2-ENGELIZER SUBGROUP OF
A 2-ENGEL TRANSITIVE GROUPS

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Abstract. A general notion of $\chi$-transitive groups was introduced by C. Delizia et al. in [6], where $\chi$ is a class of groups. In [5], Ciobanu, Fine and Rosenberger studied the relationship among the notions of conjugately separated abelian, commutative transitive and fully residually $\chi$-groups.

In this article we study the concept of 2-Engel transitive groups and among other results, its relationship with conjugately separated 2-Engel and fully residually $\chi$-groups are established. We also introduce the notion of 2-Engelizer of the element $x$ in $G$ and denote the set of all 2-Engelizers in $G$ by $E^2(G)$. Then we construct the possible values of $|E^2(G)|$.

1. Introduction

An element $x$ of a group $G$ is called a right Engel element, if for every $y \in G$, there exists a natural number $n = n(x, y)$ such that $[x, n y] = 1$. If $n$ can be chosen independent of $y$, then $x$ is called a right $n$-Engel element or simply a bounded right Engel element. We denote the sets of all right Engel elements and bounded right Engel elements of $G$ by $R(G)$ and $\overline{R}(G)$, respectively.

An element $x$ of $G$ is called a left Engel element, if for every $y \in G$, there exists a natural number $n = n(x, y)$ such that $[y, n x] = 1$. If $n$ can be chosen independent of $y$, then $x$ is called a left $n$-Engel element or simply a bounded left Engel element. We denote the sets of all left Engel elements and bounded left Engel elements of $G$ by $L(G)$ and $\overline{L}(G)$, respectively. For any positive integer $n$, a group $G$ is called an $n$-Engel group, if $[x, n y] = [y, n x] = 1$ for all $x, y \in G$.

A proper subset $E$ of a group $G$ is said to be $n$-Engel set, whenever $[x, n y] = [y, n x] = 1$ for all $x, y \in E$.

Let $\chi$ be a class of groups. Then a group $G$ is residually $\chi$ if for every non-trivial element $g \in G$, there is a homomorphism $\phi : G \to \tilde{H}$, where $\tilde{H}$
is a $\chi$-group such that $\phi(g) \neq 1$. Also a group $G$ is fully residually $\chi$ if for finitely many non-trivial elements $g_1, \ldots, g_n$ in $G$ there exists a homomorphism $\phi : G \to H$ where $H$ is a $\chi$-group such that $\phi(g_i) \neq 1$ for all $i = 1, \ldots, n$.

**Definition 1.1.** A subgroup $H$ of a group $G$ is called malnormal or conjugately separated, if $H \cap H^x = 1$ for all $x \in G \setminus H$.

It is clear that the intersection of a family of malnormal subgroups of a given group $G$ is again malnormal, which allows us to define the malnormal closure of a subgroup $H$ of $G$. Clearly the intersection of all malnormal subgroups of $G$ contains $H$ is malnormal.

2. 2-Engel transitive groups

A group $G$ is called a conjugately separated 2-Engel (henceforth CSE$^2$-group) if all of its maximal 2-Engel subgroups are malnormal. In the following, we discuss the notion of 2-Engel transitive group and then give its relationship with CSE$^2$-group and fully residually $\chi$-groups.

**Definition 2.1.** (a) A group $G$ is 2-Engel transitive (henceforth 2-ET), when $[x, y, y] = 1$ and $[y, z, z] = 1$ imply that $[x, z, z] = 1$ for every non-trivial elements $x, y, z$ in $G$.

(b) For a given element $x$ of $G$, we call $E^2_G(x) = \{y \in G : [x, y, y] = 1, [y, x, x] = 1\}$ to be the set of 2-Engelizer of $x$ in $G$. The family of all 2-Engelizers in $G$ is denoted by $E^2(G)$ and $|E^2(G)|$ denotes the number of distinct 2-Engelizers in $G$.

As an example consider $Q_{16} = \langle a, b : a^8 = 1, a^4 = b^4, b^{-1}ab = a^{-1}\rangle$, the Quaternion group of order 16 and take the element $b$ in $Q_{16}$. Then one can easily check that the 2-Engelizer set of $b$ is as follows:

$$E^2_{Q_{16}}(b) = \{1, a^2, a^4, a^6, b, a^2b, a^4b, a^6b\}.$$ 

The following lemma is useful for our further investigations.

