Korean J. Math. **24** (2016), No. 2, pp. 273–296
http://dx.doi.org/10.11568/kjm.2016.24.2.273

SOME TRACE INEQUALITIES FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

Silvestru Sever Dragomir

Abstract. Some new trace inequalities for convex functions of selfadjoint operators in Hilbert spaces are provided. The superadditivity and monotonicity of some associated functionals are investigated. Some trace inequalities for matrices are also derived. Examples for the operator power and logarithm are presented as well.

1. Introduction

Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle ., .\rangle)$ with the spectrum $Sp(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [29, p. 257]):

$$\langle f(A) x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y\rangle), \tag{1.1}$$

Received May 24, 2016. Revised June 14, 2016. Accepted June 18, 2016.
2010 Mathematics Subject Classification: 47A63, 47A99.
Key words and phrases: Trace class operators, Hilbert-Schmidt operators, Trace, Convex functions, Jensen's inequality, Trace inequalities for matrices, Power series of operators.

and

$$\|f(A)x\|^2 = \int_{m-0}^{M} |f(\lambda)|^2 \, d\|E_\lambda x\|^2, \tag{1.2}$$

for any x, $y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ while } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$ for any $x \in H$.

The following result that provides an operator version for the Jensen inequality may be found for instance in Mond & Pečarić [40] (see also [28, p. 5]):

Theorem 1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If h is a convex function on $[m, M]$, then*

$$h(\langle Ax, x\rangle) \le \langle h(A)x, x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1 we have the following Hölder-McCarthy inequality:

Theorem 2 (Hölder-McCarthy, 1967, [38]). *Let A be a selfadjoint positive operator on a Hilbert space H. Then for all $x \in H$ with $\|x\| = 1$,*

(i) $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ *for all $r > 1$;*

(ii) $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r$ *for all $0 < r < 1$;*

(iii) If A is invertible, then $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r$ for all $r < 0$.

The following reverse for (MP) that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [28, p. 57]:

Theorem 3. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If h is a convex function on $[m, M]$, then*

$$\langle h(A)x, x\rangle \le \frac{M - \langle Ax, x\rangle}{M - m} h(m) + \frac{\langle Ax, x\rangle - m}{M - m} h(M) \tag{LR}$$

for each $x \in H$ with $\|x\| = 1$.

For some inequalities for convex functions see [8]- [12], [26] and [43]. For inequalities for functions of selfadjoint operators, see [14]- [23], [37], [39], [40], [41], [42] and the books [24], [25] and [28].

In order to state our new results concerning some trace inequalities for convex functions of selfadjoint operators on Hilbert space $(H, \langle .,. \rangle)$ we need some preparations as follows.

2. Trace of Operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{2.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{2.2}$$

showing that the definition (2.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{2.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (2.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 4. *We have*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^* Ae_i, e_i\rangle \tag{2.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{2.5}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{2.6}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, *where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H.

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{2.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$;

(ii) A *(or* $|A|$*) is the product of two elements of* $\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 5. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \leq \|A\|_1 \tag{2.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H),\ \|B\| \leq 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H) \cong K(H)^* \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),$$

where $K(H)^*$ *is the dual space of* $K(H)$ *and* $\mathcal{B}_1(H)^*$ *is the dual space of* $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{2.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (2.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 6. *We have*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{2.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{2.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any A, $B \in \mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [36].

For some classical trace inequalities see [5], [7], [33] and [47], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [5], [27], [30], [31], [32], [34] and [44].

3. Some Trace Inequalities for Convex Functions

Consider the orthonormal basis $\mathcal{E} := \{e_i\}_{i\in I}$ in the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and for a nonzero operator $B \in \mathcal{B}_2(H)$ let introduce the subset of indices from I defined by

$$I_{\mathcal{E},B} := \{i \in I : \; Be_i \neq 0\}.$$

We observe that $I_{\mathcal{E},B}$ is non-empty for any nonzero operator B and if $\ker(B) = 0$, i.e. B is injective, then $I_{\mathcal{E},B} = I$. We also have for $B \in \mathcal{B}_2(H)$ that

$$\operatorname{tr}\left(|B|^2\right) = \operatorname{tr}(B^*B) = \sum_{i\in I}\langle B^*Be_i, e_i\rangle = \sum_{i\in I}\|Be_i\|^2 = \sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2.$$

Theorem 7. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$, $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $B \in \mathcal{B}_2(H) \setminus \{0\}$, then $\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)} \in [m, M]$ and*

(3.1)
$$\begin{aligned} & f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\operatorname{tr}\left(|B|^2\right) \\ & \leq J_{\mathcal{E}}(f; A, B) \leq \operatorname{tr}\left(|B|^2 f(A)\right) \\ & \leq \frac{1}{M-m}\left(f(m)\operatorname{tr}\left[|B|^2(M1_H - A)\right] + f(M)\operatorname{tr}\left[|B|^2(A - m1_H)\right]\right), \end{aligned}$$

where

$$J_{\mathcal{E}}(f;A,B) := \sum_{i\in I_{\mathcal{E},B}} f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2. \tag{3.2}$$

Proof. Since $Sp(A) \subseteq [m, M]$, then $m\|y\|^2 \le \langle Ay, y\rangle \le M\|y\|^2$ for any $y \in H$. Therefore

$$m\|Be_i\|^2 \le \langle ABe_i, Be_i\rangle \le M\|Be_i\|^2,$$

for any $i \in I$, which implies that

$$m\sum_{i\in I}\|Be_i\|^2 \le \sum_{i\in I}\langle ABe_i, Be_i\rangle \le M\sum_{i\in I}\|Be_i\|^2$$

and we conclude that $\frac{\operatorname{tr}(|B|^2A)}{\operatorname{tr}(|B|^2)} \in [m, M]$.

By Jensen's inequality (MP) we have

$$f\left(\frac{\langle Ay, y\rangle}{\|y\|^2}\right) \le \frac{\langle f(A)y, y\rangle}{\|y\|^2} \tag{3.3}$$

for any $y \in H\setminus\{0\}$.

