ADDITIVE $\rho$-FUNCTIONAL INEQUALITIES

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Abstract. In this paper, we solve the additive $\rho$-functional inequalities

\[ \|f(x+y) + f(x-y) - 2f(x)\| \leq \|\rho \left( 2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x) \right)\|, \]  

(0.1)

where $\rho$ is a fixed complex number with $|\rho| < 1$, and

\[ \|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\| \leq \|\rho(f(x+y) + f(x-y) - 2f(x))\|, \]

(0.2)

where $\rho$ is a fixed complex number with $|\rho| < 1$.

Furthermore, we prove the Hyers-Ulam stability of the additive $\rho$-functional inequalities (0.1) and (0.2) in complex Banach spaces.

1. Introduction and Preliminaries


The functional equation $f(x+y) = f(x) + f(y)$ is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [6] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [1] for additive mappings and by Rassias [8] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.

The stability of quadratic functional equation was proved by Skof [10] for mappings $f : E_1 \rightarrow E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group. See [2, 4, 7, 9, 12] for more information on the stability problems of functional equations.

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In Section 2, we solve the additive $\rho$-functional inequality (0.1) and prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (0.1) in complex Banach spaces.

In Section 3, we solve the additive $\rho$-functional inequality (0.2) and prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (0.2) in complex Banach spaces.

Throughout this paper, let $G$ be a 2-divisible abelian group. Assume that $X$ is a real or complex normed space with norm $\| \cdot \|$ and that $Y$ is a complex Banach space with norm $\| \cdot \|$.

2. ADDITIVE $\rho$-FUNCTIONAL INEQUALITY (0.1)

Throughout this section, assume that $\rho$ is a fixed complex number with $|\rho| < 1$.

In this section, we solve and investigate the additive $\rho$-functional inequality (0.1) in complex Banach spaces.

Lemma 2.1. If a mapping $f : G \to Y$ satisfies $f(0) = 0$ and

\[
(2.1) \quad \|f(x + y) + f(x - y) - 2f(x)\| \leq \left\| \rho \left( 2f \left( \frac{x + y}{2} \right) + f (x - y) - 2f(x) \right) \right\|
\]

for all $x, y \in G$, then $f : G \to Y$ is additive.

Proof. Assume that $f : G \to Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $\|f(2x) - 2f(x)\| \leq 0$ and so $f(2x) = 2f(x)$ for all $x \in G$. Thus

\[
(2.2) \quad f \left( \frac{x}{2} \right) = \frac{1}{2}f(x)
\]

for all $x \in G$.

It follows from (2.1) and (2.2) that

\[
\|f(x + y) + f(x - y) - 2f(x)\| \leq \left\| \rho \left( 2f \left( \frac{x + y}{2} \right) + f (x - y) - 2f(x) \right) \right\|
\]

\[
= \|\rho\|\|f(x + y) + f(x - y) - 2f(x)\|
\]

and so $f(x + y) + f(x - y) = 2f(x)$ for all $x, y \in G$. It is easy to show that $f$ is additive.

We prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (2.1) in complex Banach spaces.
Theorem 2.2. Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$
\|f(x + y) + f(x - y) - 2f(x)\| \\
\leq \|\rho \left( 2f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) \right)\| + \theta (\|x\|^r + \|y\|^r)
$$

for all $x, y \in X$. Then there exists a unique additive mapping $h : X \to Y$ such that

$$
\|f(x) - h(x)\| \leq \frac{2\theta}{2^r - 2}\|x\|^r
$$

for all $x \in X$.

Proof. Letting $y = x$ in (2.3), we get

$$
\|f(2x) - 2f(x)\| \leq 2\theta\|x\|^r
$$

for all $x \in X$. So

$$
\|f(x) - 2f \left( \frac{x}{2} \right)\| \leq \frac{2\theta}{2^r}\|x\|^r
$$

for all $x \in X$. Hence

$$
\left\|2^j f \left( \frac{x}{2^j} \right) - 2^m f \left( \frac{x}{2^m} \right)\right\| \leq \sum_{j=1}^{m-1} \left\|2^j f \left( \frac{x}{2^j} \right) - 2^{j+1} f \left( \frac{x}{2^{j+1}} \right)\right\|
$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.6) that the sequence $\{2^m f \left( \frac{x}{2^m} \right)\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{2^m f \left( \frac{x}{2^m} \right)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$
h(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)
$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.4).