**Lemma 2.2.** Let $G$ be a 2-ET group. Then 2-Engelizer of each non-trivial element of $G$ is 2-Engel set.

**Proof.** Let $G$ be a 2-Engel transitive group, then $[x, y, y] = 1$ and $[y, z, z] = 1$ imply $[x, z, z] = 1$ for all non-trivial elements $x, y, z$ in $G$. Clearly using the definition, for $y, z \in E^2_G(x)$, it follows that $[z, y, y] = 1$ and $[y, z, z] = 1$. Thus $E^2_G(x)$ is 2-Engel set. 

We remark that for the identity element $e$ of $G$, we have $G = E^2_G(e)$ and hence $G \in E^2(G)$. Clearly in general, the 2-Engelizer of each non-trivial element of an arbitrary group $G$ does not form a subgroup. The following example shows our claim.
Example 2.3. Let $G$ be a finitely presented group of the following form:

$$G = \langle a_1, a_2, a_3, a_4 : a_3^3 = a_4^3 = 1, [a_1, a_2] = 1, [a_1, a_3] = a_4, [a_1, a_4] = 1, [a_2, a_3] = 1, [a_2, a_4] = a_2, [a_3, a_4] = 1 \rangle.$$ 

Using GAP [7] implies that $G$ is an infinite group. One can easily check that $G$ is not 2-ET, as $[a_2, a_1, a_2] = 1$ and $[a_1, a_4, a_4] = 1$, while $[a_2, a_4, a_4] = a_2$.

Moreover, $E^2_G(a_1)$ is not a subgroup of $G$, since it is easily calculated that $a_2, a_3 \in E^2_G(a_1)$ but $a_2 a_3 \not\in E^2_G(a_1)$.

Here, we state an interesting property of 2-Engel transitive groups.

**Proposition 2.4.** Let $G$ be a 2-ET group. Then $x E^2_G(x)$ is nilpotent of class at most 3, for every non-trivial element $x$ of $G$.

**Proof.** Note that $x E^2_G(x) = \langle y \in E^2_G(x) \rangle$. Now, for every $y \in E^2_G(x)$:

$$[x, y, x] = [[x, y], y^{-1}, x] = [x, y][x, y, y^{-1}], x]$$

On the other hand $[x, x, x] = 1$ and $[x, x, x^2] = 1$ imply that $[x, y, x^2, x^2] = 1$, as $G$ is 2-ET. Hence $x E^2_G(x)$ is 2-Engel group and so nilpotent of class at most 3. □

Now, we discuss the condition under which the 2-Engelizer of each non-trivial element of $G$ is a subgroup.

**Theorem 2.5.** Let $G$ be an arbitrary group. Then the set of each 2-Engelizer of a non-trivial element in $G$ forms a subgroup if and only if the group $x E^2_G(x)$ is abelian for all non-trivial element $x$ of $G$.

**Proof.** Let $y \in E^2_G(x)$. Then one can easily see that

$$[y^{-1}, x, x] = [[x, y], y^{-1}, x] = [y, y][x, y, y^{-1}], x]$$

and also

$$[x, y^{-1}, y^{-1}] = [[y, x], y^{-1}, y^{-1}] = [[y, x][y, x, y^{-1}], y^{-1}]$$

Thus $y^{-1} \in E^2_G(x)$.

Now, for every $y, z \in E^2_G(x)$ we have:

$$[y z, x, x] = [[y, x], y z, x, x]$$

Thus $y^{-1} \in E^2_G(x)$. 

Now, for every $y, z \in E^2_G(x)$ we have:

$$[y z, x, x] = [[y, x], y z, x, x]$$

Thus $y^{-1} \in E^2_G(x)$. 

Now, for every $y, z \in E^2_G(x)$ we have:

$$[y z, x, x] = [[y, x], y z, x, x]$$

Thus $y^{-1} \in E^2_G(x)$. 

Now, for every $y, z \in E^2_G(x)$ we have:

$$[y z, x, x] = [[y, x], y z, x, x]$$

Thus $y^{-1} \in E^2_G(x)$. 

Now, for every $y, z \in E^2_G(x)$ we have:

$$[y z, x, x] = [[y, x], y z, x, x]$$

Thus $y^{-1} \in E^2_G(x)$.
Clearly, using Witt identity and the same technique in the proof of Theorem 7.13 in [8], we may have \([y, x, z, x] = 1\) if and only if \(x^E_{2G}(x)\) is abelian. Similarly, \([x, yz, yz] = 1\) and the proof is complete. □

The proof of the following lemma is a routine argument by using Zorn’s Lemma.