Let F be a finite part of $I_{\mathcal{E},B}$. Then for any $i \in F$ we have from (3.3) that

$$f\left(\frac{\langle ABe_i, Be_i\rangle}{\|Be_i\|^2}\right) \le \frac{\langle f(A)Be_i, Be_i\rangle}{\|Be_i\|^2},$$

which is equivalent to

$$f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2 \le \langle B^*f(A)Be_i, e_i\rangle. \tag{3.4}$$

Summing over $i \in F$ we get

$$\sum_{i\in F} f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2 \le \sum_{i\in F}\langle B^*f(A)Be_i, e_i\rangle. \tag{3.5}$$

Using Jensen's discrete inequality for finite sums and for the positive weights w_i

$$f\left(\frac{\sum_{i\in F} w_iu_i}{\sum_{i\in F} w_i}\right) \le \frac{\sum_{i\in F} w_if(u_i)}{\sum_{i\in F} w_i},$$

we have

$$f\left(\frac{\sum_{i\in F}\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\|Be_i\|^2}{\sum_{i\in F}\|Be_i\|^2}\right) \le \frac{\sum_{i\in F} f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2}{\sum_{i\in F}\|Be_i\|^2},$$

which is equivalent to

(3.6)
$$f\left(\frac{\sum_{i\in F}\langle B^*ABe_i,e_i\rangle}{\sum_{i\in F}\|Be_i\|^2}\right)\sum_{i\in F}\|Be_i\|^2\le\sum_{i\in F}f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2.$$

Therefore, for any F a finite part of $I_{\mathcal{E},B}$ we have from (3.5) that

(3.7)
$$\begin{aligned}f\left(\frac{\sum_{i\in F}\langle B^*ABe_i,e_i\rangle}{\sum_{i\in F}\|Be_i\|^2}\right)\sum_{i\in F}\|Be_i\|^2&\le\sum_{i\in F}f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2\\&\le\sum_{i\in F}\langle B^*f(A)Be_i,e_i\rangle.\end{aligned}$$

By the continuity of f we then have from (3.7) that

$$\begin{aligned}&f\left(\frac{\sum_{i\in I_{\mathcal{E},B}}\langle B^*ABe_i,e_i\rangle}{\sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2}\right)\sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2\\&\le\sum_{i\in I_{\mathcal{E},B}}f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2\le\sum_{i\in I_{\mathcal{E},B}}\langle B^*f(A)Be_i,e_i\rangle\end{aligned}\tag{3.8}$$

and since $B\in\mathcal{B}_2(H)\setminus\{0\}$, then also

$$\sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2=\sum_{i\in I}\|Be_i\|^2=\operatorname{tr}\left(|B|^2\right),$$

$$\sum_{i\in I_{\mathcal{E},B}}\langle B^*ABe_i,e_i\rangle=\sum_{i\in I}\langle B^*ABe_i,e_i\rangle=\operatorname{tr}\left(|B|^2A\right)$$

and

$$\sum_{i\in I_{\mathcal{E},B}}\langle B^*f(A)Be_i,e_i\rangle=\sum_{i\in I}\langle B^*f(A)Be_i,e_i\rangle=\operatorname{tr}\left(|B|^2f(A)\right).$$

From (3.8) we then get the first and the second inequality in (3.1).

From (LR) we also have

(3.9)
$$\langle f(A)y,y\rangle\le\frac{1}{M-m}\left[\langle(M1_H-A)y,y\rangle f(m)+\langle(A-m1_H)y,y\rangle f(M)\right]$$

for any $y\in H$.

This implies that

(3.10)

$$\langle f(A)Be_i, Be_i\rangle \leq \frac{1}{M-m}\left[\langle (M1_H - A)Be_i, Be_i\rangle f(m) + \langle (A - m1_H)Be_i, Be_i\rangle f(M)\right]$$

for any $i \in I$.

By summation we have

$$\sum_{i\in I}\langle f(A)Be_i, Be_i\rangle \leq \frac{1}{M-m}\left[f(m)\sum_{i\in I}\langle (M1_H - A)Be_i, Be_i\rangle + f(M)\sum_{i\in I}\langle (A - m1_H)Be_i, Be_i\rangle\right]$$

and the last part of (3.1) is proved. □

Remark 1. We observe that the quantities

$$J_s(f;A,B) = \sup_{\mathcal{E}} J_{\mathcal{E}}(f;A,B) \text{ and } J_i(f;A,B) = \inf_{\mathcal{E}} J_{\mathcal{E}}(f;A,B)$$

are finite and satisfy the bounds

$$f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\operatorname{tr}\left(|B|^2\right) \leq J_i(f;A,B) \leq J_s(f;A,B) \leq \operatorname{tr}\left(|B|^2 f(A)\right). \tag{3.11}$$

We have the following version for nonnegative operators $P \geq 0$, i.e. P satisfies the condition $\langle Px, x\rangle \geq 0$ for any $x \in H$.

Corollary 1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$, $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $P \in \mathcal{B}_1(H)\setminus\{0\}$, $P \geq 0$ then $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in [m, M]$ and*

$$\begin{aligned} f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\operatorname{tr}(P) &\leq K_{\varepsilon}(f;A,P) \leq \operatorname{tr}(Pf(A)) \\ &\leq \frac{1}{M-m}\left(f(m)\operatorname{tr}\left[P(M1_H - A)\right] + f(M)\operatorname{tr}\left[P(A - m1_H)\right]\right), \end{aligned} \tag{3.12}$$

where

$$K_{\mathcal{E}}\left(f;A,P\right):=\sum_{i\in I_{\mathcal{E},P}}f\left(\frac{\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle}{\left\langle Pe_i,e_i\right\rangle}\right)\left\langle Pe_i,e_i\right\rangle$$

and

$$I_{\mathcal{E},P}:=\left\{i\in I:\ P^{1/2}e_i\neq 0\right\}$$

Moreover, the quantities

$$K_i\left(f;A,P\right):=\inf_{\mathcal{E}}K_{\mathcal{E}}\left(f;A,P\right)\ \text{ and } K_s\left(f;A,P\right):=\sup_{\mathcal{E}}K_{\mathcal{E}}\left(f;A,P\right)$$

are finite and satisfy the bounds

(3.13)
$$f\left(\frac{\operatorname{tr}\left(PA\right)}{\operatorname{tr}\left(P\right)}\right)\operatorname{tr}\left(P\right)\leq K_i\left(f;A,P\right)\leq K_s\left(f;A,P\right)\leq\operatorname{tr}\left(Pf\left(A\right)\right).$$

The finite dimensional case is of interest.

