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CONVERGENCE OF PARALLEL ITERATIVE ALGORITHMS FOR A SYSTEM OF NONLINEAR VARIATIONAL INEQUALITIES IN BANACH SPACES†

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Abstract. In this paper, we consider the problems of convergence of parallel iterative algorithms for a system of nonlinear variational inequalities and nonexpansive mappings. Strong convergence theorems are established in the frame work of real Banach spaces.

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1. Introduction

Let $(E, \|\cdot\|)$ be a Banach space and C be a nonempty closed convex subset of E. This paper deals with the problems of convergence of iterative algorithms for a system of nonlinear variational inequalities: Find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho_1 T_1(y^*, x^*) + g_1(x^*) - g_1(y^*), j(g_1(x) - g_1(x^*))\rangle \geq 0, & \forall g_1(x) \in C, \\ \langle \rho_2 T_2(x^*, y^*) + g_2(y^*) - g_2(x^*), j(g_2(x) - g_2(y^*))\rangle \geq 0, & \forall g_2(x) \in C, \end{cases} \tag{1.1}$$

where $T_1, T_2 : C \times C \to E$, $g_1, g_2 : C \to C$ are nonlinear mappings, J is the normalized duality mapping, $j \in J$ and ρ_1, ρ_2 are two positive real numbers.

If $T_1, T_2 : C \to E$ are nonlinear mappings and $g_1 = g_2 = I$ (I denotes the identity mapping), then (1.1) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho_1 T_1(y^*) + x^* - y^*, j(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \rho_2 T_2(x^*) + y^* - x^*, j(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.2}$$

which was considered by Yao et al. [13].

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If $E = H$ is a real Hilbert space and $T_1, T_2 : C \to H$ are nonlinear mappings and $g_1 = g_2 = g$, then (1.1) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho_1 T_1(y^*) + g(x^*) - g(y^*), g(x) - g(x^*) \rangle \geq 0, & \forall g(x) \in C, \\ \langle \rho_2 T_2(x^*) + g(y^*) - g(x^*), g(x) - g(y^*) \rangle \geq 0, & \forall g(x) \in C, \end{cases} \tag{1.3}$$

which was studied by Yang et al. [12].

If $g = I$, then (1.3) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho_1 T_1(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \rho_2 T_2(x^*) + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.4}$$

which was introduced by Ceng et al. [2].

In particular, if $T_1 = T_2 = T$, then (1.4) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho_1 T(y^*) + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \rho_2 T(x^*) + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.5}$$

which is defined by Verma [9].

Further, if $x^* = y^*$, then (1.5) reduces to the following classical variational inequality (VI(T, C)) of finding $x^* \in C$ such that

$$\langle T(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.6}$$

We can see easily that the variational inequality (1.6) is equivalent to a fixed point problem. An element $x^* \in C$ is a solution of the variational inequality (1.6) if and only if $x^* \in C$ is a fixed point of the mapping $P_C(I - \lambda T)$, where P_C is the metric projection and λ is a positive real number. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Recent development of the variational inequality is to design efficient iterative algorithms to compute approximate solutions for variational inequalities and their generalization. Up to now, many authors have presented implementable and significant numerical methods such as projection method and it's variant forms, linear approximation, descent method, Newton's method and the method based on auxiliary principle technique.

However, these sequential iterative methods are only suitable for implementing on the traditional single-processor computer. To satisfy practical requirements of modern multiprocessor systems, efficient iterative methods having parallel characteristics need to be further developed for the system of variational inequalities (see [1,4,5,6,12,14]).

Motivated and inspired by the research work going on this field, in this paper, we construct an parallel iterative algorithm for approximating the solution of a new system of variational inequalities involving four different nonlinear mappings. Finally, we prove the strong convergence of the purposed iterative scheme in 2-uniformly smooth Banach spaces.

2. Preliminaries

Let C be a nonempty closed convex subset of a Banach space E with the dual space E^*. Let $\langle \cdot, \cdot \rangle$ denote the dual pair between E and E^*. Let 2^E denote the family of all the nonempty subsets of E. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2} J(x)$ for all $x \in E$ and J_q is single-valued if E^* is strictly convex or E is uniformly smooth. If $E = H$ is a Hilbert space, $J = I$, the identity mappings.

Let $B = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in B$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space E is called uniformly smooth if $\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$. E is called q-uniformly smooth if there exists a constant $c > 0$ such that

$$\rho_E(t) \leq ct^q, \quad q > 1.$$

If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth.

