of $R$. If $a_1 \cdots a_{n+1} \notin J$, then $(a_1 + J) \cdots (a_{n+1} + J) \in (I/J) \backslash \phi(I)$ and so either $(a_1 + J) \cdots (a_i + J) \cdots (a_{n+1} + J) \in I/J$ or $(a_1 + J) \cdots (a_{i+1} + J) \cdots (a_{n+1} + J) \in \sqrt{I/J}$ for some $1 \leq i \leq n$. Therefore, either $a_1 \cdots a_n \notin I$ or $a_1 \cdots \hat{a}_i \cdots a_{n+1} \notin \sqrt{I}$ for some $1 \leq i \leq n$. Consequently $I$ is a weakly $\phi$-primary ideal of $R$.

Corollary 2.22. Let $R$ be a ring, and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. An ideal $I$ of $R$ is a weakly $\phi$-primary ideal if and only if $I/\phi(I)$ is a weakly $\phi$-primary ideal of $R/\phi(I)$.

Proof. In parts (2) and (3) of Theorem 2.21 set $J = \phi(I)$. 

Corollary 2.23. Let $R$ be a ring, $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function and $L$ be a proper ideal of $R$. Suppose that $\phi(J) \subseteq \phi(L, X) \subseteq \langle X \rangle$. Assume that $(L, X)$ is a weakly $\phi$-primary ideal of $R[X]$. Then $L$ is a weakly $\phi$-primary ideal of $R$. The converse holds if in addition $R$ is an integral domain.

Proof. Consider the isomorphism $\langle L, X \rangle / \langle X \rangle \simeq L$ in $R[X]/\langle X \rangle \simeq R$. Set $I := \langle L, X \rangle$ and $J := \langle X \rangle$. Assume that $(L, X)$ is a weakly $\phi$-primary ideal of $R[X]$. So, by part (3) of Theorem 2.21, $I/J \simeq L$ is a weakly $\phi$-primary ideal of $R[X]/J \simeq R$. Now, suppose that $R$ is an integral domain and $L$ is a weakly $\phi$-primary ideal of $R$. Since $J = \langle X \rangle$ is a prime ideal of $R[X]$, then it is a weakly $\phi$-primary ideal. On the other hand $I/J \simeq L$ is a weakly $\phi$-primary ideal of $R[X]/J \simeq R$. Hence, part (4) of Theorem 2.21 implies that $I = \langle L, X \rangle$ is a weakly $\phi$-primary ideal of $R[X]$.

Let $S$ be a multiplicatively closed subset of a ring $R$. Let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function and define $\phi_S : \mathfrak{J}(R_S) \to \mathfrak{J}(R_S) \cup \{\emptyset\}$ by $\phi_S(J) = (\phi(J \cap R_S))_S$ (and $\phi_S(J) = \emptyset$ if $\phi(J \cap R_S) = \emptyset$) for every ideal $J$ of $R_S$. Note that $\phi_S(J) \subseteq J$. Let $M$ be an $R$-module. The set of all zero divisors on $M$ is:

$$Z_R(M) = \{r \in R \mid \text{there exists an element } 0 \neq x \in M \text{ such that } rx = 0\}.$$ 

Proposition 2.24. Let $R$ be a ring and $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Suppose that $S$ is a multiplicatively closed subset of $R$ and $I$ is a proper ideal of $R$. 


(1) If \( I \) is a \( \phi \)-\( n \)-absorbing primary ideal of \( R \) with \( I \cap S = \emptyset \) and \( \phi(I)_S \subseteq \phi_S(I_S) \), then \( I_S \) is a \( \phi_S \)-\( n \)-absorbing primary ideal of \( R_S \).

(2) If \( I_S \) is a \( \phi_S \)-\( n \)-absorbing primary ideal of \( R_S \) with \( \phi_S(I_S) \subseteq \phi(I)_S \), then \( I \cap \bigcap_{i=1}^{n-1} I_S \neq \emptyset \) if \( n \) is an absorbing primary ideal of \( R \).

Proof. (1) Assume that \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_{n-1}}{s_{n-1}} = \frac{a_n}{s_n} \in I_S \setminus \phi_S(I_S) \) for some \( \frac{a_1}{s_1}, \frac{a_2}{s_2}, \cdots, \frac{a_n}{s_n} \in R_S \) such that \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \notin I_S \). Since \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \in I_S \), then there is \( s \in S \) such that \( sa_1 a_2 \cdots a_{n+1} \in I \). If \( sa_1 a_2 \cdots a_{n+1} \notin \phi(I) \), then \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_{n-1}}{s_{n-1}} = \frac{sa_1 a_2 \cdots a_{n+1}}{s} \in \phi(I)_S \subseteq \phi_S(I_S) \), a contradiction. Hence \( a_1 a_2 \cdots a_n (sa_{n+1}) \in I \setminus \phi(I) \). As \( I \) is \( \phi \)-\( n \)-absorbing primary, we get either \( a_1 a_2 \cdots a_n \in I \) or \( a_1 \cdots a_1 (sa_{n+1}) \in \sqrt{I} \) for some \( 1 \leq i \leq n \). The first case implies that \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \in I \), which is a contradiction, and the second case implies that \( \frac{sa_1}{s_1} \frac{sa_2}{s_2} \cdots \frac{a_n}{s_n} \in I_S \), which is a contradiction. Consequently \( I_S \) is a \( \phi_S \)-\( n \)-absorbing primary ideal of \( R_S \).

(2) Let \( a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I) \) for some \( a_1, a_2, \ldots, a_n \in R \) and let \( a_1 a_2 \cdots a_n \notin I \). Then \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \in I_S \). Assume that \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \in \phi_S(I_S) \). Since \( \phi_S(I_S) \subseteq \phi(I)_S \), then there exists a \( s \in S \) such that \( sa_1 a_2 \cdots a_{n+1} \in \phi(I) \). Since \( S \cap Z_R(\frac{1}{\sqrt{I}}) = \emptyset \), we have that \( a_1 a_2 \cdots a_{n+1} \in I \), which is a contradiction. Therefore \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \in I_S \setminus \phi_S(I_S) \). Hence, either \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \in I \) or \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \in \sqrt{I} = (\sqrt{I})_S = (\sqrt{I})_S \) for some \( 1 \leq i \leq n \). If \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_n}{s_n} \in I \), then there exists \( u \in S \) such that \( ua_1 a_2 \cdots a_n \in I \) and so the assumption \( S \cap Z_R(\frac{1}{\sqrt{I}}) = \emptyset \) shows that \( a_1 a_2 \cdots a_n \in I \), a contradiction. Therefore, there is \( 1 \leq i \leq n \) such that \( \frac{a_1}{s_1} \frac{a_2}{s_2} \cdots \frac{a_{i-1}}{s_{i-1}} \frac{a_{i+1}}{s_{i+1}} \cdots \frac{a_n}{s_n} \in \sqrt{I} \), and thus there exists a \( t \in I \) such that \( ta_1 a_2 \cdots a_{n+1} \in \sqrt{I} \). Note that \( S \cap Z_R(\frac{1}{\sqrt{I}}) = \emptyset \) implies that \( S \cap \bigcap_{i=1}^{n-1} I_S = \emptyset \), and thus \( a_1 \cdots a_n \in \sqrt{I} \). Consequently \( I \) is a \( \phi \)-\( n \)-absorbing primary ideal of \( R \). \( \square \)

Let \( f : R \rightarrow T \) be a homomorphism of rings and let \( \phi_T : \mathfrak{J}(T) \rightarrow \mathfrak{J}(T) \cup \{\emptyset\} \) be a function. Define \( \phi_R : \mathfrak{J}(R) \rightarrow \mathfrak{J}(R) \cup \{\emptyset\} \) by \( \phi_R(I) = \phi_T(I^c) \) (and \( \phi_R(\emptyset) = \emptyset \) if \( \phi_T(I^c) = \emptyset \)). We recall that if \( R \) is a Prüfer domain or \( T = R_S \) for some multiplicatively closed subset \( S \) of \( R \), then for every ideal \( J \) of \( T \) we have \( J^c = J \).

**Theorem 2.25.** Let \( f : R \rightarrow T \) be a homomorphism of rings. If \( J \) is a \( \phi_T \)-\( n \)-absorbing primary ideal of \( T \) such that \( \phi_T(J) \subseteq \phi_T(J^c) \) (e.g. where \( J = J^c \)), then \( J^c \subseteq \phi_R(J^c) \) is an absorbing primary ideal of \( R \).

Proof. Let \( a_1 a_2 \cdots a_{n+1} \in J^c \setminus \phi_R(J^c) \) for some \( a_1, a_2, \ldots, a_{n+1} \in R \). If \( a_1 a_2 \cdots a_{n+1} = \phi_T(J^c) \), then \( a_1 a_2 \cdots a_{n+1} \in \phi_T(J^c) \subseteq \phi_T(J^c)^c = \phi(R(J^c)) \), which is a contradiction. Therefore \( \phi_T(J^c) \subseteq J \setminus \phi_R(J^c) \). Hence, either \( \phi(f(a_1)) \phi(f(a_2)) \cdots \phi(f(a_{n+1})) \in J \) or \( f(a_1) \cdots f(a_{n+1}) \in \sqrt{J} \) for some \( 1 \leq i \leq n \). Thus, either
Let $R$, $T$ be rings and $\psi: \mathfrak{J}(R) \to \mathfrak{J}(T) \cup \{\emptyset\}$ be a function. Define $\psi_T: \mathfrak{J}(T) \to \mathfrak{J}(T) \cup \{\emptyset\}$ by $\psi_T(J) = \psi_R(J^c)^c$ (and $\psi_T(\emptyset) = \emptyset$ if $\psi_R(J^c) = \emptyset$).

We recall that if $f: R \to T$ is a faithfully flat homomorphism of rings, then for every ideal $I$ of $R$ we have $I^c = I$.

**Theorem 2.26.** Let $f: R \to T$ be a faithfully flat homomorphism of rings.

1. If $J$ is a $\psi_T$-n-absorbing primary ideal of $T$, then $J^c$ is a $\psi_R$-n-absorbing primary ideal of $R$.

2. If $I^c$ is a $\psi_T$-n-absorbing primary ideal of $T$ for some ideal $I$ or $R$, then $I$ is a $\psi_R$-n-absorbing primary ideal of $R$.

**Proof.** (1) Suppose that $J$ is a $\psi_T$-n-absorbing primary ideal of $T$. In Theorem 2.25 get $\phi_T := \psi_T$. Let $I$ be an ideal of $R$. Then

$$\phi_R(I) = \phi_T(I^c)^c = \psi_T(I)^c = \psi_R(J^c)^c = \psi_R(I).$$

So $\phi_R = \psi_R$. Moreover, $\psi_T(J) = \psi_R(J^c)^c = \psi_R(J^c)$. Therefore $J^c$ is a $\psi_R$-n-absorbing primary ideal of $R$.

(2) By part (1). □

**Proposition 2.27.** Let $I$ be an ideal of a ring $R$ such that $\phi(I)$ be an $n$-absorbing primary ideal of $R$. If $I$ is a $\phi$-n-absorbing primary ideal of $R$, then $I$ is an $n$-absorbing primary ideal of $R$.

**Proof.** Assume that $a_1a_2\cdots a_{n+1} \in I$ for some elements $a_1, a_2, \ldots, a_{n+1} \in R$ such that $a_1a_2\cdots a_n \notin I$. If $a_1a_2\cdots a_{n+1} \in \phi(I)$, then $\phi(I)$ $n$-absorbing primary and $a_1a_2\cdots a_n \notin \phi(I)$ implies that $a_1\cdots a_i\cdots a_{n+1} \in \sqrt{\phi(I)} \subseteq \sqrt{I}$ for some $1 \leq i \leq n$, and so we are done. When $a_1a_2\cdots a_{n+1} \notin \phi(I)$ clearly the result follows. □

We say that a $\phi$-prime ideal $P$ of a ring $R$ is a divided $\phi$-prime ideal if $P \subseteq xR$ for every $x \in R \setminus P$; thus a divided $\phi$-prime ideal is comparable to every ideal of $R$.