Let $\mathcal{M}_n\left(\mathbb{C}\right)$ be the space of all square matrices of order n with complex elements.

Corollary 2. *Let $A\in\mathcal{M}_n\left(\mathbb{C}\right)$ be a Hermitian matrix and assume that $Sp\left(A\right)\subseteq\left[m,M\right]$ for some scalars m, M with $m<M$. If f is a continuous convex function on $\left[m,M\right]$, $\mathcal{E}:=\left\{e_i\right\}_{i\in\{1,\dots,n\}}$ is an orthonormal basis in $\mathbb{C}^n$, then $\frac{1}{n}\operatorname{tr}\left(A\right)\in\left[m,M\right]$ and*

(3.14)
$$\begin{aligned}nf\left(\frac{\operatorname{tr}\left(A\right)}{n}\right)&\leq J_{\mathcal{E}}\left(f;A\right)\leq\operatorname{tr}\left(f\left(A\right)\right)\\&\leq\frac{1}{M-m}\left[f\left(m\right)\operatorname{tr}\left(MI_n-A\right)+f\left(M\right)\operatorname{tr}\left(A-mI_n\right)\right],\end{aligned}$$

where

$$J_{\mathcal{E}}\left(f;A\right):=\sum_{i=1}^{n}f\left(\left\langle Ae_i,e_i\right\rangle\right),$$

and I_n is the identity matrix in $\mathcal{M}_n\left(\mathbb{C}\right)$.

Remark 2. The second inequality in (3.14), namely

$$\sum_{i=1}^{n}f\left(\left\langle Ae_i,e_i\right\rangle\right)\leq\operatorname{tr}\left(f\left(A\right)\right)$$

for any $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthonormal basis in $\mathbb{C}^n$, is known in literature as Peierls Inequality. For a different proof and some applications, see, for instance [4].

4. Some Functional Properties

If we denote by $\mathcal{B}_1^+(H)$ the convex cone of nonnegative operators from $\mathcal{B}_1(H)$ we can consider the functional $\sigma_{f,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0,\infty)$ defined by

$$\sigma_{f,A}(P) := \operatorname{tr}(Pf(A)) - \operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \geq 0, \tag{4.1}$$

where A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m,M]$ for some scalars m, M $(m < M)$ and f is a continuous convex function on $[m,M]$.

One can easily observe that, if f is a continuous strictly convex function on $[m,M]$, then the inequality is strict in (4.1).

Theorem 8. *Let A be a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m,M]$ for some scalars m, M with $m < M$ and f is a continuous convex function on $[m,M]$.*

(i) For any P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have

$$\sigma_{f,A}(P+Q) \geq \sigma_{f,A}(P) + \sigma_{f,A}(Q)\,(\geq 0), \tag{4.2}$$

i.e. $\sigma_{f,A}(\cdot)$ is a superadditive functional on $\mathcal{B}_1^+(H) \setminus \{0\}$;

(ii) For any P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$ with $P \geq Q$ we have

$$\sigma_{f,A}(P) \geq \sigma_{f,A}(Q)\,(\geq 0), \tag{4.3}$$

i.e. $\sigma_{f,A}(\cdot)$ is a monotonic nondecreasing functional on $\mathcal{B}_1^+(H) \setminus \{0\}$;

(iii) If there exists the real numbers γ, $\Gamma > 0$ such that $\Gamma Q \geq P \geq \gamma Q$ with P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$\Gamma\sigma_{f,A}(Q) \geq \sigma_{f,A}(P) \geq \gamma\sigma_{f,A}(Q)\,(\geq 0). \tag{4.4}$$

Proof. (i) Let $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$. Then we have

$$\begin{aligned}\sigma_{f,A}(P+Q) &= \operatorname{tr}((P+Q)f(A)) - \operatorname{tr}(P+Q)\, f\left(\frac{\operatorname{tr}((P+Q)A)}{\operatorname{tr}(P+Q)}\right) \\ &= \operatorname{tr}(Pf(A)) + \operatorname{tr}(Pf(A)) \\ &\quad - [\operatorname{tr}(P) + \operatorname{tr}(Q)]\, f\left(\frac{\operatorname{tr}(PA) + \operatorname{tr}(QA)}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right).\end{aligned} \tag{4.5}$$

By the convexity of f we have

$$\begin{aligned} f\left(\frac{\operatorname{tr}(PA) + \operatorname{tr}(QA)}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right) &= f\left(\frac{\operatorname{tr}(P)\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \operatorname{tr}(Q)\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right) \\ &\le \frac{\operatorname{tr}(P)\, f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) + \operatorname{tr}(Q)\, f\left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right)}{\operatorname{tr}(P) + \operatorname{tr}(Q)}.\end{aligned} \tag{4.6}$$

Making use of (4.5) and (4.6) we have

$$\begin{aligned}\sigma_{f,A}(P+Q) &\ge \operatorname{tr}(Pf(A)) + \operatorname{tr}(Pf(A)) \\ &\quad - [\operatorname{tr}(P) + \operatorname{tr}(Q)]\, \frac{\operatorname{tr}(P)\, f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) + \operatorname{tr}(Q)\, f\left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right)}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \\ &= \operatorname{tr}(Pf(A)) + \operatorname{tr}(Pf(A)) \\ &\quad - \operatorname{tr}(P)\, f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \operatorname{tr}(Q)\, f\left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right) \\ &= \sigma_{f,A}(P) + \sigma_{f,A}(Q)\end{aligned}$$

and the inequality (4.2) is proved.

(ii) Let $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$ with $P \ge Q$. Then on applying the superadditivity property of $\sigma_{f,A}$ for $P - Q \ge 0$ and $Q \ge 0$ we have

$$\sigma_{f,A}(P) = \sigma_{f,A}(P - Q + Q) \ge \sigma_{f,A}(P - Q) + \sigma_{f,A}(Q) \ge \sigma_{f,A}(Q)$$

and the inequality (4.3) is proved.