It follows from (2.3) that

$$
\|h(x + y) + h(x - y) - 2h(x)\|
$$

$$
= \lim_{n \to \infty} 2^n \left\| f \left( \frac{x + y}{2^n} \right) + f \left( \frac{x - y}{2^n} \right) - 2f \left( \frac{x}{2^n} \right)\right\|
$$

$$
\leq \lim_{n \to \infty} 2^n \|\rho\| \left\|2f \left( \frac{x + y}{2^{n+1}} \right) + f \left( \frac{x - y}{2^{n+1}} \right) - 2f \left( \frac{x}{2^{n+1}} \right)\right\| + \lim_{n \to \infty} \frac{2^n \theta}{2^{nr}} (\|x\|^r + \|y\|^r)
$$

$$
= \|\rho\| \left\|2h \left( \frac{x + y}{2} \right) + h(x - y) - 2h(x)\right\|
$$
for all $x, y \in X$. So
\[
\|h(x + y) + h(x - y) - 2h(x)\| \leq \|\rho \left(2h \left(\frac{x + y}{2} \right) + h(x - y) - 2h(x)\right)\|
\]
for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is additive.

Now, let $T : X \to Y$ be another additive mapping satisfying (2.4). Then we have
\[
\|h(x) - T(x)\| = 2^n \left\|h \left(\frac{x}{2^n}\right) - T \left(\frac{x}{2^n}\right)\right\|
\leq 2^n \left(\left\|h \left(\frac{x}{2^n}\right) - f \left(\frac{x}{2^n}\right)\right\| + \|T \left(\frac{x}{2^n}\right) - f \left(\frac{x}{2^n}\right)\|\right)
\leq \frac{4 \cdot 2^n}{(2^r - 2)2^{nr}} \|x\|^r,
\]
which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $h$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (2.4).

**Theorem 2.3.** Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique additive mapping $h : X \to Y$ such that
\[
\|f(x) - h(x)\| \leq \frac{2\theta}{2 - 2^r} \|x\|^r
\]
for all $x \in X$.

**Proof.** It follows from (2.5) that
\[
\left\|f(x) - \frac{1}{2} f(2x)\right\| \leq \theta \|x\|^r
\]
for all $x \in X$. Hence
\[
\left\|\frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x)\right\| \leq \sum_{j=l}^{m-1} \left\|\frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x)\right\|
\leq \sum_{j=l}^{m-1} 2^{rj} \|x\|^r
\]
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.8) that the sequence $\left\{\frac{1}{2^n} f(2^n x)\right\}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\left\{\frac{1}{2^n} f(2^n x)\right\}$ converges. So one can define the mapping $h : X \to Y$ by
\[
h(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]
for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.8), we get (2.7).

The rest of the proof is similar to the proof of Theorem 2.2. □

**Remark 2.4.** If $\rho$ is a real number such that $-1 < \rho < 1$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.

3. **ADDITIVE $\rho$-FUNCTIONAL INEQUALITY (0.2)**

Throughout this section, assume that $\rho$ is a fixed complex number with $|\rho| < 1$.

In this section, we solve and investigate the additive $\rho$-functional inequality (0.2) in complex Banach spaces.

**Lemma 3.1.** If a mapping $f : G \to Y$ satisfies
\[
\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\| \leq \|\rho(f(x+y) + f(x-y) - 2f(x))\|
\]
for all $x, y \in G$, then $f : G \to Y$ is additive.

**Proof.** Assume that $f : G \to Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|f(0)\| \leq 0$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get $\|2f\left(\frac{x}{2}\right) - f(x)\| \leq 0$ and so
\[
2f\left(\frac{x}{2}\right) = f(x)
\]
for all $x \in G$.