Definition 2.1. Let $T : C \times C \to E$ be a mapping. T is said to be

(i) (δ, ξ)-relaxed cocoercive with respect to the first argument if there exist $j(x - y) \in J(x - y)$ and constants $\delta, \xi > 0$ such that

$$\langle T(x, \cdot) - T(y, \cdot), j(x - y) \rangle \geq -\delta \|T(x, \cdot) - T(y, \cdot)\|^2 + \xi \|x - y\|^2$$

for all $x, y \in C$;

(ii) μ-Lipschitz continuous with respect to the first argument if there exists a constant $\mu > 0$ such that

$$\|T(x, \cdot) - T(y, \cdot)\| \leq \mu \|x - y\|$$

for all $x, y \in C$;

(iii) γ-Lipschitz continuous with respect to the second argument if there exists a constant $\gamma > 0$ such that

$$\|T(\cdot, x) - T(\cdot, y)\| \leq \gamma \|x - y\|$$

for all $x, y \in C$.

Definition 2.2. Let $g : C \to C$ be a mapping. g is said to be
(i) ζ-strongly accretive if there exists a constant $\zeta > 0$ such that

$$\langle g(x) - g(y), j(x - y)\rangle \geq \zeta\|x - y\|^2$$

for all $x, y \in C$.
(ii) η-Lipschitz continuous if there exists a constant $\eta > 0$ such that

$$\|g(x) - g(y)\| \leq \eta\|x - y\|$$

for all $x, y \in C$.

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be sunny if

$$Q[Q(x) + t(x - Q(x))] = Q(x)$$

whenever $Q(x) + t(x - Q(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Q(z) = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

In order to prove our main results, we also need the following lemmas.

Lemma 2.3 ([11]). *Let E be a real 2-uniformly smooth Banach space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\|Ky\|^2, \quad \forall x, y \in E,$$

where K is the 2-uniformly smooth constant of E.

Lemma 2.4 ([7]). *Let C be a nonempty closed convex subset of a smooth Banach space E and let Q_C be a retraction from E onto C. Then the following are equivalent:*
(i) Q_C is both sunny and nonexpansive;
(ii) $\langle x - Q_C(x), j(y - Q_C(x))\rangle \leq 0$ for all $x \in E$ and $y \in C$.

Lemma 2.5 ([10]). *Suppose $\{\delta_n\}$ is a nonnegative sequence satisfying the following inequality:*

$$\delta_{n+1} \leq (1 - \lambda_n)\delta_n + \sigma_n, \quad \forall n \geq 0,$$

with $\lambda_n \in [0, 1]$, $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\sigma_n = 0(\lambda_n)$. Then $\lim_{n\to\infty} \delta_n = 0$.

Lemma 2.6 ([3]). *Let $\{c_n\}$ and $\{k_n\}$ be two real sequences of nonnegative numbers that satisfy the following conditions:*
(i) $0 \leq k_n \leq 1$ for $n = 1, 2, \cdots$ and $\limsup_n k_n < 1$;
(ii) $c_{n+1} \leq k_n c_n$ for $n = 1, 2, \cdots$.
Then c_n converges to 0 as $n \to \infty$.

3. Iterative algorithms

In this section, we suggest a parallel iterative algorithm for solving the system of nonlinear variational inequality (1.1). First of all, we establish the equivalence between the system of variational inequalities and fixed point problems.

Lemma 3.1. *Let C be a nonempty closed convex subset of a smooth Banach space E. Let $Q_C : E \to C$ be a sunny nonexpansive retraction, $T_i : C \times C \to E$ and $g_i : C \to C$ be mappings for $i = 1, 2$. Then (x^*, y^*) with $x^*, y^* \in C$ is a solution of problem (1.1) if and only if*

$$\begin{cases} x^* = x^* - g_1(x^*) + Q_C[g_1(y^*) - \rho_1 T_1(y^*, x^*)], \\ y^* = y^* - g_2(y^*) + Q_C[g_2(x^*) - \rho_2 T_2(x^*, y^*)]. \end{cases}$$