**Theorem 2.28.** Let $P$ be a divided $\phi$-prime ideal of a ring $R$. Suppose that $I$ is a $\phi$-n-absorbing ideal of $R$ with $\sqrt{I} = P$ and $\phi(P) \subseteq \phi(I)$. Then $I$ is a $\phi$-primary ideal of $R$.

**Proof.** Let $xy \in \Gamma \setminus \phi(I)$ for $x, y \in R$ and $y \notin P$. Since $xy \in P \setminus \phi(P)$, then $x \in P$. If $y^{n-1} \in \phi(P)$, then $y \in \sqrt{P}$, which is a contradiction. Therefore $y^n \notin \phi(P)$, and so $y^{n-1} \notin P$. Thus $P \subseteq y^{n-1}R$, because $P$ is a divided $\phi$-prime ideal of $R$. Hence $x = y^{n-1}z$ for some $z \in R$. As $y^nz = yz \in \Gamma \setminus \phi(I)$, $y^n \notin I$, and $I$ is a $\phi$-n-absorbing ideal of $R$, we have $x = y^{n-1}z \in I$. Hence $I$ is a $\phi$-primary ideal of $R$.
Let $I$ be an ideal of a ring $R$ and $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Assume that $I$ is a $\phi$-$n$-absorbing primary ideal of $R$ and $a_1, \ldots, a_{n+1} \in R$. We say that $(a_1, \ldots, a_{n+1})$ is an $\phi$-$(n+1)$-tuple of $I$ if $a_1 \cdots a_{n+1} \in \phi(I)$, $a_1 a_2 \cdots a_n \notin I$ and for each $1 \leq i \leq n$, $a_1 \cdots \hat{a}_i \cdots a_n a_{n+1} \notin \sqrt{I}$.

In the following theorem $a_1 \cdots \hat{a}_i \cdots a_j \cdots a_n$ denotes that $a_i$ and $a_j$ are eliminated from $a_1 \cdots a_n$.

**Theorem 2.29.** Let $I$ be a $\phi$-$n$-absorbing primary ideal of a ring $R$ and suppose that $(a_1, \ldots, a_{n+1})$ is a $\phi$-$(n+1)$-tuple of $I$ for some $a_1, \ldots, a_{n+1} \in R$. Then for every elements $\alpha_1, \alpha_2, \ldots, \alpha_m \in \{1, 2, \ldots, n+1\}$ which $1 \leq m \leq n$,

$$a_1 \cdots \hat{a}_{\alpha_1} \cdots \hat{a}_{\alpha_m} \cdots \hat{a}_{\alpha_{m+1}} \cdots a_{n+1} \subseteq \phi(I).$$

**Proof.** We use induction on $m$. Let $m = 1$ and suppose that $a_1 \cdots \hat{a}_{\alpha_1} \cdots a_{n+1} x \notin \phi(I)$ for some $x \in I$. Then $a_1 \cdots \hat{a}_{\alpha_1} \cdots a_{n+1} (a_{\alpha_1} + x) \notin \phi(I)$. Since $I$ is a $\phi$-$n$-absorbing primary ideal of $R$ and $a_1 \cdots \hat{a}_{\alpha_1} \cdots a_{n+1} \notin I$, we conclude that $a_1 \cdots \hat{a}_{\alpha_1} \cdots a_{n+1} (a_{\alpha_1} + x) \in \sqrt{I}$, for some $1 \leq \alpha_1 \leq n+1$ different from $1$. Hence $a_1 \cdots \hat{a}_{\alpha_2} \cdots a_{n+1} \in \sqrt{I}$, a contradiction. Thus $a_1 \cdots \hat{a}_{\alpha_1} \cdots a_{n+1} I \subseteq \phi(I)$.

Now suppose $m > 1$ and assume that for all integers less than $m$ the claim holds. Let $a_1 \cdots \hat{a}_{\alpha_1} \cdots \hat{a}_{\alpha_m} \cdots a_{n+1} x_1 x_2 \cdots x_m \notin \phi(I)$ for some $x_1, x_2, \ldots, x_m \in I$. By induction hypothesis, we conclude that there exists $\zeta \in \phi(I)$ such that

$$\begin{align*}
a_1 \cdots \hat{a}_{\alpha_1} \cdots \hat{a}_{\alpha_2} \cdots \hat{a}_{\alpha_m} \cdots a_{n+1} (a_{\alpha_1} + x_1) (a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \\
= \zeta + a_1 \cdots \hat{a}_{\alpha_1} \cdots \hat{a}_{\alpha_2} \cdots \hat{a}_{\alpha_m} \cdots a_{n+1} x_1 x_2 \cdots x_m \notin \phi(I).
\end{align*}$$

Now, we consider two cases.

**Case 1.** Assume that $\alpha_m < n + 1$. Since $I$ is $\phi$-$n$-absorbing primary, then either

$$a_1 \cdots \hat{a}_{\alpha_1} \cdots \hat{a}_{\alpha_2} \cdots \hat{a}_{\alpha_m} \cdots a_n (a_{\alpha_1} + x_1) (a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \in I,$$

or

$$a_1 \cdots \hat{a}_{\alpha_1} \cdots \hat{a}_{\alpha_2} \cdots \hat{a}_{\alpha_3} \cdots \hat{a}_j \cdots a_{n+1} (a_{\alpha_1} + x_1) (a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \in \sqrt{I}$$

for some $j < n + 1$ distinct from $\alpha_i$'s; or

$$a_1 \cdots \hat{a}_{\alpha_1} \cdots \hat{a}_{\alpha_2} \cdots \hat{a}_{\alpha_3} \cdots a_{n+1} (a_{\alpha_1} + x_1) \cdots (a_{\alpha_i} + x_i) \cdots (a_{\alpha_m} + x_m) \in \sqrt{I}$$

for some $1 \leq i \leq m$. Thus either $a_1 a_2 \cdots a_n \notin I$ or $a_1 \cdots \hat{a}_j \cdots a_{n+1} \in \sqrt{I}$ or $a_1 \cdots \hat{a}_{\alpha_1} \cdots a_{n+1} \in \sqrt{I}$, which any of these cases has a contradiction.

**Case 2.** Assume that $\alpha_m = n + 1$. Since $I$ is $\phi$-$n$-absorbing primary, then either

$$a_1 \cdots \hat{a}_{\alpha_1} \cdots \hat{a}_{\alpha_2} \cdots \hat{a}_{\alpha_m-1} \cdots a_{n+1} (a_{\alpha_1} + x_1) (a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \in I,$$

or

$$a_1 \cdots \hat{a}_{\alpha_1} \cdots \hat{a}_{\alpha_2} \cdots \hat{a}_{\alpha_m-1} \cdots \hat{a}_j \cdots a_{n+1} (a_{\alpha_1} + x_1) (a_{\alpha_2} + x_2) \cdots (a_{\alpha_m} + x_m) \in \sqrt{I}$$

and the claim holds.
for some $j < n + 1$ different from $\alpha_i$’s; or
$$a_1 \cdots \hat{a}_1 \cdots \hat{a}_i \cdots \hat{a}_{m+1}(a_{\alpha_1} + x_1) \cdots (a_{\alpha_m} + x_m) \in \sqrt{I}$$
for some $1 \leq i \leq m - 1$. Thus either $a_1 a_2 \cdots a_n \in I$ or $a_1 \cdots \hat{a}_j \cdots a_{n+1} \in \sqrt{I}$ or $a_1 \cdots \hat{a}_i \cdots a_{n+1} \in \sqrt{I}$, which any of these cases has a contradiction. Thus $a_1 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_{n+1}I^{n+1} \subseteq \phi(I)$.

**Theorem 2.30.** Let $I$ be an $\phi$-$n$-absorbing primary ideal of $R$ that is not an $n$-absorbing primary ideal. Then

1. $I^{n+1} \subseteq \phi(I)$.
2. $\sqrt{I} = \sqrt{\phi(I)}$.

**Proof.** (1) Since $I$ is not an $n$-absorbing primary ideal of $R$, $I$ has an $\phi(\n+1)$-triple-zero $(a_1, \ldots, a_{n+1})$ for some $a_1, \ldots, a_{n+1} \in R$. Suppose that $x_1 x_2 \cdots x_{n+1} \notin \phi(I)$ for some $x_1, x_2, \ldots, x_{n+1} \in I$. Then by Theorem 2.29, there is $\zeta \in \phi(I)$ such that $(a_1+x_1) \cdots (a_{n+1}+x_{n+1}) = \zeta x_1 x_2 \cdots x_{n+1} \notin \phi(I)$. Hence either $(a_1+x_1) \cdots (a_n+x_n) \in I$ or $(a_1+x_1) \cdots (a_i+x_i) \cdots (a_{n+1}+x_{n+1}) \in \sqrt{I}$ for some $1 \leq i \leq n$. Thus either $a_1 \cdots a_n \in I$ or $a_1 \cdots \hat{a}_i \cdots a_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$, a contradiction. Hence $I^{n+1} \subseteq \phi(I)$.

(2) Clearly, $\sqrt{\phi(I)} \subseteq \sqrt{I}$. As $I^{n+1} \subseteq \phi(I)$, we get $\sqrt{I} \subseteq \sqrt{\phi(I)}$, as required.

**Corollary 2.31.** Let $I$ be an ideal of a ring $R$ that is not $n$-absorbing primary.

1. If $I$ is weakly $n$-absorbing primary, then $I^{n+1} = \{0\}$ and $\sqrt{I} = Nil(R)$.
2. If $I$ is $\phi$-$n$-absorbing primary where $\phi \leq \phi_{n+2}$, then $I^{n+1} = I^{n+2}$.

**Corollary 2.32.** Let $I$ be a $\phi$-$n$-absorbing primary ideal where $\phi \leq \phi_{n+2}$. Then $I$ is $\omega$-$n$-absorbing primary.

**Proof.** If $I$ is $n$-absorbing primary, then it is $\omega$-$n$-absorbing primary. So assume that $I$ is not $n$-absorbing primary. Then $I^{n+1} = I^{n+2}$ by Corollary 2.31(2). By hypothesis $I$ is $\phi$-$n$-absorbing primary and $\phi \leq \phi_{n+1}$. So $I$ is $\phi_{n+1}$-$n$-absorbing primary. On the other hand $\phi_\omega(I) = I^{n+1} = \phi(I)$. Therefore $I$ is $\omega$-$n$-absorbing primary.

**Theorem 2.33.** Let $R$ be a ring and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R)$ be a function. Suppose that $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of ideals of $R$ such that for every $\lambda, \lambda' \in \Lambda$, $\sqrt{\phi(I_\lambda)} = \sqrt{I_\lambda}$ and $\phi(I_\lambda) \subseteq \phi(I)$ where $I = \bigcap_{\lambda \in \Lambda} I_\lambda$. If for every $\lambda \in \Lambda$, $I_\lambda$ is a $\phi$-$n$-absorbing primary ideal of $R$ that is not $n$-absorbing primary, then $I$ is a $\phi$-$n$-absorbing primary ideal of $R$.