(iii) If $P \ge \gamma Q$, then by the monotonicity property of $\sigma_{f,A}$ we have

$$\sigma_{f,A}(P) \ge \sigma_{f,A}(\gamma Q) = \gamma \sigma_{f,A}(Q)$$

and a similar inequality for Γ. □

We have the following particular case of interest:

Corollary 3. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$, there exists the real numbers γ, $\Gamma > 0$ such that $\Gamma I_n \geq P \geq \gamma I_n$ with P positive definite, where I_n is the identity matrix, then*

$$\Gamma\left[\operatorname{tr}(f(A)) - nf\left(\frac{\operatorname{tr}(A)}{n}\right)\right] \geq \operatorname{tr}(Pf(A)) - \operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \geq \gamma\left[\operatorname{tr}(f(A)) - nf\left(\frac{\operatorname{tr}(A)}{n}\right)\right] (\geq 0). \tag{4.7}$$

The following result also holds:

Theorem 9. *Let A be a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$ and f is a continuous convex function on $[m, M]$. For $p \geq 1$, the functional $\psi_{p,f,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0, \infty)$ defined by*

$$\psi_{p,f,A}(P) := [\operatorname{tr}(P)]^{1-\frac{1}{p}} \sigma_{f,A}(P)$$

is superadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. First of all we observe that the following elementary inequality holds:

$$(\alpha + \beta)^p \geq (\leq) \alpha^p + \beta^p \tag{4.8}$$

for any $\alpha, \beta \geq 0$ and $p \geq 1$ $(0 < p < 1)$.

Indeed, if we consider the function $f_p : [0, \infty) \to \mathbb{R}$, $f_p(t) = (t+1)^p - t^p$ we have $f_p'(t) = p\left[(t+1)^{p-1} - t^{p-1}\right]$. Observe that for $p > 1$ and $t > 0$ we have that $f_p'(t) > 0$ showing that f_p is strictly increasing on the interval $[0, \infty)$. Now for $t = \frac{\alpha}{\beta}$ $(\beta > 0, \alpha \geq 0)$ we have $f_p(t) > f_p(0)$ giving that $\left(\frac{\alpha}{\beta} + 1\right)^p - \left(\frac{\alpha}{\beta}\right)^p > 1$, i.e., the desired inequality (4.8).

For $p \in (0, 1)$ we have that f_p is strictly decreasing on $[0, \infty)$ which proves the second case in (4.8).

Now, since $\sigma_{f,A}(\cdot)$ is superadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$ and $p \geq 1$ then by (4.8) we have

$$\sigma_{f,A}^p(P + Q) \geq [\sigma_{f,A}(P) + \sigma_{f,A}(Q)]^p \geq \sigma_{f,A}^p(P) + \sigma_{f,A}^p(Q) \tag{4.9}$$

for any P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Utilising (4.9) and the additivity property of $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1^+(H)\setminus\{0\}$ we have

$$\begin{aligned}
(4.10)\qquad \frac{\sigma_{f,A}^p(P+Q)}{\operatorname{tr}(P+Q)} &\geq \frac{\sigma_{f,A}^p(P)+\sigma_{f,A}^p(Q)}{\operatorname{tr}(P)+\operatorname{tr}(Q)} \\
&= \frac{\operatorname{tr}(P)\frac{\sigma_{f,A}^p(P)}{\operatorname{tr}(P)}+\operatorname{tr}(Q)\frac{\sigma_{f,A}^p(Q)}{\operatorname{tr}(Q)}}{\operatorname{tr}(P)+\operatorname{tr}(Q)} \\
&= \frac{\operatorname{tr}(P)\left(\frac{\sigma_{f,A}(P)}{\operatorname{tr}^{1/p}(P)}\right)^p+\operatorname{tr}(Q)\left(\frac{\sigma_{f,A}(Q)}{\operatorname{tr}^{1/q}(Q)}\right)^p}{\operatorname{tr}(P)+\operatorname{tr}(Q)} =: I,
\end{aligned}$$

for any $P,Q\in\mathcal{B}_1^+(H)\setminus\{0\}$.

Since for $p\geq 1$ the power function $g(t)=t^p$ is convex, then

$$\begin{aligned}
(4.11)\qquad I &\geq \left(\frac{\operatorname{tr}(P)\frac{\sigma_{f,A}(P)}{\operatorname{tr}^{1/p}(P)}+\operatorname{tr}(Q)\frac{\sigma_{f,A}(Q)}{\operatorname{tr}^{1/q}(Q)}}{\operatorname{tr}(P)+\operatorname{tr}(Q)}\right)^p \\
&= \left(\frac{\operatorname{tr}^{1-1/p}(P)\,\sigma_{f,A}(P)+\operatorname{tr}^{1-1/q}(Q)\,\sigma_{f,A}(Q)}{\operatorname{tr}(P+Q)}\right)^p
\end{aligned}$$

for any $P,\ Q\in\mathcal{B}_1^+(H)\setminus\{0\}$.

By combining (4.10) with (4.11) we get

$$\frac{\sigma_{f,A}^p(P+Q)}{\operatorname{tr}(P+Q)} \geq \left(\frac{\operatorname{tr}^{1-1/p}(P)\,\sigma_{f,A}(P)+\operatorname{tr}^{1-1/q}(Q)\,\sigma_{f,A}(Q)}{\operatorname{tr}(P+Q)}\right)^p,$$

which is equivalent to

$$(4.12)\qquad \frac{\sigma_{f,A}(P+Q)}{\operatorname{tr}^{1/p}(P+Q)} \geq \frac{\operatorname{tr}^{1-1/p}(P)\,\sigma_{f,A}(P)+\operatorname{tr}^{1-1/q}(Q)\,\sigma_{f,A}(Q)}{\operatorname{tr}(P+Q)},$$

for any $P,\ Q\in\mathcal{B}_1^+(H)\setminus\{0\}$.

Finally, if we multiply (4.12) by $\operatorname{tr}(P+Q)>0$ we get

$$\psi_{p,f,A}(P+Q)\geq\psi_{p,f,A}(P)+\psi_{p,f,A}(Q)$$

for any $P,\ Q\in\mathcal{B}_1^+(H)\setminus\{0\}$ and the proof is complete. □

Corollary 4. *With the assumptions of Theorem 9, the two parameters $p,\ q\geq 1$ functional $\psi_{p,q,f,A}:\mathcal{B}_1^+(H)\setminus\{0\}\to[0,\infty)$ defined by*

$$\psi_{p,q,f,A}(P):=[\operatorname{tr}(P)]^{q\left(1-\frac{1}{p}\right)}\sigma_{f,A}^q(P)$$

is superadditive on $\mathcal{B}_1^+(H)\setminus\{0\}$.