Proof. Applying Lemma 2.4, we have that

$$\begin{cases} \langle \rho_1 T_1(y^*, x^*) + g_1(x^*) - g_1(y^*), j(g_1(x) - g_1(x^*))\rangle \geq 0, & \forall g_1(x) \in C, \\ \langle \rho_2 T_2(x^*, y^*) + g_2(y^*) - g_2(x^*), j(g_2(x) - g_2(y^*))\rangle \geq 0, & \forall g_2(x) \in C. \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \langle g_1(y^*) - \rho_1 T_1(y^*, x^*) - g_1(x^*), j(g_1(x) - g_1(x^*))\rangle \leq 0, & \forall g_1(x) \in C, \\ \langle g_2(x^*) - \rho_2 T_2(x^*, y^*) - g_2(y^*), j(g_2(x) - g_2(y^*))\rangle \leq 0, & \forall g_2(x) \in C. \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} g_1(x^*) = Q_C[g_1(y^*) - \rho_1 T_1(y^*, x^*)], \\ g_2(y^*) = Q_C[g_2(x^*) - \rho_2 T_2(x^*, y^*)]. \end{cases}$$

That is,

$$\begin{cases} x^* = x^* - g_1(x^*) + Q_C[g_1(y^*) - \rho_1 T_1(y^*, x^*)], \\ y^* = y^* - g_2(y^*) + Q_C[g_2(x^*) - \rho_2 T_2(x^*, y^*)]. \end{cases}$$

This completes the proof. □

This fixed point formulation allow us to suggest the following parallel iterative algorithms.

Algorithm 3.1. For any given $x_0, y_0 \in C$, computer the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$\begin{cases} x_{n+1} = x_n - g_1(x_n) + Q_C[g_1(y_n) - \rho_1 T_1(y_n, x_n)], \\ y_{n+1} = y_n - g_2(y_n) + Q_C[g_2(x_n) - \rho_2 T_2(x_n, y_n)], \end{cases}$$

where ρ_1, ρ_2 are positive real numbers.

Also, we propose a relaxed parallel algorithm which can be applied to the approximation of solution of the problem (1.1) and common fixed point of two mappings.

Algorithm 3.2. For any given $x_0, y_0 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$\begin{cases} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n[\kappa S_1(x_n) \\ & \quad +(1-\kappa)(x_n - g_1(x_n) + Q_C(g_1(y_n) - \rho_1 T_1(y_n, x_n)))], \\ y_{n+1} &= (1-\beta_n)y_n + \beta_n[\kappa S_2(y_n) \\ & \quad +(1-\kappa)(y_n - g_2(y_n) + Q_C(g_2(x_n) - \rho_2 T_2(x_n, y_n)))], \end{cases}$$

where $S_1, S_2 : C \to C$ are nonexpansive mappings, $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0,1], $\kappa \in (0,1)$ and ρ_1, ρ_2 are positive real numbers.

If $T_1, T_2 : C \to E$ are nonlinear mappings and $g_1 = g_2 = I$, then the algorithm 3.1 reduces to the following parallel iterative method for solving problem (1.2).

Algorithm 3.3. For any given $x_0, y_0 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$\begin{cases} x_{n+1} = Q_C[y_n - \rho_1 T_1(y_n)], \\ y_{n+1} = Q_C[x_n - \rho_2 T_2(x_n)], \end{cases}$$

where ρ_1, ρ_2 are positive real numbers.

If $E = H$ is a Hilbert space, $T_1, T_2 : C \to H$ are nonlinear mappings and $g_1 = g_2 = g$, Algorithm 3.1 reduces to the following parallel iterative method for solving problem (1.3).

Algorithm 3.4. For any given $x_0, y_0 \in C$, compute the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$\begin{cases} x_{n+1} = x_n - g(x_n) + P_C[g(y_n) - \rho_1 T_1(y_n)], \\ y_{n+1} = y_n - g(y_n) + P_C[g(x_n) - \rho_2 T_2(x_n)], \end{cases}$$

where ρ_1, ρ_2 are positive real numbers.

4. Main results

We now state and prove the main results of this paper.

Theorem 4.1. *Let E be a 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, C be a nonempty closed convex subset of E and Q_C be a sunny nonexpansive retraction from E onto C. Let $T_i : C \times C \to E$ be a nonlinear mapping such that (δ_i, ξ_i)-relaxed cocoercive, μ_i-Lipschitz continuous with respect to the first argument and γ_i-Lipschitz continuous with respect to the second argument for $i = 1, 2$. Let $g_i : C \to C$ be a η_i-Lipschitz continuous and ζ_i-strongly accretive mapping for $i = 1, 2$. Assume that the following assumptions hold:*

$$\left|\rho_1 - \frac{\xi_1 - \delta_1\mu_1^2}{2K^2\mu_1^2}\right| < \frac{\sqrt{(\xi_1 - \delta_1\mu_1^2)^2 - 2K^2\mu_1^2\tau_1(2-\tau_1)}}{2K^2\mu_1^2}, \tag{4.1}$$