**Lemma 2.6.** Every 2-Engel subgroup \(H\) of a given group \(G\) is contained in a maximal 2-Engel subgroup.

The following fact is needed in proving our main result.

**Proposition 2.7.** Let \(G\) be a CSE\(^2\)-group. Then every non-trivial 2-Engel normal subgroup of \(G\) is maximal.

**Proof.** Let \(G\) be a CSE\(^2\)-group and \(K\) a non-trivial 2-Engel normal subgroup of \(G\). Then by Lemma 2.6, \(K\) is contained in a maximal 2-Engel subgroup \(M\) of \(G\). Let \(1 \neq k \in K\), then for each \(x \in G\) we have \(k^x \in M\). Since \(G\) is CSE\(^2\), it follows that \(M\) is malnormal and therefore \(x \in M\). Thus \(G = M\), which implies that \(K\) is maximal. □

Using the above proposition, we obtain the following useful result.

**Corollary 2.8.** Let \(G\) be a CSE\(^2\)-group. Then every 2-Engel normal subgroup of \(G\) is equal to the second centre of \(G\).

**Lemma 2.9.** Let \(\chi\) be a class of groups such that each non-2-Engel \(\chi\)-group \(H \in \chi\) is CSE\(^2\)-group. Let \(N\) be a 2-Engel normal subgroup of a non-2-Engel residually \(\chi\)-group \(G\). Then \(N\) is contained in the second centre of \(G\).

**Proof.** Let \(G \in \chi\), then by the assumption \(G\) is CSE\(^2\) and therefore by Corollary 2.8, every 2-Engel normal subgroup of \(G\) is equal to the second centre of \(G\). Now let \(N\) be a 2-Engel normal subgroup of a non-2-Engel residually \(\chi\)-group \(G\) so that \(N\) is not contained in the second centre of \(G\). Then there exist elements \(n \in N\) and \(g_1, g_2 \in G\) such that \([n, g_1, g_2] = x \neq 1\), say. Since \(G\) is residually \(\chi\), there exists a normal subgroup \(N_x\) of \(G\) such that \(G/N_x \in \chi\) and \(x \notin N_x\). Clearly \(NN_x/N_x\) is a non-trivial 2-Engel normal subgroup of \(G/N_x\). Then \(NN_x/N_x = Z_2(G/N_x)\) and this contradicts that \(x \notin N_x\). Therefore \(N\) is contained in the second centre of \(G\). □

**Remark 2.10.** Let \(G\) be a 2-ET and non 2-Engel group, then it is clear that \(Z_2(G) = 1\). So it follows from the above lemma that any normal 2-Engel subgroup of \(G\) must be trivial.

Now we study the relationship between the non 2-Engel CSE\(^2\), 2-ET and fully residually \(\chi\)-groups.

**Theorem 2.11.** Let \(\chi\) be a class of groups such that each non 2-Engel \(\chi\)-group is CSE\(^2\) and \(G\) be a non 2-Engel and residually \(\chi\)-group. Then

(i) \(G\) is a CSE\(^2\).

(ii) If \(G\) is a 2-Engel transitive, then \(G\) is fully residually \(\chi\)-group.
Proof. (i) Let $G$ be a non 2-Engel group. Then there exist $x, y \in G$ such that $[x, y, y] \neq 1$. On the other hand, there is a normal subgroup $N$ of $G$, for which $[x, y, y] \notin N$ and $G/N \in \chi$, as $G$ is residually $\chi$. Clearly, $x, y \notin N$ and $G/N$ is non 2-Engel. Hence $G/N$ is CSE$^2$ and so every maximal 2-Engel subgroup in $G/N$ is malnormal. Suppose $M/N$ is a maximal 2-Engel subgroup of $G/N$. Then $\frac{M}{N} \cap (\frac{M}{N})g^N = N$, for all $gN \in \frac{G}{N}$, which implies that $M \cap Mg = 1$ for every $g \in G\setminus M$, and hence $G$ is CSE$^2$.