**Proof.** Since $I_\lambda$’s are $\phi$-$n$-absorbing primary but are not $n$-absorbing primary, then for every $\lambda \in \Lambda$, $\sqrt{I_\lambda} = \sqrt{\phi(I_\lambda)}$, by Theorem 2.30. On the other hand $\phi(I_\lambda) \subseteq \phi(I)$ for every $\lambda \in \Lambda$, and so $\sqrt{\phi(I_\lambda)} \subseteq \sqrt{I}$. Hence $\sqrt{I} = \bigcap_{\lambda \in \Lambda} \sqrt{I_\lambda} = \sqrt{\phi(I_\lambda)}$ for every $\lambda \in \Lambda$. Let $a_1 a_2 \cdots a_{n+1} \in I \setminus \phi(I)$ for some $a_1, a_2, \ldots, a_{n+1} \in I$.

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$R$, and let $a_1a_2 \cdots a_n \notin I$. Therefore there is a $\lambda \in \Lambda$ such that $a_1a_2 \cdots a_n \notin I_\lambda$. Since $I_\lambda$ is $\phi$-$n$-absorbing primary and $a_1a_2 \cdots a_{n+1} \in I_\lambda \setminus \phi(I_\lambda)$, then $a_1 \cdots \hat{a_i} \cdots a_{n+1} \in \sqrt{I_\lambda} = \sqrt{T}$ for some $1 \leq i \leq n$. Consequently $I$ is a $\phi$-$n$-absorbing primary ideal of $R$. \hfill $\square$

**Corollary 2.34.** Let $R$ be a ring, $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function and $I$ be an ideal of $R$. Suppose that $\sqrt{\phi(I)} = \phi(\sqrt{T})$ that is an $n$-absorbing ideal of $R$. If $I$ is a $\phi$-$n$-absorbing primary ideal of $R$, then $\sqrt{T}$ is an $n$-absorbing ideal of $R$.

**Proof.** Assume that $I$ is a $\phi$-$n$-absorbing primary ideal of $R$. If $I$ is an $n$-absorbing primary ideal of $R$, then $\sqrt{T}$ is an $n$-absorbing ideal, by Theorem 2.6. If $I$ is not an $n$-absorbing primary ideal of $R$, then by Theorem 2.30 and by our hypothesis, $\sqrt{T} = \sqrt{\phi(I)}$ which is an $n$-absorbing ideal. \hfill $\square$

**Theorem 2.35.** Let $I$ be a $\phi$-$n$-absorbing primary ideal of a ring $R$ that is not $n$-absorbing primary and let $J$ be a $\phi$-$n$-absorbing primary ideal of $R$ that is not $m$-absorbing primary, and $n \geq m$. Suppose that the two ideals $\phi(I)$ and $\phi(J)$ are not coprime. Then

(1) $\sqrt{I + J} = \sqrt{\phi(I) + \phi(J)}$.

(2) If $\phi(I) \subseteq J$ and $\phi(J) \subseteq \phi(I + J)$, then $I + J$ is a $\phi$-$n$-absorbing primary ideal of $R$.

**Proof.** (1) By Theorem 2.30, we have $\sqrt{T} = \sqrt{\phi(I)}$ and $\sqrt{J} = \sqrt{\phi(J)}$. Now, by [24, 2.25(i)] the result follows.

(2) Assume that $\phi(I) \subseteq J$ and $\phi(J) \subseteq \phi(I + J)$. Since $\phi(I) + \phi(J) \neq R$, then $I + J$ is a proper ideal of $R$, by part (1). Since $(I + J)/J \cong I/(I \cap J)$ and $I$ is $\phi$-$n$-absorbing primary, we get that $(I + J)/J$ is a weakly $n$-absorbing primary ideal of $R/J$, by Theorem 2.21(3). On the other hand $J$ is also $\phi$-$n$-absorbing primary, by Remark 2.1(6). Now, the assertion follows from Theorem 2.21(4).

Let $R$ be a ring and $M$ an $R$-module. A submodule $N$ of $M$ is called a pure submodule if the sequence $0 \to N \otimes_R E \to M \otimes_R E$ is exact for every $R$-module $E$.

As another consequence of Theorem 2.30 we have the following corollary.

**Corollary 2.36.** Let $R$ be a ring.

(1) If $I$ is a pure $\phi$-$n$-absorbing primary ideal of $R$ that is not $n$-absorbing primary, then $I = \phi(I)$.

(2) If $R$ is von Neumann regular ring, then every $\phi$-$n$-absorbing primary ideal of $R$ that is not $n$-absorbing primary is of the form $\phi(I)$ for some ideal $I$ of $R$.

**Proof.** Note that every pure ideal is idempotent (see [12]), also every ideal of a von Neumann regular ring is idempotent. \hfill $\square$
Corollary 2.31(1). Hence $J$ is a $\phi$-primary ideal of $R$. Therefore, $x^\prime \in J$ clearly if $x^\prime$ is a $\phi$-primary ideal of $R$.

Suppose that $a_1, a_2, \ldots, a_{n+1} \in R$ such that $a_1a_2 \cdots a_{n+1} \notin J$. Since $J \subseteq I$, we have $a_1a_2 \cdots a_{n+1} \in I \setminus \phi(I)$. Consider two cases.

**Case 1.** Assume that $a_1a_2 \cdots a_n \notin I$. Since $I$ is a $\phi$-primary ideal, then it is a $\phi$-absorbing primary ideal, by Remark 2.1(6). Hence $a_1 \cdots a_n \in I \phi(I)$. On the other hand $I$ is a $\phi$-primary ideal, so either $a_1 \cdots a_n \in I \subseteq \sqrt{I}$ or $a_1 \cdots a_n \notin \sqrt{I}$. For some $1 \leq i \leq n$. Consequently $J$ is a $\phi$-primary ideal of $R$.

**3. $\phi$-absorbing primary ideals in direct products of commutative rings**

**Theorem 3.1.** Let $R_1$ and $R_2$ be rings, and let $I$ be a weakly $n$-absorbing primary ideal of $R_1$. Then $J = I \times R_2$ is a $\phi$-absorbing primary ideal of $R = R_1 \times R_2$ for each $\phi$ with $\phi_\omega \leq \phi \leq \phi_1$.

**Proof.** Suppose that $I$ is a weakly $n$-absorbing primary ideal of $R_1$. If $I$ is $n$-absorbing primary, then $J$ is $n$-absorbing primary and hence is $\phi$-absorbing primary, for all $\phi$. Assume that $I$ is not $n$-absorbing primary. Then $I^{n+1} = \{0\}$. Corollary 2.31(1). Hence $J^{n+1} = \{0\} \times R_2$ and hence $\phi_\omega(J) = \{0\} \times R_2$. Therefore, $J \setminus \phi_\omega(J) = \{I \setminus \{0\}\} \times R_2$. Let $(x_1, y_1)(x_2, y_2) \cdots (x_{n+1}, y_{n+1}) \in J \setminus \phi_\omega(J)$ for some $x_1, x_2, \ldots, x_{n+1} \in R_1$ and $y_1, y_2, \ldots, y_{n+1} \in R_2$. Then clearly $x_1x_2 \cdots x_{n+1} \in I \setminus \{0\}$. Since $J$ is weakly $n$-absorbing primary, either $x_1 \cdots x_n \in I$ or $x_1 \cdots x_{n+1} \in \sqrt{I}$ for some $1 \leq i \leq n$. Therefore, either $(x_1, y_1) \cdots (x_i, y_i) \cdots (x_{n+1}, y_{n+1}) \in J = I \times R_2$ or $(x_1, y_1) \cdots (x_i, y_i) \cdots (x_{n+1}, y_{n+1}) \in \sqrt{I} = \sqrt{I \times R_2}$ for some $1 \leq i \leq n$. Consequently $J$ is a $\omega$-absorbing primary and hence $\phi$-absorbing primary.

**Theorem 3.2.** Let $R$ be a ring and $J$ be a finitely generated proper ideal of $R$. Suppose that $J$ is a $\phi$-absorbing primary, where $\phi \leq \phi_{n+2}$. Then, either $J$ is weakly $n$-absorbing primary or $J^{n+1} = \{0\}$ is idempotent and $R$ decomposes as $R_1 \times R_2$ where $R_2 = J^{n+1}$ and $J = I \times R_2$, where $I$ is weakly $n$-absorbing primary.

**Proof.** If $J$ is $n$-absorbing primary, then $J$ is weakly $n$-absorbing primary. So we can assume that $J$ is not $n$-absorbing primary. Then by Corollary 2.31(2),
\(J^{n+1} = J^{n+2}\) and hence \(J^{n+1} = J^{2(n+1)}\). Thus \(J^{n+1}\) is idempotent, since \(J^{n+1}\) is finitely generated, \(J^{n+1} = \langle e \rangle\) for some idempotent element \(e \in R\). Suppose \(J^{n+1} = 0\). So \(\phi(J) = 0\), and hence \(J\) is weakly \(n\)-absorbing primary. Assume that \(J^{n+1} \neq 0\). Put \(R_2 = J^{n+1} = Re\) and \(R_1 = R(1-e)\); hence \(R = R_1 \times R_2\). Let \(I = J(1-e)\), so \(J = I \times R_2\), where \(I^{n+1} = 0\). We show that \(I\) is weakly \(n\)-absorbing primary. Let \(x_1, x_2, \ldots, x_{n+1} \in R\) and \(x_1x_2 \cdots x_{n+1} \in I \setminus \{0\}\) such that \(x_1x_2 \cdots x_n \notin I\). So \((x_1, 0)(x_2, 0) \cdots (x_{n+1}, 0) = (x_1x_2 \cdots x_{n+1}, 0) \in I \times R_2 = J\). Since \(J^{n+1} = \{0\} \times R_2\) and \(\phi(J) \subseteq J^{n+1}\), then \((x_1, 0)(x_2, 0) \cdots (x_{n+1}, 0) = (x_1x_2 \cdots x_{n+1}, 0) \in J \setminus \phi(J)\). Since \(J = \phi\)-\(n\)-absorbing primary, so either \((x_1, 0)(x_2, 0) \cdots (x_{n+1}, 0) \in I \times R_2 = J\) or \((x_1, 0)(x_2, 0) \cdots (x_{n+1}, 0) = (x_1 \cdots x_i \cdots x_{n+1}, 0) \in \sqrt{I} \times R_2 = \sqrt{I}J\) for some \(1 \leq i \leq n\). The first case implies that \(x_1x_2 \cdots x_n \notin I\), which is a contradiction. The second case implies that \(x_1 \cdots x_i \cdots x_{n+1} \in \sqrt{I}\) for some \(1 \leq i \leq n\). Consequently \(I\) is weakly \(n\)-absorbing primary.

**Corollary 3.3.** Let \(R\) be an indecomposable ring and \(J\) a finitely generated \(\phi\)-\(n\)-absorbing primary ideal of \(R\), where \(\phi \leq \phi_{n+2}\). Then \(J\) is weakly \(n\)-absorbing primary. Furthermore, if \(R\) is an integral domain, then \(J\) is actually \(n\)-absorbing primary.

**Corollary 3.4.** Let \(R\) be a Noetherian integral domain. A proper ideal \(J\) of \(R\) is \(n\)-absorbing primary if and only if it is \((n+2)\)-almost \(n\)-absorbing primary.

**Theorem 3.5.** Let \(R = R_1 \times \cdots \times R_s\) be a decomposable ring and \(\psi_i : \mathcal{I}(R_i) \rightarrow \mathcal{I}(R_i) \cup \{\emptyset\}\) be a function for \(i = 1, 2, \ldots, s\). Set \(\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_s\). Suppose that

\[L = I_1 \times \cdots \times I_{\alpha_1-1} \times R_{\alpha_1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times R_{\alpha_j} \times I_{\alpha_j+1} \times \cdots \times I_s\]

be an ideal of \(R\) in which \(\{\alpha_1, \ldots, \alpha_j\} \subset \{1, \ldots, s\}\). Moreover, suppose that \(\psi_{\alpha_i}(R_{\alpha_i}) \neq R_{\alpha_i}\) for some \(\alpha_i \in \{\alpha_1, \ldots, \alpha_j\}\). The following conditions are equivalent:

1. \(L\) is a \(\phi\)-\(n\)-absorbing primary ideal of \(R\);
2. \(L\) is an \(n\)-absorbing primary ideal of \(R\);
3. \(L' := I_1 \times \cdots \times I_{\alpha_1-1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_j-1} \times I_{\alpha_j+1} \times \cdots \times I_s\) is an \(n\)-absorbing primary ideal of \(R' := R_1 \times \cdots \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times \cdots \times R_{\alpha_j-1} \times R_{\alpha_j+1} \times \cdots \times R_s\).