Proof. Observe that $\psi_{p,q,f,A}(P) = [\psi_{p,f,A}(P)]^q$ for $P \in \mathcal{B}_1^+(H) \setminus \{0\}$. Therefore, by Theorem 9 and the inequality (4.8) for $q \geq 1$ we have that

$$\begin{aligned}\psi_{p,q,f,A}(P+Q) &= [\psi_{p,f,A}(P+Q)]^q \\ &\geq [\psi_{p,f,A}(P) + \psi_{p,f,A}(Q)]^q \\ &\geq [\psi_{p,f,A}(P)]^q + [\psi_{p,f,A}(Q)]^q \\ &= \psi_{p,q,f,A}(P) + \psi_{p,q,f,A}(Q)\end{aligned}$$

for any P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$ and the statement is proved. □

REMARK 3. If we consider the functional

$$\tilde{\psi}_{p,f,A}(P) := [\mathrm{tr}(P)]^{p-1} \sigma_{f,A}^p(P)$$

then, for $p \geq 1$, $\tilde{\psi}_{p,f,A}(\cdot)$ is superadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

COROLLARY 5. *With the assumptions of Theorem 9 and for parameter $p \geq 1$, if there exists the real numbers γ, $\Gamma > 0$ such that $\Gamma Q \geq P \geq \gamma Q$ with $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$, then*

$$\begin{aligned}\Gamma^{2-\frac{1}{p}} [\mathrm{tr}(Q)]^{1-\frac{1}{p}} \sigma_{f,A}(Q) &\geq [\mathrm{tr}(P)]^{1-\frac{1}{p}} \sigma_{f,A}(P) \\ &\geq \gamma^{2-\frac{1}{p}} [\mathrm{tr}(Q)]^{1-\frac{1}{p}} \sigma_{f,A}(Q) \, (\geq 0).\end{aligned} \tag{4.13}$$

The case of finite-dimensional spaces is as follows:

COROLLARY 6. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$, there exists the real numbers γ, $\Gamma > 0$ such that $\Gamma I_n \geq P \geq \gamma I_n$ with P positive definite, then*

$$\begin{aligned}&\Gamma^{2-\frac{1}{p}} n^{1-\frac{1}{p}} \left[\mathrm{tr}(f(A)) - nf\left(\frac{\mathrm{tr}(A)}{n}\right)\right] \\ &\geq [\mathrm{tr}(P)]^{1-\frac{1}{p}} \left[\mathrm{tr}(Pf(A)) - \mathrm{tr}(P) f\left(\frac{\mathrm{tr}(PA)}{\mathrm{tr}(P)}\right)\right] \\ &\geq \gamma^{2-\frac{1}{p}} n^{1-\frac{1}{p}} \left[\mathrm{tr}(f(A)) - nf\left(\frac{\mathrm{tr}(A)}{n}\right)\right] (\geq 0)\end{aligned} \tag{4.14}$$

for any $p \geq 1$.

The following result also holds:

THEOREM 10. *Let A be a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$ and f*

is a continuous strictly convex function on $[m, M]$. *For* $p \in (0, 1)$, *the functional* $\chi_{p,f,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0, \infty)$ *defined by*

$$\chi_{p,f,A}(P) := \frac{[\operatorname{tr}(P)]^{1-\frac{1}{p}}}{\sigma_{f,A}(P)}$$

is subadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. Let $s := -p \in (-1, 0)$. For $s < 0$ we have the following inequality

$$(\alpha + \beta)^s \leq \alpha^s + \beta^s \tag{4.15}$$

for any α, $\beta > 0$.

Indeed, by the convexity of the function $f_s(t) = t^s$ on $(0, \infty)$ with $s < 0$ we have that

$$(\alpha + \beta)^s \leq 2^{s-1}(\alpha^s + \beta^s)$$

for any α, $\beta > 0$ and since, obviously, $2^{s-1}(\alpha^s + \beta^s) \leq \alpha^s + \beta^s$, then (4.15) holds true.

Taking into account that $\sigma_{f,A}(\cdot)$ is superadditive and $s \in (-1, 0)$ we have

$$\sigma_{f,A}^s(P + Q) \leq [\sigma_{f,A}(P) + \sigma_{f,A}(Q)]^s \leq \sigma_{f,A}^s(P) + \sigma_{f,A}^s(Q) \tag{4.16}$$

for any P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Since $\operatorname{tr}(\cdot)$ is additive on $\mathcal{B}_1^+(H) \setminus \{0\}$, then by (4.17) we have

$$\begin{aligned} \frac{\sigma_{f,A}^s(P+Q)}{\operatorname{tr}(P+Q)} &\leq \frac{\sigma_{f,A}^s(P) + \sigma_{f,A}^s(Q)}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \\ &= \frac{\operatorname{tr}(P)\left(\frac{\sigma_{f,A}(P)}{\operatorname{tr}^{1/s}(P)}\right)^s + \operatorname{tr}(Q)\left(\frac{\sigma_{f,A}(Q)}{\operatorname{tr}^{1/s}(Q)}\right)^s}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \\ &= \frac{\operatorname{tr}(P)\left(\frac{\operatorname{tr}^{1/s}(P)}{\sigma_{f,A}(P)}\right)^{-s} + \operatorname{tr}(Q)\left(\frac{\operatorname{tr}^{1/s}(Q)}{\sigma_{f,A}(Q)}\right)^{-s}}{\operatorname{tr}(P) + \operatorname{tr}(Q)} =: J \end{aligned} \tag{4.17}$$

for any P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

By the concavity of the function $g(t) = t^{-s}$ with $s \in (-1, 0)$ we also have

$$J \leq \left[\frac{\operatorname{tr}(P)\frac{\operatorname{tr}^{1/s}(P)}{\sigma_{f,A}(P)} + \operatorname{tr}(Q)\frac{\operatorname{tr}^{1/s}(Q)}{\sigma_{f,A}(Q)}}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right]^{-s} \tag{4.18}$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Making use of (4.17) and (4.18) we get