(ii) Let $G$ be a 2-ET, non 2-Engel and residually $\chi$-group. Then we show that $G$ is fully residually $\chi$. In order to do this, we prove that for given non-trivial elements $g_1, \ldots, g_n$ in $G$ there is a normal subgroup $N$ such that $g_1, \ldots, g_n$ are not in $N$ and $G/N \in \chi$. This is equivalent to showing that given non-trivial elements $g_1, \ldots, g_n \in G$ there exists a non-trivial element $g \in G$ such that for any normal subgroup $N$ of $G$ if $g \notin N$, then $g \notin N$ for $i = 1, \ldots, n$. We proceed by induction on $n$. This is true for $n = 1$, as $G$ is residually $\chi$. Now assume the result holds for $n - 1$. Let $[g_i^x, g, g] = 1$ be a non-trivial commutator for any $x \in G$. Then by 2-Engel transitivity, the normal closure $g_i^n$ is 2-Engel and hence by Remark 2.11 it is trivial, but $g_i$ is in $g_i^n$, which is non-trivial. Therefore either $[g_i^x, g, g] \neq 1$ or $[g, g_i^n, g_i^n] \neq 1$, for some $x \in G$. Then either of the latest commutators is not in some normal subgroup $N$ of $G$. This follows that $g_1, \ldots, g_n \notin N$, which gives the proof. □

In 1967, B. Baumslag [3] introduced the notion of fully residually free groups and proved that a residually free group is fully residually free if and only if it is commutative transitive. A group $G$ is commutative transitive, if $[x, y] = 1$ and $[y, z] = 1$ implies that $[x, z] = 1$ for nontrivial elements $x, y, z \in G$.

Here we show that Baumslag’s theorem is also true in the case of 2-Engel transitive groups.

**Theorem 2.12.** Let $G$ be a residually free group. Then $G$ is fully residually free if and only if $G$ is 2-Engel transitive.

**Proof.** Let $G$ be a fully residually free group. Assume $[x, y, y] = [y, z, z] = 1$, for every non-trivial elements $x, y, z \in G$. We must show that $[x, z, z] = 1$. If $[x, z, z] \neq 1$, there exists a homomorphism $\phi : G \to F$, where $F$ is a free group and

$$\phi([x, z, z]) = [\phi(x), \phi(z), \phi(z)] \neq 1, \phi(x) \neq 1, \phi(y) \neq 1, \phi(z) \neq 1.$$  

Hence, $\phi([x, y, y]) \neq 1$ and $\phi([y, z, z]) \neq 1$ in $F$, which contradict the assumptions that $[x, y, y] = 1$ and $[y, z, z] = 1$ in $G$. Thus $G$ is 2-ET.

Conversely, without loss of generality we may assume that $G$ is non-abelian. Also if $G$ is non 2-Engel and residually free, then the result holds by Theorem 2.11(ii). Now, let $G$ be a non-abelian 2-Engel residually free group. Then $[x, y, y] = 1$ for all $x, y \in G$ and for some non-trivial elements $x_0, y_0 \in G$, we have $[x_0, y_0] \neq 1$. Hence there is a homomorphism $\phi : G \to F$, where $F$ is a
free group such that $\phi([x_0,y_0]) \neq 1$. Thus

$$\phi([x_0,y_0]) = [\phi(x_0), \phi(y_0)] \neq 1 \Rightarrow \phi(x_0) \neq 1, \ \phi(y_0) \neq 1.$$  

On the other hand, since $F$ is free we must have $\phi([x_0,y_0,y_0]) \neq 1$ in $F$, which contradicts that $[x_0,y_0,y_0] = 1$ in $G$. Therefore $G$ is not 2-Engel and the required result is obtained from Theorem 2.11(ii), when we take $\chi$ to be the class of all free groups. \hfill \Box

3. The number of 2-Engelizers

As in the previous section, $E^2(G)$ denotes the set of all 2-Engelizers in the group $G$. Now for a given group $G$, one may ask about the size of $E^2(G)$. So our goal in this section is to study the possible values of $|E^2(G)|$. Note that in this section, we assume that the 2-Engelizer of each element of $G$ is a subgroup. Indeed $E^2_G(x)$ is abelian, for every non-trivial element $x$ of $G$.

One can easily check that $G$ is 2-Engel group if and only if $|E^2(G)| = 1$. Moreover, $Z_2(G) \subseteq \cap_{x \in G} E^2_G(x)$.

**Lemma 3.1.** A group $G$ is the union of 2-Engelizers of all elements of $G \setminus Z_2(G)$, that is to say $G = \cup_{x \in G \setminus Z_2(G)} E^2_G(x)$.