**Proof.** (1) \(\Rightarrow\) (2) Since \(\psi_{\alpha_i}(R_{\alpha_i}) \neq R_{\alpha_i}\) for some \(\alpha_i \in \{\alpha_1, \ldots, \alpha_j\}\), then clearly \(L \nsubseteq \sqrt{\phi(L)}\). So by Theorem 2.30(2), \(L\) is an \(n\)-absorbing primary ideal of \(R\).

(2) \(\Rightarrow\) (3) Assume that \(L\) is an \(n\)-absorbing primary ideal of \(R\) and

\[(a_1^{(1)}, \ldots, a_{\alpha_1-1}^{(1)}, a_{\alpha_1+1}^{(1)}, \ldots, a_{\alpha_j-1}^{(1)}, a_{\alpha_j+1}^{(1)}, \ldots, a_s^{(1)})
\cdots (a_1^{(n+1)}, \ldots, a_{\alpha_1-1}^{(n+1)}, a_{\alpha_1+1}^{(n+1)}, \ldots, a_{\alpha_j-1}^{(n+1)}, a_{\alpha_j+1}^{(n+1)}, \ldots, a_s^{(n+1)}) \in L',\]
in which \(a_i^{(t)}\)'s are in \(R_t\), respectively. Then

\[
(a_1^{(1)}, \ldots, a_{\alpha_i^{(1)}}^{(1)}, 1, a_{\alpha_i^{(1)}}^{(1)}, \ldots, a_s^{(1)}) \\
\vdots \\
(a_1^{(n+1)}, \ldots, a_{\alpha_i^{(n+1)}}^{(n+1)}, 1, a_{\alpha_i^{(n+1)}}^{(n+1)}, \ldots, a_s^{(n+1)}) \in L.
\]

So, either

\[
(a_1^{(1)}, \ldots, a_{\alpha_i^{(1)}}^{(1)}, 1, a_{\alpha_i^{(1)}}^{(1)}, \ldots, a_s^{(1)}) \\
\vdots \\
(a_1^{(n)}, \ldots, a_{\alpha_i^{(n)}}^{(n)}, 1, a_{\alpha_i^{(n)}}^{(n)}, \ldots, a_s^{(n)}) \in L,
\]

or there exists \(1 \leq i \leq n\) such that

\[
(a_1^{(1)}, \ldots, a_{\alpha_i^{(1)}}^{(1)}, 1, a_{\alpha_i^{(1)}}^{(1)}, \ldots, a_s^{(1)}) \\
\vdots \\
(a_1^{(i-1)}, \ldots, a_{\alpha_i^{(i-1)}}^{(i-1)}, 1, a_{\alpha_i^{(i-1)}}^{(i-1)}, \ldots, a_s^{(i-1)}) \\
(a_1^{(i)}, \ldots, a_{\alpha_i^{(i)}}^{(i)}, 1, a_{\alpha_i^{(i)}}^{(i)}, \ldots, a_s^{(i)}) \\
\vdots \\
(a_1^{(n)}, \ldots, a_{\alpha_i^{(n)}}^{(n)}, 1, a_{\alpha_i^{(n)}}^{(n)}, \ldots, a_s^{(n)}) \in \sqrt{L},
\]

because \(L\) is an \(n\)-absorbing primary ideal of \(R\). Hence, either

\[
(a_1^{(1)}, \ldots, a_{\alpha_i^{(1)}}^{(1)}, 1, a_{\alpha_i^{(1)}}^{(1)}, \ldots, a_s^{(1)}) \\
\vdots \\
(a_1^{(n)}, \ldots, a_{\alpha_i^{(n)}}^{(n)}, 1, a_{\alpha_i^{(n)}}^{(n)}, \ldots, a_s^{(n)}) \in L',
\]

or there exists \(1 \leq i \leq n\) such that

\[
(a_1^{(1)}, \ldots, a_{\alpha_i^{(1)}}^{(1)}, 1, a_{\alpha_i^{(1)}}^{(1)}, \ldots, a_s^{(1)}) \\
\vdots \\
(a_1^{(i-1)}, \ldots, a_{\alpha_i^{(i-1)}}^{(i-1)}, 1, a_{\alpha_i^{(i-1)}}^{(i-1)}, \ldots, a_s^{(i-1)}) \\
(a_1^{(i)}, \ldots, a_{\alpha_i^{(i)}}^{(i)}, 1, a_{\alpha_i^{(i)}}^{(i)}, \ldots, a_s^{(i)}) \\
\vdots \\
(a_1^{(n)}, \ldots, a_{\alpha_i^{(n)}}^{(n)}, 1, a_{\alpha_i^{(n)}}^{(n)}, \ldots, a_s^{(n)}) \in \sqrt{L'}.
\]

Consequently, \(L'\) is an \(n\)-absorbing primary ideal of \(R'\).

(3) \(\Rightarrow\) (1) Let \(L'\) is an \(n\)-absorbing primary ideal of \(R'\). It is routine to see that \(L\) is an \(n\)-absorbing primary ideal of \(R\). Consequently, \(L\) is a \(\phi\)-\(n\)-absorbing primary ideal of \(R\).

\[\square\]

**Theorem 3.6.** Let \(n \geq 2\) be a positive integer, \(R = R_1 \times \cdots \times R_n\) be a ring with identity and let \(\psi : \mathcal{J}(R_n) \to \mathcal{J}(R_i) \cup \{\emptyset\}\) be a function for \(i = 1, 2, \ldots, n\) such that \(\psi_n(R_n) \neq R_n\). Set \(\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_n\). Suppose that \(I_1 \times I_2 \times \cdots \times I_n\) is an ideal of \(R\) which \(\psi_1(I_1) \neq I_1\), and for some \(2 \leq j \leq n\), \(\psi_j(I_j) \neq I_j\), and \(I_i\) is a proper ideal of \(R_i\) for each \(1 \leq i \leq n - 1\). The following conditions are equivalent:

1. \(I_1 \times I_2 \times \cdots \times I_n\) is a \(\phi\)-\(n\)-absorbing primary ideal of \(R\);
2. \(I_i = R_n\) and \(I_1 \times I_2 \times \cdots \times I_{n-1}\) is an \(n\)-absorbing primary ideal of \(R_1 \times \cdots \times R_{n-1}\) or \(I_i\) is a primary ideal of \(R_i\) for every \(1 \leq i \leq n\), respectively.
(3) $I_1 \times I_2 \times \cdots \times I_n$ is an $n$-absorbing primary ideal of $R$.

Proof. (1)$\Rightarrow$(2) Suppose that $I_1 \times I_2 \times \cdots \times I_n$ is a $n$-absorbing primary ideal of $R$. First assume that $I_n = R_n$. Since $\psi_n(R_n) \neq R_n$, then $I_1 \times I_2 \times \cdots \times I_{n-1}$ is an $n$-absorbing primary ideal of $R_1 \times \cdots \times R_{n-1}$ by Theorem 3.5. Now, suppose that $I_n \neq R_n$. Fix $2 \leq i \leq n$. We show that $I_i$ is a primary ideal of $R_i$. Suppose that $ab \in I_i$ for some $a, b \in R_i$. Let $x \in I_1 \setminus \psi_1(I_1)$. Then

$(x, 1, \ldots, 1)(1, 0, 1, \ldots, 1, 1, 1, 0, 1, \ldots, 1) \cdots$

$(1, \ldots, 1, a, 1, \ldots, 1)(1, \ldots, 1, 1, 0, 1, \ldots, 1) \cdots (1, \ldots, 1, 0)$

$= (x, 0, \ldots, 0, ab, 0, \ldots, 0) \in I_1 \times \cdots \times I_n \setminus \psi_1(I_1) \times \cdots \times \psi_n(I_n)$.

Since $I_1 \times I_2 \times \cdots \times I_n$ is $n$-absorbing primary and $I_i$'s are proper, then either

$(x, 1, \ldots, 1)(1, 0, 1, \ldots, 1, 1, 1, 0, 1, \ldots, 1) \cdots$

$(1, \ldots, 1, a, 1, \ldots, 1)(1, \ldots, 1, 1, 0, 1, \ldots, 1) \cdots (1, \ldots, 1, 0)$

$(1, 1, 1, a, 1, \ldots, 1) = (x, 0, \ldots, 0, \frac{a}{a}, 0, \ldots, 0) \in I_1 \times \cdots \times I_n$,

or

$(x, 1, \ldots, 1)(1, 0, 1, \ldots, 1, 1, 1, 0, 1, \ldots, 1) \cdots$

$(1, \ldots, 1, 0, 1, \ldots, 1, 1, 1, 0, 1, \ldots, 1) \cdots (1, \ldots, 1, 0)$

$(1, 1, 1, b, 1, \ldots, 1) = (x, 0, \ldots, 0, \frac{b}{b}, 0, \ldots, 0) \in \sqrt{I_1 \times \cdots \times I_n}$,

and thus either $a \in I_i$ or $b \in \sqrt{I_i}$. Consequently $I_i$ is a primary ideal of $R_i$. Since for some $2 \leq j \leq n$, $\psi_j(I_j) \neq I_j$, similarly we can show that $I_i$ is a primary ideal of $R_i$.

(2)$\Rightarrow$(3) If $I_n = R_n$ and $I_1 \times I_2 \times \cdots \times I_{n-1}$ is an $n$-absorbing primary ideal of $R_1 \times \cdots \times R_{n-1}$, then $I_1 \times I_2 \times \cdots \times I_n$ is an $n$-absorbing primary ideal of $R$, by Theorem 3.5. Now, assume that $I_n$ is a primary ideal of $R_n$ and for each $1 \leq i \leq n-1$, $I_i$ is a primary ideal of $R_i$. Suppose that

$(a_1^{(1)}, \ldots, a_n^{(1)})(a_1^{(2)}, \ldots, a_n^{(2)}) \cdots (a_1^{(n+1)}, \ldots, a_n^{(n+1)})$

$\in I_1 \times I_2 \times \cdots \times I_n \setminus \psi_1(I_1) \times \cdots \times \psi_n(I_n)$,

in which for every $1 \leq j \leq n + 1$, $a_{i}^{(j)}$'s are in $R_i$, respectively. Suppose that

$(a_1^{(1)}, \ldots, a_n^{(1)})(a_1^{(2)}, \ldots, a_n^{(2)}) \cdots (a_1^{(n)}, \ldots, a_n^{(n)}) \notin I_1 \times I_2 \times \cdots \times I_n$. 

Without loss of generality we may assume that \( a_1^{(1)} \cdots a_n^{(n)} \notin I_1 \). Since \( I_1 \) is primary, we deduce that \( a_i^{(n+1)} \in \sqrt{I_1} \). On the other hand \( \sqrt{I_i} \) is a prime ideal, for any \( 2 \leq i \leq n \), then at least one of the \( a_i^{(j)} \)'s is in \( \sqrt{I_i} \), say \( a_i^{(j)} \in \sqrt{I_i} \). Thus \( (a_1^{(2)} , \ldots , a_n^{(2)}) \cdots (a_1^{(n+1)} , \ldots , a_n^{(n+1)}) \in \sqrt{I_1} \times I_2 \times \cdots \times I_n \).