$$\frac{\sigma_{f,A}^s(P+Q)}{\operatorname{tr}(P+Q)} \leq \left[\frac{\operatorname{tr}(P)\frac{\operatorname{tr}^{1/s}(P)}{\sigma_{f,A}(P)} + \operatorname{tr}(Q)\frac{\operatorname{tr}^{1/s}(Q)}{\sigma_{f,A}(Q)}}{\operatorname{tr}(P)+\operatorname{tr}(Q)}\right]^{-s}$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$, and by taking the power $-1/s > 0$ we get

$$\frac{\sigma_{f,A}^{-1}(P+Q)}{\operatorname{tr}^{-1/s}(P+Q)} \leq \frac{\frac{\operatorname{tr}^{1+1/s}(P)}{\sigma_{f,A}(P)} + \frac{\operatorname{tr}^{1+1/s}(Q)}{\sigma_{f,A}(Q)}}{\operatorname{tr}(P)+\operatorname{tr}(Q)},$$

which is equivalent to

$$\frac{\operatorname{tr}^{1+1/s}(P+Q)}{\sigma_{f,A}(P+Q)} \leq \frac{\operatorname{tr}^{1+1/s}(P)}{\sigma_{f,A}(P)} + \frac{\operatorname{tr}^{1+1/s}(Q)}{\sigma_{f,A}(Q)}$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

This completes the proof. □

The following result may be stated as well:

Corollary 7. *With the assumptions of Theorem 10, the two parameters* $0 < p, q < 1$ *functional* $\chi_{p,q,f,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0, \infty)$ *defined by*

$$\chi_{p,q,f,A}(P) = \frac{\operatorname{tr}^{q\left(1-\frac{1}{p}\right)}(P)}{\sigma_{f,A}^q(P)}$$

is subadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

Remark 4. *If we consider the functional* $\tilde{\chi}_{p,f,A}(P) = \frac{\operatorname{tr}^{p-1}(P)}{\sigma_{f,A}^p(P)}$ *for* $0 < p < 1$, *then* $\tilde{\chi}_{p,f,A}(\cdot)$ *is also subadditive on* $\mathcal{B}_1^+(H) \setminus \{0\}$.

5. Some Examples

We consider the power function $f : (0, \infty) \to (0, \infty)$, $f(t) = t^r$ with $t \in \mathbb{R} \setminus \{0\}$. For $r \in (-\infty, 0) \cup [1, \infty)$, f is convex while for $r \in (0, 1)$, f is concave.

Let $r \geq 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 \leq m < M$.

If $\mathcal{E} := \{e_i\}_{i \in I}$ is an orthonormal basis in H and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ then

$$
\begin{aligned}
(5.1) \qquad & [\operatorname{tr}(PA)]^r [\operatorname{tr}(P)]^{1-r} \\
& \le K_{\varepsilon}(r; A, P) \le \operatorname{tr}(PA^r) \\
& \le \frac{1}{M-m} \left(m^r \operatorname{tr}[P(M1_H - A)] + M^r \operatorname{tr}[P(A - m1_H)] \right),
\end{aligned}
$$

where

$$K_{\varepsilon}(r; A, P) := \sum_{i \in I_{\mathcal{E},P}} \left\langle P^{1/2} A P^{1/2} e_i, e_i \right\rangle^r \langle P e_i, e_i \rangle^{1-r}.$$

Moreover, the quantities

$$K_i(r; A, P) := \inf_{\varepsilon} K_{\varepsilon}(r; A, P) \text{ and } K_s(r; A, P) := \sup_{\varepsilon} K_{\varepsilon}(r; A, P)$$

are finite and satisfy the bounds

$$(5.2) \qquad [\operatorname{tr}(PA)]^r [\operatorname{tr}(P)]^{1-r} \le K_i(r; A, P) \le K_s(r; A, P) \le \operatorname{tr}(PA^r).$$

Now, if we take $A = P$, $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then by (5.1) we have

$$(5.3) \qquad \left[\operatorname{tr}\left(P^2\right)\right]^r [\operatorname{tr}(P)]^{1-r} \le K_{\varepsilon}(r; P) \le \operatorname{tr}\left(P^{r+1}\right)$$

where

$$K_{\varepsilon}(r; P) := \sum_{i \in I_{\mathcal{E},P}} \left\langle P^2 e_i, e_i \right\rangle^r \langle P e_i, e_i \rangle^{1-r}.$$

If we consider the functional $\sigma_{r,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0, \infty)$ defined by

$$(5.4) \qquad \sigma_{r,A}(P) := \operatorname{tr}(PA^r) - [\operatorname{tr}(PA)]^r [\operatorname{tr}(P)]^{1-r} \ge 0,$$

where A is a selfadjoint operator on the Hilbert space H with $Sp(A) \subseteq [m, M] \subset [0, \infty)$, then $\sigma_{r,A}(\cdot)$ is superadditive, monotonic nondecreasing and if there exists the real numbers γ, $\Gamma > 0$ such that $\Gamma Q \ge P \ge \gamma Q$ with P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$(5.5) \qquad \Gamma \sigma_{r,A}(Q) \ge \sigma_{r,A}(P) \ge \gamma \sigma_{r,A}(Q) \, (\ge 0).$$

Consider the convex function $f : (0, \infty) \to (0, \infty)$, $f(t) = -\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $\mathcal{E} := \{e_i\}_{i \in I}$

is an orthonormal basis in H and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ then

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} \geq L_{\mathcal{E}}(A,P) \geq \exp[\operatorname{tr}(P\ln A)] \geq m^{\frac{\operatorname{tr}[P(M1_H-A)]}{M-m}} M^{\frac{\operatorname{tr}[P(A-m1_H)]}{M-m}}, \tag{5.6}$$

where

$$L_{\mathcal{E}}(A,P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\langle P^{1/2}AP^{1/2}e_i, e_i\rangle}{\langle Pe_i, e_i\rangle}\right)^{\langle Pe_i, e_i\rangle}.$$