$$\left|\rho_2 - \frac{\xi_2 - \delta_2\mu_2^2}{2K^2\mu_2^2}\right| < \frac{\sqrt{(\xi_2 - \delta_2\mu_2^2)^2 - 2K^2\mu_2^2\tau_2(2-\tau_2)}}{2K^2\mu_2^2}, \tag{4.2}$$

$$\xi_1 > \delta_1\mu_1^2 + K\mu_1\sqrt{2\tau_1(2-\tau_1)},$$

$$\xi_2 > \delta_2\mu_2^2 + K\mu_2\sqrt{2\tau_2(2-\tau_2)},$$

where $\tau_1 = m_1 + m_2 + \rho_2\gamma_2$, $\tau_2 = m_1 + m_2 + \rho_1\gamma_1$, $m_1 = \sqrt{1 - 2\zeta_1 + 2K^2\eta_1^2}$ *and* $m_2 = \sqrt{1 - 2\zeta_2 + 2K^2\eta_2^2}$.

Then there exist $x^*, y^* \in E$, *which solves the problem (1.1). Moreover, the parallel iterative sequences* $\{x_n\}$ *and* $\{y_n\}$ *generated by the Algorithm 3.1 converge to* x^* *and* y^*, *respectively.*

Proof. To proof the result, we first need to evaluate $\|x_{n+1} - x_n\|$ for all $n \geq 0$. From Algorithm 3.1 and the nonexpansive property of the sunny nonexpansive retraction Q_C, we can get

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&= \|x_n - g_1(x_n) + Q_C[g_1(y_n) - \rho_1 T_1(y_n, x_n)] \\
&\quad - (x_{n-1} - g_1(x_{n-1}) + Q_C[g_1(y_{n-1}) - \rho_1 T_1(y_{n-1}, x_{n-1})])\| \\
&\leq \|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\| \\
&\quad + \|Q_C[g_1(y_n) - \rho_1 T_1(y_n, x_n)] - Q_C[g_1(y_{n-1}) - \rho_1 T_1(y_{n-1}, x_{n-1})]\| \\
&\leq \|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\| \\
&\quad + \|y_n - y_{n-1} - (g_1(y_n) - g_1(y_{n-1}))\| \\
&\quad + \|y_n - y_{n-1} - \rho_1(T_1(y_n, x_n) - T_1(y_{n-1}, x_n))\| \\
&\quad + \rho_1\|T_1(y_{n-1}, x_n) - T_1(y_{n-1}, x_{n-1})\|.
\end{aligned} \tag{4.3}$$

Using the strongly accretivity and Lipschitz continuity of g_1 and Lemma 2.3, we find that

$$\begin{aligned}
&\|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\|^2 \\
&\leq \|x_n - x_{n-1}\| - 2\langle g_1(x_n) - g_1(x_{n-1}), j(x_n - x_{n-1})\rangle \\
&\quad + 2K^2\|g_1(x_n) - g_1(x_{n-1})\|^2 \\
&\leq \|x_n - x_{n-1}\|^2 - 2\zeta_1\|x_n - x_{n-1}\|^2 + 2K^2\eta_1^2\|x_n - x_{n-1}\|^2 \\
&= (1 - 2\zeta_1 + 2K^2\eta_1^2)\|x_n - x_{n-1}\|^2
\end{aligned}$$

and

$$\|y_n - y_{n-1} - (g_1(y_n) - g_1(y_{n-1}))\|^2 \leq (1 - 2\zeta_1 + 2K^2\eta_1^2)\|y_n - y_{n-1}\|^2,$$

which imply that

$$\|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\| \leq m_1\|x_n - x_{n-1}\| \tag{4.4}$$

and

$$\|y_n - y_{n-1} - (g_1(y_n) - g_1(y_{n-1}))\| \leq m_1\|y_n - y_{n-1}\|, \tag{4.5}$$

where $m_1 = \sqrt{1 - 2\zeta_1 + 2K^2\eta_1^2}$. Since T_1 is (δ_1, ξ_1)-relaxed cocoercive and μ_1-Lipschitz continuous with respect to the first argument, we have