Consequently \( I_1 \times I_2 \times \cdots \times I_n \) is an \( n \)-absorbing primary ideal of \( R \).

(3)\( \Rightarrow \)(1) is obvious.

\( \square \)

**Theorem 3.7.** Let \( R = R_1 \times \cdots \times R_n \) be a ring with identity and let \( \psi_i : \mathfrak{I}(R_i) \to \mathfrak{I}(R_i) \cup \{\emptyset\} \) be a function for \( i = 1, \ldots , n \) such that \( \psi_i(R_i) \neq R_i \). Set \( \phi = \psi_1 \times \psi_2 \times \cdots \times \psi_n \), and suppose that for every \( 1 \leq i \leq n - 1 \), \( I_i \) is a proper ideal of \( R_i \) such that \( \psi_1(I_1) \neq I_1 \) and \( I_n \) is an ideal of \( R_n \). The following conditions are equivalent:

(1) \( I_1 \times \cdots \times I_n \) is a \( \phi \)-\( n \)-absorbing primary ideal of \( R \) that is not an \( n \)-absorbing primary ideal of \( R \).

(2) \( I_1 \) is a \( \psi_1 \)-primary ideal of \( R_1 \) that is not a primary ideal and for every \( 2 \leq i \leq n \), \( I_i = \psi_i(I_i) \) is a primary ideal of \( R_i \), respectively.

**Proof.** (1)\( \Rightarrow \)(2) Assume that \( I_1 \times \cdots \times I_n \) is a \( \phi \)-\( n \)-absorbing primary ideal of \( R \) that is not an \( n \)-absorbing primary ideal. If for some \( 2 \leq i \leq n \) we have \( \psi_i(I_i) \neq I_i \), then \( I_1 \times \cdots \times I_n \) is an \( n \)-absorbing primary ideal of \( R \) by Theorem 3.6, which contradicts our assumption. Thus for every \( 2 \leq i \leq n \), \( \psi_i(I_i) = I_i \) and so \( I_n \neq R_n \). A proof similar to part (1)\( \Rightarrow \)(2) of Theorem 3.6 shows that for every \( 2 \leq i \leq n \), \( \psi_i(I_i) = I_i \) is a primary ideal of \( R_i \). Now, we show that \( I_1 \) is a \( \psi_1 \)-primary ideal of \( R_1 \). Consider \( a , b \in R_1 \) such that \( ab \in I_1 \backslash \psi_1(I_1) \). Note that

\[
(1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1) \cdots (1, 0, 0, 1, \ldots, 1)\times (a, 1, \ldots, 1)(b, 1, \ldots, 1)
\neq (ab, 0, 0, 0, 0) \in (I_1 \times I_2 \times \cdots \times I_n)\backslash (\psi_1(I_1) \times \cdots \times \psi_n(I_n)).
\]

Because \( I_i \)'s are proper, the product of \((a, 1, \ldots, 1)(b, 1, \ldots, 1)\) with \( n - 2 \) of \((1, 0, 1, \ldots, 1), (1, 1, 0, 1, \ldots, 1), \ldots, (1, 0, 0, 1, \ldots, 1)\) is not in \( \sqrt{I_1} \times I_2 \times \cdots \times I_n \).

Since \( I_1 \times I_2 \times \cdots \times I_n \) is a \( \phi \)-\( n \)-absorbing primary ideal of \( R \), we have either

\[
(1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1) \cdots (1, 1, 0, 1, \ldots, 1)\times (a, 1, \ldots, 1)
\neq (a, 0, \ldots, 0) \in I_1 \times I_2 \times \cdots \times I_n,
\]

or

\[
(1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1) \cdots (1, 1, 0, 1, \ldots, 1)\times (b, 1, \ldots, 1)
\neq (b, 0, \ldots, 0) \in \sqrt{I_1} \times I_2 \times \cdots \times I_n.
\]

So either \( a \in I_1 \) or \( b \in \sqrt{I_1} \). Thus \( I_1 \) is a \( \psi_1 \)-primary ideal of \( R_1 \). Assume \( I_1 \) is a primary ideal of \( R_1 \), since for every \( 2 \leq i \leq n \), \( I_i \) is a primary ideal of \( R_i \), it is easy to see that \( I_1 \times \cdots \times I_n \) is an \( n \)-absorbing primary ideal of \( R \), which is a contradiction.
(2) ⇒ (1) It is clear that $I_1 \times \cdots \times I_n$ is a $\phi$-$n$-absorbing primary ideal of $R$. Since $I_1$ is not a primary ideal of $R_1$, there exist elements $a, b \in R_1$ such that $ab \in \psi_1(I_1)$, but $a \not{\in} I_1$ and $b \not{\in} \sqrt{I_1}$. Hence
\[
(1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1) \cdots (1, 0, 1, \ldots, 1)(a, 1, \ldots, 1)(b, 1, \ldots, 1)
\]
\[
= (ab, 0, \ldots, 0) \in \psi_1(I_1) \times \cdots \times \psi_n(I_n),
\]
but neither
\[
(1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1) \cdots (1, 0, 1, \ldots, 1)(a, 1, \ldots, 1)
\]
\[
= (a, 0, \ldots, 0) \in I_1 \times \cdots \times I_n,
\]
or
\[
(1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1) \cdots (1, 0, 1, \ldots, 1)(b, 1, \ldots, 1)
\]
\[
= (b, 0, \ldots, 0) \in \sqrt{I_1} \times \cdots \times I_n.
\]
Also the product of $(a, 1, \ldots, 1)(b, 1, \ldots, 1)$ with $n-2$ of elements $(1, 0, 1, \ldots, 1)$, $(1, 1, 0, 1, \ldots, 1), (1, \ldots, 1, 0)$ is not in $\sqrt{I_1} \times \cdots \times I_n$. Consequently $I_1 \times \cdots \times I_n$ is not an $n$-absorbing primary ideal of $R$.

**Theorem 3.8.** Let $R = R_1 \times \cdots \times R_{n+1}$ where $R_i$’s are rings with identity and let for $i = 1, 2, \ldots, n+1$, $\psi_i : \mathfrak{J}(R_i) \rightarrow \mathfrak{J}(R_i) \cup \{\emptyset\}$ be a function such that $\psi_i(R_i) \neq R_i$. Set $\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_{n+1}$.

1. For every ideal $I$ of $R$, $\phi(I)$ is not an $n$-absorbing primary ideal of $R$.
2. If $I$ is a $\phi$-$n$-absorbing primary ideal of $R$, then either $I = \phi(I)$, or $I$ is an $n$-absorbing primary ideal of $R$.

**Proof.** Let $I$ be an ideal of $R$. We know that the ideal $I$ is of the form $I_1 \times \cdots \times I_{n+1}$ where $I_i$’s are ideals of $R_i$’s, for $i = 1, \ldots, n+1$.

1. Suppose that $\phi(I)$ is an $n$-absorbing primary ideal of $R$. Since
\[
(0, 1, \ldots, 1)(1, 0, 1, \ldots, 1) \cdots (1, 0, 1, \ldots, 1)(0, 1, \ldots, 0)
\]
\[
\in \phi(I) = \psi_1(I_1) \times \cdots \times \psi_{n+1}(I_{n+1}),
\]
we have that either
\[
(0, 1, \ldots, 1)(1, 0, 1, \ldots, 1) \cdots (1, 0, 1, 1) = (0, 0, 0, 1)
\]
\[
\in \psi_1(I_1) \times \cdots \times \psi_{n+1}(I_{n+1}),
\]
or the product of $(1, 1, 1, 0)$ with $n-1$ of $(0, 1, \ldots, 1), (1, 0, 1, \ldots, 1), \ldots, (1, 1, 1, 0)$ is in $\sqrt{\phi(I)}$. Hence, for some $1 \leq i \leq n+1$, $1 \in \psi_i(I_i)$ which implies that $\psi_i(R_i) = R_i$, a contradiction. Consequently $\phi(I)$ is not an $n$-absorbing primary ideal of $R$.

2. Let $I \neq \phi(I)$. So we have $I = I_1 \times \cdots \times I_{n+1} \neq \psi_1(I_1) \times \psi_2(I_2) \times \cdots \times \psi_{n+1}(I_{n+1})$. Hence, there is an element $(a_1, \ldots, a_{n+1}) \in I \setminus (\psi_1(I_1) \times \psi_2(I_2) \times \cdots \times \psi_{n+1}(I_{n+1}))$. Then $(a_1, 1, 1, \ldots, 1)(1, a_2, 1, \ldots, 1) \cdots (1, 1, 1, a_{n+1}) \in I \setminus \phi(I)$. Since $I$ is a $\phi$-$n$-absorbing primary ideal of $R$, then either
\[
(a_1, 1, 1, \ldots, 1)(1, a_2, 1, \ldots, 1) \cdots (1, 1, 1, a_{n+1}) = (a_1, a_2, \ldots, a_{n+1}) \in I,
\]
or, for some 1 \leq i \leq n we have

\((a_1, 1, \ldots, 1) \cdots (1, \ldots, 1, a_{i-1}, 1, \ldots, 1)(1, \ldots, 1, a_{i+1}, 1, \ldots, 1) \cdots (1, \ldots, 1, a_{n+1}) = (a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_{n+1}) \in \sqrt{I}.

Then \(I_t = R_t\), for some 1 \leq i \leq n+1 and so \(I = I_1 \times \cdots \times I_{i-1} \times R_t \times I_{i+1} \times \cdots \times I_{n+1}\).

If \(I \subseteq \sqrt{\phi(I)}\), then \(\psi_i(R_t) = R_t\) which is a contradiction. Therefore, by Theorem 2.30, \(I\) must be an \(n\)-absorbing primary ideal of \(R\). □

Theorem 3.9. Let \(R = R_1 \times \cdots \times R_{n+1}\) where \(R_i\)'s are rings with identity and let for \(i = 1, 2, \ldots, n + 1\), \(\psi_i : \mathfrak{J}(R_i) \rightarrow \mathfrak{J}(R_t) \cup \{\emptyset\}\) be a function such that \(\psi_i(R_t) \neq R_t\). Set \(\phi = \psi_1 \times \psi_2 \times \cdots \times \psi_{n+1}\). Let \(L = I_1 \times \cdots \times I_{n+1}\) be a proper ideal of \(R\) with \(L \neq \phi(L)\). The following conditions are equivalent:

1. \(L = I_1 \times \cdots \times I_{n+1}\) is a \(\phi\)-\(n\)-absorbing primary ideal of \(R\);
2. \(L = I_1 \times \cdots \times I_{n+1}\) is an \(n\)-absorbing primary ideal of \(R\);
3. \(L = I_1 \times \cdots \times I_{i-1} \times R_i \times I_{i+1} \times \cdots \times I_{n+1}\) for some 1 \leq i \leq n+1 such that
   
   - for each 1 \leq t \leq n+1 different from \(i\), \(I_t\) is a primary ideal of \(R_t\);
   - or \(L = I_1 \times \cdots \times I_{i-1} \times R_{\alpha_1} \times I_{i+1} \times \cdots \times I_{n+1}\) for some 1 \leq i \leq n+1 such that for each 1 \leq t \leq n+1 different from \(i\), \(I_t\) is a proper ideal of \(R_t\).