*Proof.* Clearly, $\cup_{x \in G \setminus Z_2(G)} E^2_G(x) \subseteq G$. By the definition, if $g \in Z_2(G)$, then $g \in E^2_G(x)$ for every $x \in G$ and hence $g \in \cup_{x \in G \setminus Z_2(G)} E^2_G(x)$. In the case $g \in G \setminus Z_2(G)$, then clearly $g \in E^2_G(g)$ and so

$$g \in \cup_{x \in G \setminus Z_2(G)} E^2_G(x).$$

Therefore $G \subseteq \cup_{x \in G \setminus Z_2(G)} E^2_G(x)$ and the proof is complete. \hfill \Box

**Lemma 3.2.** A group $G$ can not be written as the union of two proper subgroups of $G$.

*Proof.* Suppose $H$ and $K$ are two proper subgroups of $G$ such that $G = H \cup K$. Let $x \in H \setminus K$ and $y \in K \setminus H$. If $xy \in H$, then $x^{-1}xy = y \in H$, which gives a contradiction. Similarly, $xy$ can not be in $K$ and hence the claim is proved. \hfill \Box

Using the above lemmas we prove the following:

**Theorem 3.3.** Let $G$ be any group. Then $|E^2(G)| \geq 4$.

*Proof.* Using Lemma 3.1, the group $G$ is the union of its proper 2-Engelizers, i.e., $G = \cup_{x \in G \setminus Z_2(G)} E^2_G(x)$. If $|E^2(G)| = 1$, then $G$ is 2-Engel, which contradicts the assumption. If $|E^2(G)| = 2$, then $G$ is the proper subgroup of itself, which is impossible. Assume $|E^2(G)| = 3$. Then $E^2(G) = \{G, E^2_G(x), E^2_G(y)\}$, where $E^2_G(x)$ and $E^2_G(y)$ are proper 2-Engelizers of $G$. Therefore $G = E^2_G(x) \cup E^2_G(y)$, which contradicts Lemma 3.2. Hence $|E^2(G)| \geq 4$ and this completes the proof. \hfill \Box
Part (i) of the following example shows that the lower bound obtained in the above theorem is attained. Also one notes that the number of 2-Engelizers of a given group is always less than or equal to the number of centralizers.

**Example 3.4.** (i) Consider $D_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^{-1} \rangle$, the dihedral group of order 16. It can be easily calculated that all 2-Engelizers of $D_{16}$ are precisely as follows:

$$D_{16}, \quad E_{D_{16}}^2(a) = \langle a \rangle, \quad E_{D_{16}}^2(b) = \{1, a^2, a^4, a^6, b, a^2b, a^4b, a^6b\},$$

$$E_{D_{16}}^2(ab) = \{1, a^2, a^4, a^6, ab, a^3b, a^5b, a^7b\}.$$

Hence $|E_2^2(D_{16})| = 4$.

(ii) All 2-Engelizers of the symmetric group $S_3 = \langle a, b : b^3 = a^2 = 1, aba^{-1} = b^{-1} \rangle$ are as follows:

$$S_3, \quad E_{S_3}^2(a) = \{1, a\}, \quad E_{S_3}^2(b) = \{1, b, b^2\}, \quad E_{S_3}^2(ab) = \{1, ab\}, \quad E_{S_3}^2(ab^2) = \{1, ab^2\}.$$

Therefore $|E_2^2(S_3)| = 5$.

**Lemma 3.5.** Let $|E_{G/Z_2(G)}^2(xZ_2(G))| = p$ for some non second central element $x$ of a group $G$ and $p$ be an any prime number. For all $y \in G \setminus Z_2(G)$, if $E_{G/Z_2(G)}^2(xZ_2(G)) = E_{G/Z_2(G)}^2(yZ_2(G))$, then $E_{G}^2(x) = E_{G}^2(y)$.

**Proof.** Clearly,

$$E_{G}^2(x)/Z_2(G) \leq E_{G/Z_2(G)}^2(xZ_2(G)).$$

Assume that $E_{G}^2(x)/Z_2(G) < E_{G/Z_2(G)}^2(xZ_2(G))$. As $|E_{G/Z_2(G)}^2(xZ_2(G))| = p$ and $|E_{G}^2(x)/Z_2(G)|$ divides $|E_{G/Z_2(G)}^2(xZ_2(G))|$, we get $|E_{G}^2(x)/Z_2(G)| = 1$ and so $E_{G}^2(x) = Z_2(G)$. Thus $x \in Z_2(G)$ which is a contradiction. Therefore $E_{G}^2(x)/Z_2(G) = E_{G/Z_2(G)}^2(xZ_2(G))$. Clearly for all $y \in G \setminus Z_2(G)$,