$$\begin{aligned}
&\|y_n - y_{n-1} - \rho_1(T_1(y_n, x_n) - T_1(y_{n-1}, x_n))\|^2 \\
&\leq \|y_n - y_{n-1}\|^2 - 2\rho_1\langle T_1(y_n, x_n) - T_1(y_{n-1}, x_n), j(y_n - y_{n-1})\rangle \\
&\quad + 2K^2\rho_1^2\|T_1(y_n, x_n) - T_1(y_{n-1}, x_n)\|^2 \\
&\leq \|y_n - y_{n-1}\|^2 - 2\rho_1[-\delta_1\|T_1(y_n, x_n) - T_1(y_{n-1}, x_n)\|^2 \\
&\quad + \xi_1\|y_n - y_{n-1}\|^2] + 2K^2\rho_1^2\|T_1(y_n, x_n) - T_1(y_{n-1}, x_n)\|^2 \\
&\leq \|y_n - y_{n-1}\|^2 + 2\rho_1\delta_1\mu_1^2\|y_n - y_{n-1}\|^2 - 2\rho_1\xi_1\|y_n - y_{n-1}\|^2 \\
&\quad + 2K^2\rho_1^2\mu_1^2\|y_n - y_{n-1}\|^2 \\
&= (1 + 2\rho_1\delta_1\mu_1^2 - 2\rho_1\xi_1 + 2K^2\rho_1^2\mu_1^2)\|y_n - y_{n-1}\|^2. \qquad (4.6)
\end{aligned}$$

Also,using the Lipschitz continuity of T_1 with respect to second argument,

$$\|T_1(y_{n-1}, x_n) - T_1(y_{n-1}, x_{n-1})\| \leq \gamma_1\|x_n - x_{n-1}\|. \qquad (4.7)$$

Combining (4.3)-(4.7), we have

$$\|x_{n+1} - x_n\| \leq (m_1 + \rho_1\gamma_1)\|x_n - x_{n-1}\| + (m_1 + \theta_1)\|y_n - y_{n-1}\|, \qquad (4.8)$$

where $\theta_1 = \sqrt{1 + 2\rho_1\delta_1\mu_1^2 - 2\rho_1\xi_1 + 2K^2\rho_1^2\mu_1^2}$.

Similarly, since g_2 is η_2-Lipschitz continuous and ζ_2-strongly accretive, T_2 is (δ_2, ξ_2)-relaxed cocoercive, μ_2-Lipschitz continuous with respect to the first argument and γ_2-Lipschitz continuous with respect to the second argument, we obtain

$$\|y_{n+1} - y_n\| \leq (m_2 + \theta_2)\|x_n - x_{n-1}\| + (m_2 + \rho_2\gamma_2)\|y_n - y_{n-1}\|, \qquad (4.9)$$

where $m_2 = \sqrt{1 - 2\zeta_2 + 2K^2\eta_2^2}$ and $\theta_2 = \sqrt{1 + 2\rho_2\delta_2\mu_2^2 - 2\rho_2\xi_2 + 2K^2\rho_2^2\mu_2^2}$. It follows from (4.8) and (4.9) that

$$\begin{aligned}
&\|x_{n+1} - x_n\| + \|y_{n+1} - y_n\| \\
&\leq (m_1 + m_2 + \theta_2 + \rho_1\gamma_1)\|x_n - x_{n-1}\| + (m_1 + m_2 + \theta_1 + \rho_2\gamma_2)\|y_n - y_{n-1}\| \\
&\leq k(\|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|), \qquad (4.10)
\end{aligned}$$

where $k = \max\{m_1 + m_2 + \theta_2 + \rho_1\gamma_1, m_1 + m_2 + \theta_1 + \rho_2\gamma_2\}$. From (4.1) and (4.2), we know that $0 \leq k < 1$. Let $c_n = \|x_n - x_{n-1}\| + \|y_n - y_{n-1}\|$. Then (4.10) can be rewritten as

$$c_{n+1} \leq kc_n, \quad n = 1, 2, \cdots.$$

It follows from Lemma 2.6 that $\{x_n\}$ and $\{y_n\}$ are both Cauchy sequences in E. There exist $x^*, y^* \in E$ such that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. By continuity, we know that x^*, y^* satisfy

$$\begin{cases} x^* = x^* - g_1(x^*) + Q_C[g_1(y^*) - \rho_1 T_1(y^*, x^*)], \\ y^* = y^* - g_2(y^*) + Q_C[g_2(x^*) - \rho_2 T_2(x^*, y^*)]. \end{cases}$$

It follows from Lemma 3.1 that (x^*, y^*) is a solution of problem (1.1). This completes the proof. □

If $T_1, T_2 : C \to E$ are nonlinear mappings and $g_1 = g_2 = I$, the the following corollary follows immediately from Theorem 4.1.