   In this case \(\{\alpha_1, \ldots, \alpha_t\} \subset \{1, \ldots, n+1\}\) and

   \[I_1 \times \cdots \times I_{i-1} \times I_{\alpha_1+1} \times \cdots \times I_{\alpha_i-1} \times I_{\alpha_i+1} \times \cdots \times I_{n+1}\]

   is an \(n\)-absorbing primary ideal of

   \[R_1 \times \cdots \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times \cdots \times R_{\alpha_i-1} \times R_{\alpha_i+1} \times \cdots \times R_{n+1}\]

Proof. (1)⇒(2) Since \(L\) is a \(\phi\)-\(n\)-absorbing primary ideal of \(R\) and \(L \neq \phi(L)\), then \(L\) is an \(n\)-absorbing primary ideal of \(R\), by Theorem 3.8.

(2)⇒(3) Suppose that \(L\) is an \(n\)-absorbing primary ideal of \(R\), then for some 1 \leq i \leq n+1, \(I_i = R_i\) by the proof of Theorem 3.8. Assume that

\[L = I_1 \times \cdots \times I_{i-1} \times R_i \times I_{i+1} \times \cdots \times I_{n+1}\]

for some 1 \leq i \leq n+1 such that for each 1 \leq t \leq n+1 different from \(i\), \(I_t\) is a proper ideal of \(R_t\). Fix an \(I_t\) different from \(I_i\). We may assume that \(t > i\). Let \(ab \in I_t\) for some \(a, b \in R_t\).

In this case

\[(0, 1, \ldots, 1)(1, 0, 1, \ldots, 1) \cdots (1, 0, 1, \ldots, 1)(1, \ldots, 1) = (0, \ldots, 0, 1, 0, 1, \ldots, 1) \cdots (1, 0, 1, \ldots, 1)
\]

\[(1, 1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1)(1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1)(1, 1, 0, 1, \ldots, 1)
\]

\[(1, 1, 0) \times \cdots \times I_{n+1}\]

is \(n\)-absorbing primary ideal of

\[R_1 \times \cdots \times R_{\alpha_1-1} \times R_{\alpha_1+1} \times \cdots \times R_{\alpha_i-1} \times R_{\alpha_i+1} \times \cdots \times R_{n+1}\]

or
and hence

\[ i = \leq 1 \]

by Theorem 3.5.

**Proof.** Assume that every proper ideal of \( R \) or the product of \((\ldots)\) in which

\[ (1, 0, 1) (1, 0, 1, 0) (1, 0, 1, 0, 1) (1, 0, 1, 0) (1, 0, 1, 0) (1, 0, 1, 0) \]

is in the first form, then similar to the proof of part (2) \( \Rightarrow (3) \) of Theorem 3.6 we can verify that \( L \) is an \( n \)-absorbing primary ideal of \( R \), and hence \( L \) is a \( \phi \)-\( n \)-absorbing primary ideal of \( R \). For the second form apply Theorem 3.5.

(3) \( \Rightarrow (1) \) If \( L \) is in the first form, then similar to the proof of part (2) \( \Rightarrow (3) \)

of Theorem 3.6 we can verify that \( L \) is an \( n \)-absorbing primary ideal of \( R \), and hence \( L \) is a \( \phi \)-\( n \)-absorbing primary ideal of \( R \).

**Theorem 3.10.** Let \( R = R_1 \times \cdots \times R_{n+1} \) where \( R_i \)'s are rings with identity and let for \( i = 1, 2, \ldots, n + 1 \), \( \psi_i : \mathfrak{J}(R_i) \to \mathfrak{J}(R_i) \cup \{0\} \) be a function. Set \( \phi = \psi_1 \times \psi_2 \times \cdots \times \psi_{n+1} \). Then, every proper ideal of \( R \) is a \( \phi \)-\( n \)-absorbing primary ideal (\( \phi \)-\( n \)-absorbing ideal) of \( R \) if and only if \( I = \psi_i(I) \) for every \( 1 \leq i \leq n + 1 \) and every proper ideal \( I \) of \( R_i \).

**Proof.** Assume that every proper ideal of \( R \) is a \( \phi \)-\( n \)-absorbing primary ideal (\( \phi \)-\( n \)-absorbing ideal) of \( R \). Fix an \( i \) and let \( I \) be a proper ideal of \( R_i \). Assume that \( I \neq \psi_i(I) \), so give an element \( x \in I \setminus \psi_i(I) \). Set

\[ J := I \times \{0\} \times \cdots \times \{0\}. \]

Notice that

\[ (1, 0, 1, 0, 1, 0) (1, 0, 1, 0, 1, 0) (x, 1, 1, 0, 1, 0) \in J \setminus \phi(J). \]

Since \( I \) is \( \phi \)-\( n \)-absorbing primary, then either

\[ (1, 0, 1, 0, 1, 0) (1, 1, 0, 1, 0, 1, 0) \in J, \]

or the product of \((x, 1, 1, 0, 1, 0)\) with \( n-1 \) of \((1, 0, 1, 0, 1, 0)\), \((1, 1, 0, 1, 0, 1, 0)\), \(\ldots\), \((1, 1, 0, 1, 0)\) in \( \sqrt{J} \) which implies that either \( 1 \in I \) or \( 1 \in \{0\} \), a contradiction. Consequently \( I \neq \psi_i(I) \). The converse is obvious.

**Corollary 3.11.** Let \( n \geq 2 \) be a natural number and \( R = R_1 \times \cdots \times R_{n+1} \) be a decomposable ring with identity. The following conditions are equivalent:

1. \( R \) is a von Neumann regular ring;
2. Every proper ideal of \( R \) is an \( n \)-almost \( n \)-absorbing primary ideal of \( R \);
3. Every proper ideal of \( R \) is an \( \omega \)-\( n \)-absorbing primary ideal of \( R \);
(4) Every proper ideal of \( R \) is an \( n \)-almost \( n \)-absorbing ideal of \( R \).

**Proof.** (1)\( \Leftrightarrow \) (2), (1)\( \Leftrightarrow \) (3) and (1)\( \Leftrightarrow \) (4): Notice that, \( \phi_n(I) = I \) (or \( \phi_\omega(I) = I \)) if and only if \( I = I^2 \). By the fact that \( R \) is von Neumann regular if and only if \( I = I^2 \) for every ideal \( I \) of \( R \) and regarding Theorem 3.10 we have the implications. \( \square \)

**Corollary 3.12.** Let \( R_1, R_2, \ldots, R_{n+1} \) be rings and let \( R = R_1 \times R_2 \times \cdots \times R_{n+1} \). Then the following conditions are equivalent:

1. \( R_1, R_2, \ldots, R_{n+1} \) are fields;
2. Every proper ideal of \( R \) is a weakly \( n \)-absorbing ideal of \( R \);
3. Every proper ideal of \( R \) is a weakly \( n \)-absorbing primary ideal of \( R \).

**Proof.** (1)\( \Rightarrow \) (2) By [11, Theorem 1.10].

(2)\( \Rightarrow \) (3) is clear.

(3)\( \Rightarrow \) (1) In Theorem 3.10 assume that \( \phi = \phi_0 \). \( \square \)

4. The stability of \( \phi \)-\( n \)-absorbing primary ideals with respect to idealization

Let \( R \) be a commutative ring and \( M \) be an \( R \)-module. We recall from [14, Theorem 25.1] that every ideal of \( R(+)^{\mathcal{I}} \) is in the form of \( I^{\mathcal{I}}N \) in which \( I \) is an ideal of \( R \) and \( N \) is a submodule of \( M \) such that \( IM \subseteq N \). Moreover, if \( I_1(+)^{\mathcal{I}}N_1 \) and \( I_2(+)^{\mathcal{I}}N_2 \) are ideals of \( R(+)^{\mathcal{I}} \), then \( I_1(+)^{\mathcal{I}}N_1 \cap I_2(+)^{\mathcal{I}}N_2 = (I_1 \cap I_2)(+)^{\mathcal{I}}(N_1 \cap N_2) \).

**Theorem 4.1.** Let \( R \) be a ring, \( I \) a proper ideal of \( R \) and \( M \) an \( R \)-module. Suppose that \( \psi : \mathcal{I}(R) \to \mathcal{I}(R) \cup \{ \emptyset \} \) and \( \phi : \mathcal{I}(R(+)^{\mathcal{I}}) \to \mathcal{I}(R(+)^{\mathcal{I}}) \cup \{ \emptyset \} \) are two functions such that \( \phi(I(+)^{\mathcal{I}}M) = \psi(I(+)^{\mathcal{I}}N) \) for some submodule \( N \) of \( M \) with \( \psi(IM) \subseteq N \). Then the following conditions are equivalent:

1. \( I(+)^{\mathcal{I}}M \) is a \( \phi \)-\( n \)-absorbing primary ideal of \( R(+)^{\mathcal{I}} \);
2. \( I \) is a \( \psi \)-\( n \)-absorbing primary ideal of \( R \) and if \( (a_1, \ldots, a_{n+1}) \) is a \( \psi \)-(\( n+1 \))-tuple, then the second component of \( (a_1, m_1) \cdots (a_{n+1}, m_{n+1}) \) is in \( N \) for any elements \( m_1, \ldots, m_{n+1} \in M \).

**Proof.** (1) \( \Rightarrow \) (2) Assume that \( I(+)^{\mathcal{I}}M \) is a \( \phi \)-\( n \)-absorbing primary ideal of \( R(+)^{\mathcal{I}}M \). Let \( x_1 \cdots x_{n+1} \in I \backslash \psi(I) \) for some \( x_1, \ldots, x_{n+1} \in R \). Therefore

\[
(x_1, 0) \cdots (x_{n+1}, 0) = (x_1 \cdots x_{n+1}, 0) \in I(+)^{\mathcal{I}}M \backslash \phi(I(+)^{\mathcal{I}}M),
\]

because \( \phi(I(+)^{\mathcal{I}}M) = \psi(I(+)^{\mathcal{I}}N) \). Hence either \( (x_1, 0) \cdots (x_n, 0) = (x_1 \cdots x_n, 0) \in I(+)^{\mathcal{I}}M \) or \( (x_1, 0) \cdots (x_i, 0) \cdots (x_{n+1}, 0) = (x_1 \cdots \hat{x}_i \cdots x_{n+1}, 0) \in \sqrt{I(+)^{\mathcal{I}}}M \) for some \( 1 \leq i \leq n \). So either \( x_1 \cdots x_n \in I \) or \( x_1 \cdots \hat{x}_i \cdots x_{n+1} \in \sqrt{I} \) for some \( 1 \leq i \leq n \) which shows that \( I \) is \( \psi \)-\( n \)-absorbing primary. For the second statement suppose that \( a_1 \cdots a_{n+1} \in \psi(I), a_1 \cdots a_n \notin I \) and \( a_1 \cdots \hat{a}_i \cdots a_{n+1} \notin \sqrt{I} \).
Thus either \((a_1, m_1) \cdots (a_{n+1}, m_{n+1})\) is not in \(N\), then

\[(a_1, m_1) \cdots (a_{n+1}, m_{n+1}) \in I(+)M \psi(I)(+)N.\]

Thus either \((a_1, m_1) \cdots (a_n, m_n) \in I(+)M\) or

\[(a_1, m_1) \cdots (a_n, m_n) \in \sqrt{I(+)M}\]

for some \(1 \leq i \leq n\). So either \(a_1 \cdots a_n \in I\) or \(a_1 \cdots a_{n+1} \in \sqrt{I}\) for some \(1 \leq i \leq n\), which is a contradiction.