Moreover, the quantities

$$L_i(A,P) := \inf_{\mathcal{E}} L_{\mathcal{E}}(A,P) \text{ and } L_s(A,P) := \sup_{\mathcal{E}} L_{\mathcal{E}}(A,P)$$

are finite and satisfy the bounds

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} \geq L_s(A,P) \geq L_i(A,P) \geq \exp[\operatorname{tr}(P\ln A)]. \tag{5.7}$$

Now, if we take $A = P$, $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then by (5.6) we get

$$\left(\frac{\operatorname{tr}(P^2)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} \geq L_{\mathcal{E}}(P) \geq \exp[\operatorname{tr}(P\ln P)] \tag{5.8}$$

where

$$L_{\mathcal{E}}(P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\langle P^2e_i, e_i\rangle}{\langle Pe_i, e_i\rangle}\right)^{\langle Pe_i, e_i\rangle}.$$

Consider the functional $\delta_A : \mathcal{B}_1^+(H) \setminus \{0\} \to (0,\infty)$ defined by

$$\delta_A(P) := \frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)}}{\exp(\operatorname{tr}(P\ln A))} \geq 1,$$

where A is a selfadjoint operator on the Hilbert space H and such that $Sp(A) \subseteq [m,M]$ for some scalars m, M with $0 < m < M$.

Observe that

$$\sigma_{-\ln,A}(P) := \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} - \ln[\exp(\operatorname{tr}(P\ln A))] = \ln[\delta_A(P)]$$

for $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Utilising the properties of $\sigma_{-\ln,A}(\cdot)$ we conclude that $\delta_A(\cdot)$ is supermultiplicative, i.e.

$$\delta_A(P+Q) \geq \delta_A(P)\,\delta_A(Q) \geq 1$$

for any $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$. The functional $\delta_A(\cdot)$ is also monotonic nondecreasing on $\mathcal{B}_1(H) \setminus \{0\}$.

Consider the convex function $f(t) = t\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ then

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)} \leq I_{\mathcal{E}}(A,P) \leq \exp\left[\operatorname{tr}(PA\ln A)\right] \leq m^{\frac{m\operatorname{tr}[P(M1_H-A)]}{M-m}} M^{\frac{M\operatorname{tr}[P(A-m1_H)]}{M-m}}, \tag{5.9}$$

where

$$I_{\mathcal{E}}(A,P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\langle P^{1/2}AP^{1/2}e_i, e_i\rangle}{\langle Pe_i, e_i\rangle}\right)^{\langle P^{1/2}AP^{1/2}e_i, e_i\rangle}.$$

Moreover, the quantities

$$I_i(A,P) := \inf_{\mathcal{E}} I_{\mathcal{E}}(A,P) \text{ and } I_s(A,P) := \sup_{\mathcal{E}} I_{\mathcal{E}}(A,P)$$

are finite and satisfy the bounds

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)} \leq I_i(A,P) \leq I_s(A,P) \leq \exp\left[\operatorname{tr}(PA\ln A)\right]. \tag{5.10}$$

Now, if we take $A = P$, $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then by (5.9) we get

$$\left(\frac{\operatorname{tr}(P^2)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P^2)} \leq I_{\mathcal{E}}(P) \leq \exp\left[\operatorname{tr}\left(P^2\ln P\right)\right] \tag{5.11}$$

where

$$I_{\mathcal{E}}(P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\langle P^2e_i, e_i\rangle}{\langle Pe_i, e_i\rangle}\right)^{\langle P^2e_i, e_i\rangle}.$$

Observe that for $f(t) = t \ln t$ we have

$$\begin{aligned} \sigma_{(\cdot)\ln(\cdot),A}(P) &= \operatorname{tr}(PA\ln A) - \operatorname{tr}(PA)\ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\ &= \ln\left[\frac{\exp\left[\operatorname{tr}(PA\ln A)\right]}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)}}\right] \end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Consider the functional $\lambda_A : \mathcal{B}_1^+(H) \setminus \{0\} \to (0, \infty)$ defined by

$$\lambda_A(P) := \frac{\exp\left[\operatorname{tr}(PA\ln A)\right]}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)}} \geq 1.$$

Utilising the properties of $\sigma_{(\cdot)\ln(\cdot),A}(\cdot)$ we can conclude that $\lambda_A(\cdot)$ is supermultiplicative and monotonic nondecreasing on $\mathcal{B}_1^+(H) \setminus \{0\}$.

References

[1] T. Ando, *Matrix Young inequalities*, Oper. Theory Adv. Appl. **75** (1995), 33–38.

[2] R. Bellman, *Some inequalities for positive definite matrices, in: E.F. Beckenbach (Ed.)*, General Inequalities 2, Proceedings of the 2nd International Conference on General Inequalities, Birkhäuser, Basel, 1980, pp. 89–90.

[3] E. V. Belmega, M. Jungers and S. Lasaulce, *A generalization of a trace inequality for positive definite matrices*, Aust. J. Math. Anal. Appl. **7** (2) (2010), Art. 26, 5 pp.

[4] E. A. Carlen, *Trace inequalities and quantum entropy: an introductory course*, Entropy and the quantum, 73–140, Contemp. Math., 529, Amer. Math. Soc., Providence, RI, 2010.

[5] D. Chang, *A matrix trace inequality for products of Hermitian matrices*, J. Math. Anal. Appl. **237** (1999), 721–725.

[6] L. Chen and C. Wong, *Inequalities for singular values and traces*, Linear Algebra Appl. **171** (1992), 109–120.

[7] I. D. Coop, *On matrix trace inequalities and related topics for products of Hermitian matrix*, J. Math. Anal. Appl. **188** (1994), 999–1001.

[8] S. S. Dragomir, *A converse result for Jensen's discrete inequality via Gruss' inequality and applications in information theory*, An. Univ. Oradea Fasc. Mat. **7** (1999/2000), 178–189.

[9] S. S. Dragomir, *On a reverse of Jessen's inequality for isotonic linear functionals*, J. Ineq. Pure & Appl. Math., **2** (3) (2001), Article 36.

[10] S. S. Dragomir, *A Grüss type inequality for isotonic linear functionals and applications*, Demonstratio Math. **36** (3) (2003), 551–562. Preprint RGMIA *Res. Rep. Coll.* **5**(2002), Suplement, Art. 12. [ONLINE:http://rgmia.org/v5(E).php].