Corollary 4.2. *Let E be a 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, C be a nonempty closed convex subset of E and Q_C be a sunny nonexpansive retraction from E onto C. Let $T_i : C \to E$ be a (δ_i, ξ_i)-relaxed cocoercive and μ_i-Lipschitz continuous mapping for $i = 1, 2$. Assume that the following assumptions hold:*

$$\left|\rho_1 - \frac{\xi_1 - \delta_1\mu_1^2}{2K^2\mu_1^2}\right| < \frac{\xi_1 - \delta_1\mu_1^2}{2K^2\mu_1^2},$$

$$\left|\rho_2 - \frac{\xi_2 - \delta_2\mu_2^2}{2K^2\mu_2^2}\right| < \frac{\xi_2 - \delta_2\mu_2^2}{2K^2\mu_2^2},$$

$$\xi_1 > \delta_1\mu_1^2 \quad \text{and} \quad \xi_2 > \delta_2\mu_2^2.$$

Then there exist $x^*, y^* \in E$, which solves the problem (1.2). Moreover, the parallel iterative sequences $\{x_n\}$ and $\{y_n\}$ generated by the Algorithm 3.3 converge to x^* and y^*, respectively.

Remark 4.1. (i) We note that Hilbert spaces and $L^p (p \geq 2)$ spaces are 2-uniformly smooth.

(ii) If $E = H$ is a Hilbert space, then a sunny nonexpansive retraction Q_C is coincident with the metric projection P_C from H onto C.

(iii) It is well known that the 2-uniformly smooth constant $K = \frac{\sqrt{2}}{2}$ in Hilbert spaces.

We can obtain the following result immediately.

Corollary 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_i : C \to H$ be a (δ_i, ξ_i)-relaxed cocoercive and μ_i-Lipschitz continuous mapping for $i = 1, 2$. Let $g : C \to C$ be a η-Lipschitz continuous and ζ-strongly monotone mapping. Assume that the following assumptions hold:*

$$\left|\rho_1 - \frac{\xi_1 - \delta_1\mu_1^2}{\mu_1^2}\right| < \frac{\sqrt{(\xi_1 - \delta_1\mu_1^2)^2 - \mu_1^2\tau(2-\tau)}}{\mu_1^2},$$

$$\left|\rho_2 - \frac{\xi_2 - \delta_2\mu_2^2}{\mu_2^2}\right| < \frac{\sqrt{(\xi_2 - \delta_2\mu_2^2)^2 - \mu_2^2\tau(2-\tau)}}{\mu_2^2},$$

$$\xi_1 > \delta_1\mu_1^2 + \mu_1\sqrt{\tau(2-\tau)},$$

$$\xi_2 > \delta_2\mu_2^2 + \mu_2\sqrt{\tau(2-\tau)},$$

where $\tau = 2\sqrt{1 - 2\zeta + \eta^2}$.

Then there exist $x^*, y^* \in H$, which solve the problem (1.3). Moreover, the parallel iterative sequences $\{x_n\}$ and $\{y_n\}$ generated by the Algorithm 3.4 converge to x^* and y^*, respectively.

Let $Fix(S_i)$ denote the set of fixed points of the mapping S_i, i.e., $Fix(S_i) = \{x \in C : S_i x = x\}$ and Ω the set of solutions of the problem (1.1).

Theorem 4.4. *Let E be a 2-uniformly smooth Banach space with the 2-uniformly smooth constant K, C be a nonempty closed convex subset of E and Q_C be a sunny nonexpansive retraction from E onto C. Let $T_i : C \times C \to E$ be a nonlinear mapping such that (δ_i, ξ_i)-relaxed cocoercive, μ_i-Lipschitz continuous with respect to the first argument and γ_i-Lipschitz continuous with respect to the second argument for $i = 1, 2$. Let $g_i : C \to C$ be a η_i-Lipschitz continuous and ζ_i-strongly accretive mapping for $i = 1, 2$. Let $S_i : C \to C$ be a nonexpansive mapping with a fixed point for $i = 1, 2$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$. Assume that the following assumptions hold:*

(C1) $0 < \Theta_{1,n} = \alpha_n(1 - \kappa - (1 - \kappa)(m_1 + \rho_1\gamma_1)) - \beta_n(1 - \kappa)(m_2 + \theta_2) < 1$,