$$E_{G}^2(y)/Z_2(G) \leq E_{G/Z_2(G)}^2(yZ_2(G)) = E_{G/Z_2(G)}^2(xZ_2(G)).$$

Hence $|E_{G/Z_2(G)}^2(xZ_2(G))| = |E_{G}^2(y)/Z_2(G)|$ and so

$$E_{G}^2(x)/Z_2(G) = E_{G}^2(y)/Z_2(G).$$

Thus

$$E_{G}^2(x)/Z_2(G) = E_{G}^2(y)/Z_2(G) = \{Z_2(G), x_1Z_2(G), x_2Z_2(G), \ldots, x_{p-1}Z_2(G)\},$$

where $\{x_1, \ldots, x_{p-1}\} \in E_{G}^2(x) \cap E_{G}^2(y) \setminus Z_2(G)$. So $E_{G}^2(x) = E_{G}^2(y)$. $\square$

Characterization of finite groups in terms of the number of distinct centralizers has been an interesting topic of research in recent years (see [1, 2, 4]). In [4] Belcastro and Sherman proved that $G$ is 4-centralizer if and only if $G/Z(G) \cong C_2 \times C_2$ and $G$ is 5-centralizer if and only if $G/Z(G) \cong C_3 \times C_3$ or $S_3$. Here we calculate $|E_2^2(G)|$ in the case of $G/Z_2(G) \cong C_p \times C_p$ for any prime number $p$. 

Theorem 3.6. Let $G$ be a group such that $G/Z_2(G) \cong C_p \times C_p$ for any prime number $p$. Then $|E^2(G)| = p + 2$.

Proof. Suppose that $G/Z_2(G) \cong C_p \times C_p$, and hence

$$
\frac{G}{Z_2(G)} = \langle xZ_2(G), yZ_2(G) : x^p, y^p, [x, y] \in Z_2(G) \rangle.
$$

Clearly any non-trivial proper subgroup $H/Z_2(G)$ of $G/Z_2(G)$ has order $p$. Therefore $H = Z_2(G) \cup h_1Z_2(G) \cup h_2Z_2(G) \cup \cdots \cup h_{p-1}Z_2(G)$, where $h_i \in H \setminus Z_2(G)$ for all $1 \leq i \leq p - 1$. Thus the proper subgroups of $G$ properly containing $Z_2(G)$ are one of the following forms:

- $Z_2(G) \cup xZ_2(G) \cup x^2Z_2(G) \cup \cdots \cup x^{p-1}Z_2(G)$;
- $Z_2(G) \cup yZ_2(G) \cup y^2Z_2(G) \cup \cdots \cup y^{p-1}Z_2(G)$ or
- $Z_2(G) \cup x^iy^jZ_2(G)$, where $1 \leq i, j \leq p - 1$. Note that, for all $1 \leq i, j \leq p - 1$, it is easy to see that $x^iy^jZ_2(G) = x^iy^jZ_2(G)$ since $[x, y] \in Z_2(G)$. Hence we have only $p - 1$ proper subgroups of $G$ of latest type. For simplicity, we denote all the above subgroups by $H_1, H_2, \ldots, H_{p+1}$, respectively. Now we show that $H_1, H_2, \ldots, H_{p+1}$ are the only proper 2-Engelizers of $G$. Let $a \in G \setminus Z_2(G)$, then $aZ_2(G) = bZ_2(G)$ for some $b \in \{ x, \ldots, x^{p-1}, y, \ldots, y^{p-1}, xy, xy^2, \ldots, xy^{p-1}, \ldots, x^{-1}y, \ldots, x^{-1}y^{-1} \}$.

Therefore $E_G^2(aZ_2(G)) = E_G^2(bZ_2(G))$ and Lemma 3.5 implies that $E_G^2(a) = E_G^2(b)$. Again let $b \in H_1 \setminus Z_2(G)$ then $E_G^2(b) \subseteq \cup_{j=1}^{p+1} H_j$, as $H_1, \ldots, H_{p+1}$ are the only proper subgroups of $G$. Also $b \in E_G^2(b)$, and hence $E_G^2(b) \neq H_j$, for $1 \leq i \neq j \leq p + 1$. Therefore $E_G^2(b) = H_i$, and $H_1, H_2, \ldots, H_{p+1}$ are the only proper 2-Engelizers of $G$ and so $|E^2(G)| = p + 2$. \hfill $\square$

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