[11] S. S. Dragomir, *Bounds for the normalized Jensen functional*, Bull. Austral. Math. Soc. **74** (3) (2006), 471–476.

[12] S. S. Dragomir, *Bounds for the deviation of a function from the chord generated by its extremities*, Bull. Aust. Math. Soc. **78** (2) (2008), 225–248.

[13] S. S. Dragomir, *Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces*, Preprint, RGMIA Res. Rep. Coll., **11**(e) (2008), Art. 11. [ONLINE: http://rgmia.org/v11(E).php].

[14] S. S. Dragomir, *Some inequalities for convex functions of selfadjoint operators in Hilbert spaces*, Filomat **23** (3) (2009), 81–92. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 10.

[15] S. S. Dragomir, *Some Jensen's type inequalities for twice differentiable functions of selfadjoint operators in Hilbert spaces*, Filomat **23** (3) (2009), 211–222. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 13.

[16] S. S. Dragomir, *Some new Grüss' type inequalities for functions of selfadjoint operators in Hilbert spaces*, Sarajevo J. Math. **6(18)**, (1) (2010), 89–107. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 12. [ONLINE: http://rgmia.org/v11(E).php].

[17] S. S. Dragomir, *New bounds for the Čebyšev functional of two functions of selfadjoint operators in Hilbert spaces*, *Filomat* **24** (2) (2010), 27–39.

[18] S. S. Dragomir, *Some Jensen's type inequalities for log-convex functions of selfadjoint operators in Hilbert spaces*, Bull. Malays. Math. Sci. Soc. **34** (3) (2011), Preprint *RGMIA Res. Rep. Coll.*, **13**(2010), Sup. Art. 2.

[19] S. S. Dragomir, *Some reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces*, J. Ineq. & Appl., Vol. **2010**, Article ID 496821. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 15. [ONLINE: http://rgmia.org/v11(E).php].

[20] S. S. Dragomir, *Some Slater's type inequalities for convex functions of selfadjoint operators in Hilbert spaces*, Rev. Un. Mat. Argentina, **52** (1) (2011), 109–120. Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 7.

[21] S. S. Dragomir, *Hermite-Hadamard's type inequalities for operator convex functions*, Appl. Math. Comp. **218** (2011), 766–772. Preprint *RGMIA Res. Rep. Coll.*, **13**(2010), No. 1, Art. 7.

[22] S. S. Dragomir, *Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces*, Preprint *RGMIA Res. Rep. Coll.*, **13** (2) (2010), Art 1.

[23] S. S. Dragomir, *New Jensen's type inequalities for differentiable log-convex functions of selfadjoint operators in Hilbert spaces*, Sarajevo J. Math. **19** (1) (2011), 67–80. Preprint *RGMIA Res. Rep. Coll.*, **13**(2010), Sup. Art. 2.

[24] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* Springer Briefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[25] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[26] S. S. Dragomir and N. M. Ionescu, *Some converse of Jensen's inequality and applications*, Rev. Anal. Numér. Théor. Approx. **23** (1) (1994), 71–78. MR:1325895 (96c:26012).
[27] S. Furuichi and M. Lin, *Refinements of the trace inequality of Belmega, Lasaulce and Debbah*, *Aust. J. Math. Anal. Appl.* **7** (2) (2010), Art. 23, 4 pp.
[28] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[29] G. Helmberg,*Introduction to Spectral Theory in Hilbert Space,* John Wiley, New York, 1969.
[30] H. D. Lee, *On some matrix inequalities*, Korean J. Math. **16** (2) (2008), 565–571.
[31] L. Liu, *A trace class operator inequality*, J. Math. Anal. Appl. **328** (2007), 1484–1486.
[32] S. Manjegani, *Hölder and Young inequalities for the trace of operators*, Positivity **11** (2007), 239–250.
[33] H. Neudecker, *A matrix trace inequality*, J. Math. Anal. Appl. **166** (1992), 302–303.
[34] K. Shebrawi and H. Albadawi, *Operator norm inequalities of Minkowski type*, J. Inequal. Pure Appl. Math. **9** (1) (2008), 1–10, article 26.
[35] K. Shebrawi and H. Albadawi, *Trace inequalities for matrices*, *Bull. Aust. Math. Soc.* **87** (2013), 139–148.
[36] B. Simon, *Trace Ideals and Their Applications,* Cambridge University Press, Cambridge, 1979.
[37] A. Matković, J. Pečarić and I. Perić, *A variant of Jensen's inequality of Mercer's type for operators with applications.*, Linear Algebra Appl. **418** (2006), no. 2-3, 551–564.
[38] C. A. McCarthy, c_p, Israel J. Math., **5** (1967), 249–271.
[39] J. Mićić, Y. Seo, S.-E. Takahasi and M. Tominaga, *Inequalities of Furuta and Mond-Pečarić*, Math. Ineq. Appl., **2**(1999), 83–111.
[40] B. Mond and J. Pečarić, *Convex inequalities in Hilbert space*, Houston J. Math., **19** (1993), 405–420.
[41] B. Mond and J. Pečarić, *On some operator inequalities*, Indian J. Math. **35** (1993), 221–232.
[42] B. Mond and J. Pečarić, *Classical inequalities for matrix functions*, Utilitas Math., **46** (1994), 155–166.
[43] S. Simić, *On a global upper bound for Jensen's inequality*, *J. Math. Anal. Appl.* **343** (2008), 414–419.
[44] Z. Ulukök and R. Türkmen, *On some matrix trace inequalities. J. Inequal. Appl.* **2010**, Art. ID 201486, 8 pp.
[45] X. Yang, *A matrix trace inequality*, J. Math. Anal. Appl. **250** (2000), 372–374.
[46] X. M. Yang, X. Q. Yang and K. L. Teo, *A matrix trace inequality*, J. Math. Anal. Appl. **263** (2001), 327–331.
[47] Y. Yang, *A matrix trace inequality*, J. Math. Anal. Appl. **133** (1988), 573–574.

Silvestru Sever Dragomir
Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia.

School of Computer Science & Applied Mathematics
University of the Witwatersrand
Johannesburg 2050, South Africa
E-mail: sever.dragomir@vu.edu.au