(2) \(\Rightarrow\) (1) Suppose that \((a_1, m_1) \cdots (a_{n+1}, m_{n+1}) \in I(+)M \setminus \psi(I)(+)N\) for some \(a_1, \ldots, a_{n+1} \in R\) and some \(m_1, \ldots, m_{n+1} \in M\). Clearly \(a_1 \cdots a_{n+1} \in I\). If \(a_1 \cdots a_{n+1} \in \psi(I)\), then the second component of \((a_1, m_1) \cdots (a_{n+1}, m_{n+1})\) cannot be in \(N\). Hence either \(a_1 \cdots a_n \in I\) or \(a_1 \cdots a_i \cdots a_{n+1} \in \sqrt{I}\) for some \(1 \leq i \leq n\). If \(a_1 \cdots a_{n+1} \notin \psi(I)\), then \(I\)-\(\psi\)-absorbing primary implies that either \(a_1 \cdots a_n \in I\) or \(a_1 \cdots a_i \cdots a_{n+1} \in \sqrt{I}\) for some \(1 \leq i \leq n\). Therefore we have either \((a_1, m_1) \cdots (a_n, m_n) \in I(+)M\) or \((a_1, m_1) \cdots (a_i, m_i) \cdots (a_{n+1}, m_{n+1}) \in \sqrt{I(+)M}\) for some \(1 \leq i \leq n\). Consequently \(I(+)M\) is a \(\psi\)-\(\phi\)-absorbing primary ideal of \(R(+)M\).

**Corollary 4.2.** Let \(R\) be a ring, \(I\) be a proper ideal of \(R\) and \(M\) be an \(R\)-module. The following conditions are equivalent:

1. \(I(+)M\) is an \(n\)-absorbing primary ideal of \(R(+)M\);
2. \(I\) is an \(n\)-absorbing primary ideal of \(R\).

**Proof.** In Theorem 4.1 set \(\phi = \phi_0, \psi = \phi_0\) and \(N = M\). □

**Corollary 4.3.** Let \(R\) be a ring, \(I\) be a proper ideal of \(R\) and \(M\) be an \(R\)-module. The following conditions are equivalent:

1. \(I(+)M\) is a weakly \(n\)-absorbing primary ideal of \(R(+)M\);
2. \(I\) is a weakly \(n\)-absorbing primary ideal of \(R\) and if \((a_1, \ldots, a_{n+1})\) is an \((n+1)\)-tuple-zero, then the second component of

\[(a_1, m_1) \cdots (a_{n+1}, m_{n+1})\]

is zero for any elements \(m_1, \ldots, m_{n+1} \in M\).

**Proof.** In Theorem 4.1 set \(\phi = \phi_0, \psi = \phi_0\) and \(N = \{0\}\). □

**Corollary 4.4.** Let \(R\) be a ring, \(I\) be a proper ideal of \(R\) and \(M\) be an \(R\)-module. Then the following conditions are equivalent:

1. \(I(+)M\) is an \(n\)-almost \(n\)-absorbing primary ideal of \(R(+)M\);
2. \(I\) is an \(n\)-almost \(n\)-absorbing primary ideal of \(R\) and if \((a_1, \ldots, a_{n+1})\) is a \(\phi_n\)-(n+1)-tuple, then for any elements \(m_1, \ldots, m_{n+1} \in M\) the second component of \((a_1, m_1) \cdots (a_{n+1}, m_{n+1})\) is in \(I^n M\).

**Proof.** Notice that \((I(+)M)^n = I^n(+)I^{n-1}M\). In Theorem 4.1 set \(\phi = \phi_n, \psi = \phi_n\) and \(N = I^{n-1} M\). □
Corollary 4.5. Let $R$ be a ring, $I$ be a proper ideal of $R$ and $M$ be an $R$-module such that $IM = M$. Then $I(+)M$ is an $n$-almost $n$-absorbing primary ideal of $R(+)M$ if and only if $I$ is an $n$-almost $n$-absorbing primary ideal of $R$.

Corollary 4.6. Let $R$ be a ring, $I$ be a proper ideal of $R$ and $M$ be an $R$-module. Then $I(+)M$ is an $\omega$-$n$-absorbing primary ideal of $R(+)M$ if and only if $I$ is an $\omega$-$n$-absorbing primary ideal of $R$.

5. Strongly $\phi$-$n$-absorbing primary ideals

Proposition 5.1. Let $I$ be a proper ideal of a ring $R$. Then the following conditions are equivalent:

1. $I$ is strongly $\phi$-$n$-absorbing primary;
2. For every ideals $I_1, \ldots, I_n$ of $R$ such that $I \subseteq I_1, I \cdots I_n \subseteq I \setminus \phi(I)$ implies that either $I_1 \cdots I_n \subseteq I$ or $I_1 \cdots I_i \cdots I_n \subseteq \sqrt{T}$ for some $1 \leq i \leq n$.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let $J, I_2, \ldots, I_{n+1}$ be ideals of $R$ such that $JI_2 \cdots I_{n+1} \subseteq I$ and $JI_2 \cdots I_{n+1} \not\subseteq \phi(I)$. Then we have that

$$(J + I)I_2 \cdots I_{n+1} = (JI_2 \cdots I_{n+1}) + (II_2 \cdots I_{n+1}) \subseteq I.$$

On the other hand

$$(J + I)I_2 \cdots I_{n+1} \not\subseteq \phi(I),$$

since $JI_2 \cdots I_{n+1} \subseteq (J + I)I_2 \cdots I_{n+1}$. Set $I_1 := J + I$. Then, by the hypothesis either $I_1 \cdots I_2 \subseteq I$ or $I_2 \cdots I_{n+1} \subseteq \sqrt{T}$ or there exists $2 \leq i \leq n$ such that $(J + I)I_2 \cdots I_i \cdots I_{n+1} \subseteq \sqrt{T}$. Therefore, either $JI_2 \cdots I_i \subseteq I$ or $I_2 \cdots I_{n+1} \subseteq \sqrt{T}$ or there exists $2 \leq i \leq n$ such that $JI_2 \cdots I_i \cdots I_{n+1} \subseteq \sqrt{T}$. So $I$ is strongly $\phi$-$n$-absorbing primary. $\square$

Remark 5.2. Let $R$ be a ring. Notice that $\text{Jac}(R)$ is a radical ideal of $R$. So $\text{Jac}(R)$ is a strongly $n$-absorbing ideal of $R$ if and only if $I$ is a strongly $n$-absorbing primary ideal of $R$.

Given any set $X$, one can define a topology on $X$ where every subset of $X$ is an open set. This topology is referred to as the discrete topology on $X$, and $X$ is a discrete topological space if it is equipped with its discrete topology.

We denote by $\text{Max}(R)$ the set of all maximal ideals of $R$.

Theorem 5.3. Let $R$ be a ring and $\text{Max}(R)$ be a discrete topological space. Then $\text{Max}(R)$ is an infinite set if and only if $\text{Jac}(R)$ is not strongly $n$-absorbing for every natural number $n$.

Proof. $(\Leftarrow)$ We can verify this implication without any assumption on $\text{Max}(R)$, by [3, Theorem 2.1].

$(\Rightarrow)$ Notice that $\text{Max}(R)$ is a discrete topological space if and only if the Jacobson radical of $R$ is the irredundant intersection of the maximal ideals
of $R$, [21, Corollary 3.3]. Let $\text{Max}(R)$ be an infinite set. Assume that for some natural number $n$, $\text{Jac}(R)$ is a strongly $n$-absorbing ideal. Choose $n$ distinct elements $M_1, M_2, \ldots, M_n$ of $\text{Max}(R)$. Set $\mathcal{M} := \{M_1, M_2, \ldots, M_n\}$, and denote by $\mathcal{M}^*$ the complement of $\mathcal{M}$ in $\text{Max}(R)$. Since $\text{Jac}(R) = M_1 \cap M_2 \cap \cdots \cap M_n \cap (\bigcap_{M \in \mathcal{M}^*} M)$, then either $M_1 \cdots M_{i-1} M_{i+1} \cdots M_n (\bigcap_{M \in \mathcal{M}^*} M) \subseteq \text{Jac}(R)$ for some $1 \leq i \leq n$, or $M_1 M_2 \cdots M_n \subseteq \text{Jac}(R)$. In the first case we have $M_1 \cdots M_{i-1} M_{i+1} \cdots M_n (\bigcap_{M \in \mathcal{M}^*} M) \subseteq M_i$ and so $\bigcap_{M \in \mathcal{M}^*} M \subseteq M_i$, a contradiction. If $M_1 M_2 \cdots M_n \subseteq \text{Jac}(R)$, then $M_1 M_2 \cdots M_n \subseteq M$ for every $M \in \mathcal{M}^*$, and so again we reach a contradiction. Consequently $\text{Jac}(R)$ is not strongly $n$-absorbing.

In the next theorem we investigate $\phi$-$n$-absorbing primary ideals over $u$-rings. Notice that any Bézout ring is a $u$-ring, [22, Corollary 1.2].

**Theorem 5.4.** Let $R$ be a $u$-ring and let $\phi : \mathfrak{J}(R) \to \mathfrak{J}(R) \cup \{\emptyset\}$ be a function. Then the following conditions are equivalent:

1. $I$ is strongly $\phi$-$n$-absorbing primary;
2. $I$ is $\phi$-$n$-absorbing primary;
3. For every elements $x_1, \ldots, x_n \in R$ with $x_1 \cdots x_n \notin \sqrt{I}$ either
   \[ (I :_R x_1 \cdots x_n) = (I :_R x_1 \cdots x_{n-1}) \]
   or $$(I :_R x_1 \cdots x_n) \subseteq (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_n)$$ for some $1 \leq i \leq n-1$ or $(I :_R x_1 \cdots x_n) = (\phi(I) :_R x_1 \cdots x_n)$;
4. For every $t$ ideals $I_1, \ldots, I_t$, $1 \leq t \leq n-1$, and for every elements
   $x_1, \ldots, x_{n-t} \in R$ with $x_1 \cdots x_{n-t-1} I_1 \cdots I_t \notin \sqrt{I}$,
   \[ (I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = (I :_R x_1 \cdots x_{n-t-1} I_1 \cdots I_t) \]
   or
   \[ (I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) \subseteq (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_{n-t} I_1 \cdots I_t) \]
   for some $1 \leq i \leq n-t-1$ or
   \[ (I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) \subseteq (\sqrt{I} :_R x_1 \cdots x_{n-t} I_1 \cdots \hat{I}_j I_t) \]
   for some $1 \leq j \leq t$ or
   \[ (I :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) = (\phi(I) :_R x_1 \cdots x_{n-t} I_1 \cdots I_t) \];
5. For every ideals $I_1, I_2, \ldots, I_n$ of $R$ with $I_1 I_2 \cdots I_n \not\subseteq I$, either there is $1 \leq i \leq n$ such that $(I :_R I_1 \cdots I_n) \subseteq (\sqrt{I} :_R I_1 \cdots \hat{I}_i \cdots I_n)$ or $(I :_R I_1 \cdots I_n) = (\phi(I) :_R I_1 \cdots I_n)$.

**Proof.** (1)⇒(2) It is clear.
(2)⇒(3) Suppose that $x_1, \ldots, x_n \in R$ such that $x_1 \cdots x_n \notin \sqrt{I}$. By Theorem 2.3,
\[
(I :_R x_1 \cdots x_n) \subseteq \bigcup_{i=1}^{n-1} (\sqrt{I} :_R x_1 \cdots \hat{x}_i \cdots x_n) \\
\quad \cup (I :_R x_1 \cdots x_{n-1}) \cup (\phi(I) :_R x_1 \cdots x_n).
\]
Since $R$ is a u-ring we have either $(I : R x_1 \cdots x_n) \subseteq (\sqrt{T} : R x_1 \cdots x_n)$ for some $1 \leq i \leq n - 1$ or $(I : R x_1 \cdots x_n) = (I : R x_1 \cdots x_{n-1})$ or $(I : R x_1 \cdots x_n) = (\phi(I) : R x_1 \cdots x_n)$.