(C2) $0 < \Theta_{2,n} = \beta_n(1 - \kappa - (1 - \kappa)(m_2 + \rho_2\gamma_2)) - \alpha_n(1 - \kappa)(m_1 + \theta_1) < 1$,

(C3) $\sum_{n=0}^{\infty} \Theta_{1,n} = \infty$ *and* $\sum_{n=0}^{\infty} \Theta_{2,n} = \infty$, *where*

$$m_1 = \sqrt{1 - 2\zeta_1 + 2K^2\eta_1^2}, \quad m_2 = \sqrt{1 - 2\zeta_2 + 2K^2\eta_2^2},$$

$$\theta_1 = \sqrt{1 + 2\rho_1\delta_1\mu_1^2 - 2\rho_1\xi_1 + 2K^2\rho_1^2\mu_1^2},$$

and

$$\theta_2 = \sqrt{1 + 2\rho_2\delta_2\mu_2^2 - 2\rho_2\xi_2 + 2K^2\rho_2^2\mu_2^2}.$$

If $\Omega \cap Fix(S_1) \cap Fix(S_2) \neq \phi$, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by the Algorithm 3.2 converge to x^* and y^*, respectively, where $(x^*, y^*) \in \Omega$ and $x^*, y^* \in Fix(S_1) \cap Fix(S_2)$.

Proof. Letting $(x^*, y^*) \in \Omega$, we obtain from Lemma 3.1 that

$$\begin{cases} x^* = x^* - g_1(x^*) + Q_C[g_1(y^*) - \rho_1 T_1(y^*, x^*)], \\ y^* = y^* - g_2(y^*) + Q_c[g_2(x^*) - \rho_2 T_2(x^*, y^*)]. \end{cases}$$

Since $x^*, y^* \in Fix(S_1) \cap Fix(S_2)$, we have

$$\begin{cases} x^* = S_1(x^* - g_1(x^*) + Q_C[g_1(y^*) - \rho_1 T_1(y^*, x^*)]), \\ y^* = S_2(y^* - g_2(y^*) + Q_C[g_2(x^*) - \rho_2 T_2(x^*, y^*)]). \end{cases}$$

Putting $e_{1,n} = \kappa S_1(x_n) + (1 - \kappa)(x_n - g_1(x_n) + Q_C[g_1(y_n) - \rho_1 T_1(y_n, x_n)])$ for each $n = 0, 1, 2, \cdots$, we arrive at

$$\begin{aligned} &\|e_{1,n} - x^*\| \\ &= \|\kappa S_1(x_n) + (1 - \kappa)(x_n - g_1(x_n) + Q_C[g_1(y_n) - \rho_1 T_1(y_n, x_n)]) - x^*\| \\ &\leq \kappa\|S_1(x_n) - x^*\| + (1 - \kappa)\|x_n - g_1(x_n) + Q_C[g_1(y_n) - \rho_1 T_1(y_n, x_n)] \end{aligned}$$

$$
\begin{aligned}
&- (x^* - g_1(x^*) + Q_C[g_1(y^*) - \rho_1 T_1(y^*, x^*)])\| \\
&\le \kappa\|x_n - x^*\| + (1-\kappa)[\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \\
&\quad + \|Q_C[g_1(y_n) - \rho_1 T_1(y_n, x_n)] - Q_C[g_1(y^*) - \rho_1 T_1(y^*, x^*)]\| \\
&\le \kappa\|x_n - x^*\| + (1-\kappa)[\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \\
&\quad + \|y_n - y^* - (g_1(y_n) - g_1(y^*))\| \\
&\quad + \|y_n - y^* - \rho_1(T_1(y_n, x_n) - T_1(y^*, x_n))\| \\
&\quad + \rho_1\|T_1(y^*, x_n) - T_1(y^*, x^*)\|]. \qquad (4.11)
\end{aligned}
$$

Using the arguments as in the proof of Theorem 4.1, we obtain

$$\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \le m_1\|x_n - x^*\|,$$

$$\|y_n - y^* - (g_1(y_n) - g_1(y^*))\| \le m_1\|y_n - y^*\|,$$

$$\|y_n - y^* - \rho_1(T_1(y_n, x_n) - T_1(y^*, x_n))\| \le \theta_1\|y_n - y^*\|,$$

and

$$\|T_1(y^*, x_n) - T_1(y^*, x^*)\| \le \gamma_1\|x_n - x^*\|,$$

where $m_1 = \sqrt{1 - 2\zeta_1 + 2K^2\eta_1^2}$ and $\theta_1 = \sqrt{1 + 2\rho_1\delta_1\mu_1^2 - 2\rho_1\xi_1 + 2K^2\rho_1^2\mu_1^2}$. From (4.11), we have