(3)$\Rightarrow$(4) We use induction on $t$. For $t = 1$, consider elements $x_1, \ldots, x_{n-1} \in R$ and ideal $I$ of $R$ such that $x_1 \cdots x_{n-1}I \not\subseteq \sqrt{T}$. Let $a \in (I : R x_1 \cdots x_{n-1}I)$. Then $I \subseteq (I : R x_1 \cdots x_{n-1})$. If $ax_1 \cdots x_{n-1} \in \sqrt{T}$, then $a \in (\sqrt{T} : R x_1 \cdots x_{n-1})$. If $ax_1 \cdots x_{n-1} \notin \sqrt{T}$, then by part (3), either $I \subseteq (I : R ax_1 \cdots x_{n-2})$ or $I \subseteq (\sqrt{T} : R ax_1 \cdots x_{n-1})$ for some $1 \leq i \leq n - 2$ or $I \subseteq (\sqrt{T} : R x_1 \cdots x_{n-1})$ or $I \subseteq (\phi(I) : R ax_1 \cdots x_{n-1})$. The first case implies that $a \in (I : R x_1 \cdots x_{n-2}I)$. The second case implies that $a \in (\sqrt{T} : R x_1 \cdots x_{n-1}I)$ for some $1 \leq i \leq n - 2$. The third case cannot be happen, because $x_1 \cdots x_{n-1}I \not\subseteq \sqrt{T}$, and the last case implies that $a \in (\phi(I) : R x_1 \cdots x_{n-1}I)$. Hence

$$(I : R x_1 \cdots x_{n-1}I) \subseteq \bigcup_{i=1}^{n-2} (\sqrt{T} : R x_1 \cdots x_{n-1}I) \cup (\sqrt{T} : R x_1 \cdots x_{n-1}) \cup (I : R x_1 \cdots x_{n-2}I) \cup (\phi(I) : R x_1 \cdots x_{n-1}I).$$

Since $R$ is a u-ring, then either $(I : R x_1 \cdots x_{n-1}I) \subseteq (\sqrt{T} : R x_1 \cdots x_{n-1}I)$ for some $1 \leq i \leq n - 2$, or $(I : R x_1 \cdots x_{n-1}I) \subseteq (\sqrt{T} : R x_1 \cdots x_{n-1})$ or $(I : R x_1 \cdots x_{n-1}I) = (I : R x_1 \cdots x_{n-2}I)$ or $(I : R x_1 \cdots x_{n-1}I) = (\phi(I) : R x_1 \cdots x_{n-1}I).$ Now suppose $t > 1$ and assume that for integer $t - 1$ the claim holds. Let $x_1, \ldots, x_{n-t}$ be elements of $R$ and let $I_1, \ldots, I_t$ be ideals of $R$ such that $x_1 \cdots x_{n-t}I_1 \cdots I_t \not\subseteq \sqrt{T}$. Consider element $a \in (I : R x_1 \cdots x_{n-t}I_1 \cdots I_t).$ Thus $I \subseteq (I : R ax_1 \cdots x_{n-t}I_1 \cdots I_t).$ If $ax_1 \cdots x_{n-t}I_1 \cdots I_t \subseteq \sqrt{T}$, then $a \in (\sqrt{T} : R x_1 \cdots x_{n-t}I_1 \cdots I_t).$ If $ax_1 \cdots x_{n-t}I_1 \cdots I_t \not\subseteq \sqrt{T}$, then by induction hypothesis, either

$$(I : R ax_1 \cdots x_{n-t}I_1 \cdots I_t) \subseteq (\sqrt{T} : R x_1 \cdots x_{n-t}I_1 \cdots I_t)$$

or

$$(I : R ax_1 \cdots x_{n-t}I_1 \cdots I_t) \subseteq (\sqrt{T} : R ax_1 \cdots x_{n-t}I_1 \cdots I_t)$$

for some $1 \leq i \leq n - t - 1$ or

$$(I : R ax_1 \cdots x_{n-t}I_1 \cdots I_t) \subseteq (\sqrt{T} : R ax_1 \cdots x_{n-t}I_1 \cdots I_t)$$

for some $1 \leq j \leq t - 1$ or

$$(I : R ax_1 \cdots x_{n-t}I_1 \cdots I_t) = (I : R ax_1 \cdots x_{n-t}I_1 \cdots I_t),$$

or $(I : R ax_1 \cdots x_{n-t}I_1 \cdots I_t) = (\phi(I) : R ax_1 \cdots x_{n-t}I_1 \cdots I_t).$ Since $x_1 \cdots x_{n-t}I_1 \cdots I_t \not\subseteq \sqrt{T}$, then the first case cannot happen. Consequently, either $a \in (\sqrt{T} : R x_1 \cdots x_{n-t}I_1 \cdots I_t)$ for some $1 \leq i \leq n - t - 1$ or $a \in (\sqrt{T} : R x_1 \cdots x_{n-t}I_1 \cdots I_t)$ for some $1 \leq j \leq t - 1$ or $a \in (I : R x_1 \cdots x_{n-t}I_1 \cdots I_t)$ or $a \in (\phi(I) : R x_1 \cdots x_{n-t}I_1 \cdots I_t).$ Hence

$$(I : R x_1 \cdots x_{n-t}I_1 \cdots I_t) \subseteq \bigcup_{i=1}^{n-t-1} (\sqrt{T} : R x_1 \cdots x_{n-t}I_1 \cdots I_t).$$
Now, since \( R \) is \( u \)-ring we are done.

(4)\( \Rightarrow \)(5) Let \( I_1, I_2, \ldots, I_n \) be ideals of \( R \) such that \( I_1I_2 \cdots I_n \supseteq I \). Suppose that \( a \in (I :_R I_1I_2 \cdots I_n) \). Then \( I_n \supseteq (I :_R aI_1I_2 \cdots I_n) \). If \( aI_1I_2 \cdots I_{n-1} \subseteq \sqrt{I} \), then \( a \in (\sqrt{I} :_R I_1I_2 \cdots I_{n-1}) \). If \( aI_1I_2 \cdots I_{n-1} \not\subseteq \sqrt{I} \), then by part (4) we have either \( I_n \subseteq (I :_R I_1I_2 \cdots I_{n-1}) \) or \( I_n \subseteq (\sqrt{I} :_R aI_1 \cdots I_{n-1}) \) for some \( 1 \leq i \leq n-1 \) or \( I_n \subseteq (\phi(I) :_R aI_1I_2 \cdots I_{n-1}) \).

Remark 5.5. Note that in Theorem 5.4, for the case \( n = 2 \) and \( \phi = \emptyset \) we can omit the condition \( u \)-ring, because if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them. So we conclude that an ideal \( I \) of a ring \( R \) is \( 2 \)-absorbing primary if and only if it is strongly \( 2 \)-absorbing primary.

Let \( R \) be a ring with identity. We recall that if \( f = a_0 + a_1X + \cdots + a_tX^t \) is a polynomial on the ring \( R \), then content of \( f \) is defined as the \( R \)-ideal, generated by the coefficients of \( f \), i.e. \( c(f) = (a_0, a_1, \ldots, a_t) \). Let \( T \) be an \( R \)-algebra and \( c \) the function from \( T \) to the ideals of \( R \) defined by \( c(f) = \cap \{ I \mid I \text{ is an ideal of } R \text{ and } f \in IT \} \) known as the content of \( f \). Note that the content function \( c \) is nothing but the generalization of the content of a polynomial \( f \in R[X] \). The \( R \)-algebra \( T \) is called a content \( R \)-algebra if the following conditions hold:

1. For all \( f \in T, \ f \in c(f)T \).
2. (Faithful flatness) For any \( r \in R \) and \( f \in T \), the equation \( c(rf) = r c(f) \) holds and \( c(1_T) = R \).
3. (Dedekind-Mertens content formula) For each \( f, g \in T \), there exists a natural number \( n \) such that \( c(f^n)c(g) = c(f)^nc(fg) \).

For more information on content algebras and their examples we refer to [19], [20] and [23]. In [18] Nasehpour gave the definition of a Gaussian \( R \)-algebra as follows: Let \( T \) be an \( R \)-algebra such that \( f \in c(f)T \) for all \( f \in T \). \( T \) is said to be a Gaussian \( R \)-algebra if \( c(fg) = c(f)c(g) \) for all \( f, g \in T \).

Example 5.6 ([18]). Let \( T \) be a content \( R \)-algebra such that \( R \) is a Prüfer domain. Since every nonzero finitely generated ideal of \( R \) is a cancellation ideal of \( R \), the Dedekind-Mertens content formula causes \( T \) to be a Gaussian \( R \)-algebra.
In the following theorem we use the functions $\phi_R$ and $\phi_T$ that defined just prior to Theorem 2.25.

**Theorem 5.7.** Let $R$ be a Prüfer domain, $T$ a content $R$-algebra and $I$ an ideal of $R$. Then $I$ is a $\phi_R$-absorbing primary ideal of $R$ if and only if $IT$ is a $\phi_T$-absorbing primary ideal of $T$.

**Proof.** Assume that $I$ is a $\phi_R$-absorbing primary ideal of $R$. Let $f_1f_2\cdots f_{n+1} \in IT \setminus \phi_T(IT)$ for some $f_1, f_2, \ldots, f_{n+1} \in T$ such that $f_1f_2\cdots f_n \notin IT$. Then $c(f_1f_2\cdots f_{n+1}) \subseteq I$. Since $R$ is a Prüfer domain and $T$ is a content $R$-algebra, then $T$ is a Gaussian $R$-algebra. Therefore
\[ c(f_1f_2\cdots f_{n+1}) = c(f_1)c(f_2)\cdots c(f_{n+1}) \subseteq I. \]

If $c(f_1f_2\cdots f_{n+1}) \subseteq \phi_R(I) = \phi_T(IT) \cap R$, then $f_1f_2\cdots f_{n+1} \in c(f_1f_2\cdots f_{n+1})T \subseteq (\phi_R(I) \cap R)T \subseteq \phi_T(IT)$, which is a contradiction. Hence $c(f_1)c(f_2)\cdots c(f_{n+1}) \subseteq I$ and $c(f_1)c(f_2)\cdots c(f_{n+1}) \notin \phi_R(I)$.

Since $R$ is a $u$-domain, $I$ is a strongly $\phi_R$-absorbing primary ideal of $R$, by Theorem 5.4, and this implies either $c(f_1)c(f_2)\cdots c(f_n) \subseteq I$ or
\[ c(f_1)\cdots c(f_i)\cdots c(f_{n+1}) \subseteq \sqrt{T} \]
for some $1 \leq i \leq n$. In the first case we have $f_1f_2\cdots f_n \in c(f_1f_2\cdots f_n)T \subseteq IT$, which contradicts our hypothesis. In the second case we have $f_1\cdots f_i\cdots f_{n+1} \in (\sqrt{T})T \subseteq \sqrt{IT}$ for some $1 \leq i \leq n$. Consequently $IT$ is a $\phi_T$-absorbing primary ideal of $T$.

For the converse, note that since $T$ is a content $R$-algebra, $IT \cap R = I$ for every ideal $I$ of $R$. Now, apply Theorem 2.25. \qed

The algebra of all polynomials over an arbitrary ring with an arbitrary number of indeterminate is an example of content algebras.

**Corollary 5.8.** Let $R$ be a Prüfer domain and $I$ be an ideal of $R$. Then $I$ is a $\phi_R$-absorbing primary ideal of $R$ if and only if $I[X]$ is a $\phi_{R[X]}$-absorbing primary ideal of $R[X]$.

As two special cases of Corollary 5.8, when $\phi_R = \phi_T = 0$ and $\phi_R = \phi_T = 0$ we have the following result.

**Corollary 5.9.** Let $R$ be a Prüfer domain and $I$ be an ideal of $R$. Then $I$ is an $n$-absorbing primary ideal of $R$ if and only if $I[X]$ is an $n$-absorbing primary ideal of $R[X]$.

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