$$
\begin{aligned}
\|e_{1,n} - x^*\| &\le \kappa\|x_n - x^*\| + (1-\kappa)[m_1\|x_n - x^*\| + m_1\|y_n - y^*\| \\
&\quad + \theta_1\|y_n - y^*\| + \rho_1\gamma_1\|x_n - x^*\|] \\
&= [\kappa + (1-\kappa)(m_1 + \rho_1\gamma_1)]\|x_n - x^*\| \\
&\quad + (1-\kappa)(m_1 + \theta_1)\|y_n - y^*\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - x^*\| &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|e_{1,n} - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\{[\kappa + (1-\kappa)(m_1 + \rho_1\gamma_1)]\|x_n - x^*\| \\
&\quad + (1-\kappa)(m_1 + \theta_1)\|y_n - y^*\|\} \\
&= [1 - \alpha_n + \alpha_n(\kappa + (1-\kappa)(m_1 + \rho_1\gamma_1))]\|x_n - x^*\| \\
&\quad + \alpha_n(1-\kappa)(m_1 + \theta_1)\|y_n - y^*\|. \qquad (4.12)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\|y_{n+1} - y^*\| &= \beta_n(1-\kappa)(m_2 + \theta_2)\|x_n - x^*\| \\
&\quad + [1 - \beta_n + \beta_n(\kappa + (1-\kappa)(m_2 + \rho_2\gamma_2))]\|y_n - y^*\|. \qquad (4.13)
\end{aligned}
$$

where $m_2 = \sqrt{1 - 2\zeta_2 + 2K^2\eta_2^2}$ and $\theta_2 = \sqrt{1 + 2\rho_2\delta_2\mu_2^2 - 2\rho_2\xi_2 + 2\rho_2^2K^2\mu_2^2}$. Now (4.12) and (4.13) imply

$$
\begin{aligned}
&\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
&\le [1 - (\alpha_n(1 - \kappa - (1-\kappa)(m_1 + \rho_1\gamma_1)) - \beta_n(1-\kappa)(m_2 + \theta_2))]\|x_n - x^*\| \\
&\quad + [1 - (\beta_n(1 - \kappa - (1-\kappa)(m_2 + \rho_2\gamma_2)) - \alpha_n(1-\kappa)(m_1 + \theta_1))]\|y_n - y^*\|
\end{aligned}
$$

$$\leq \max\{(1-\Theta_{1,n}),(1-\Theta_{2,n})\}(\|x_n - x^*\| + \|y_n - y^*\|), \tag{4.14}$$

where

$$\Theta_{1,n} = \alpha_n(1-\kappa-(1-\kappa)(m_1+\rho_1\gamma_1)) - \beta_n(1-\kappa)(m_2+\theta_2),$$

$$\Theta_{2,n} = \beta_n(1-\kappa-(1-\kappa)(m_2+\rho_2\gamma_2)) - \alpha_n(1-\kappa)(m_1+\theta_1).$$

Define the norm $\|\cdot\|_*$ on $E \times E$ by

$$\|(x,y)\|_* = \|x\| + \|y\|, \quad \forall (x,y) \in E \times E.$$

Then $(E \times E, \|\cdot\|_*)$ is a Banach space. Hence, (4,14) implies that

$$\begin{aligned} &\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\|_* \\ &\leq \max\{(1-\Theta_{1,n}),(1-\Theta_{2,n})\}\|(x_n, y_n) - (x^*, y^*)\|_*. \end{aligned} \tag{4.15}$$

From the conditions (C1)-(C3) and Lemma 2.5 to (4.15), we obtain that

$$\lim_{n\to\infty} \|(x_{n+1}, y_{n+1}) - (x^*, y^*)\|_* = 0.$$

Therefore, the sequences $\{x_n\}$ and $\{y_n\}$ converge to x^* and y^*, respectively. This completes the proof. □

Remark 4.2. Theorem 4.1 and 4.4 extend the solvability of the systems of variational inequalities (1.2)-(1.6) to the more general system of variational inequalities (1.1). The underlying mapping $T_i : C \times C \to E$ $(i = 1, 2)$ in our paper needs to be relaxed (δ_i, ξ_i)-relaxed cocoercive while the underlying operators A, B in [13] needs to inverse strongly accretive. Hence, Theorem 4.1 and 4.4 extend and improve the main results of [9,12,13].

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