It follows from (3.1) and (3.2) that
\[
\|f(x+y) + f(x-y) - 2f(x)\| = \left\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\right\|
\leq |\rho|\|f(x+y) + f(x-y) - 2f(x)\|
\]
and so $f(x+y) + f(x-y) = 2f(x)$ for all $x, y \in G$. It is easy to show that $f$ is additive. □

We prove the Hyers-Ulam stability of the additive $\rho$-functional inequality (3.1) in complex Banach spaces.

**Theorem 3.2.** Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that
\[
\|2f\left(\frac{x+y}{2}\right) + f(x-y) - 2f(x)\|
\leq \|\rho(f(x+y) + f(x-y) - 2f(x))\| + \theta(\|x\|^r + \|y\|^r)
\]
for all \( x, y \in X \). Then there exists a unique additive mapping \( h : X \to Y \) such that

\[
\| f(x) - h(x) \| \leq \frac{2^r \theta}{2^r - 2} \| x \|^r
\]

for all \( x \in X \).

**Proof.** Letting \( x = y = 0 \) in (3.3), we get \( \| f(0) \| \leq 0 \). So \( f(0) = 0 \).

Letting \( y = 0 \) in (3.3), we get

\[
\| 2f\left( \frac{x}{2^j} \right) - f(x) \| \leq \theta \| x \|^r
\]

for all \( x \in X \). So

\[
\| 2^j f\left( \frac{x}{2} \right) - 2^m f\left( \frac{x}{2^m} \right) \| \leq \sum_{j=l}^{m-1} \| 2^j f\left( \frac{x}{2^j} \right) - 2^{j+1} f\left( \frac{x}{2^{j+1}} \right) \|
\]

\[
\leq \sum_{j=l}^{m-1} \frac{2^j \theta}{2^j} \| x \|^r
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (3.6) that the sequence \( \{ 2^n f\left( \frac{x}{2^n} \right) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ 2^n f\left( \frac{x}{2^n} \right) \} \) converges. So one can define the mapping \( h : X \to Y \) by

\[
h(x) := \lim_{n \to \infty} 2^n f\left( \frac{x}{2^n} \right)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.6), we get (3.4).

It follows from (3.3) that

\[
\| 2h\left( \frac{x+y}{2} \right) + h(x-y) - 2h(x) \|
\]

\[
= \lim_{n \to \infty} 2^n \left\| 2f\left( \frac{x+y}{2^{n+1}} \right) + f\left( \frac{x-y}{2^n} \right) - 2 f\left( \frac{x}{2^n} \right) \right\|
\]

\[
\leq \lim_{n \to \infty} 2^n \left\| \rho \left( f\left( \frac{x+y}{2^n} \right) + f\left( \frac{x-y}{2^n} \right) - 2 f\left( \frac{x}{2^n} \right) \right) \right\| + \lim_{n \to \infty} \frac{2^n \theta}{2^{n+1}} (\| x \|^r + \| y \|^r)
\]

\[
= \| \rho(h(x+y) + h(x-y) - 2h(x)) \|
\]

for all \( x, y \in X \). So

\[
\| 2h\left( \frac{x+y}{2} \right) + h(x-y) - 2h(x) \| \leq \| \rho(h(x+y) + h(x-y) - 2h(x)) \|
\]

for all \( x, y \in X \). By Lemma 3.1, the mapping \( h : X \to Y \) is additive.
Now, let $T : X \to Y$ be another additive mapping satisfying (3.4). Then we have
\[
\|h(x) - T(x)\| = 2^n \left\| h \left( \frac{x}{2^n} \right) - T \left( \frac{x}{2^n} \right) \right\|
\leq 2^n \left( \|h \left( \frac{x}{2^n} \right) - f \left( \frac{x}{2^n} \right)\| + \|T \left( \frac{x}{2^n} \right) - f \left( \frac{x}{2^n} \right)\| \right)
\leq \frac{2 \cdot 2^n \cdot 2^r}{(2^r - 2)2^{nr}} \theta \|x\|^r,
\]
which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of $h$. Thus the mapping $h : X \to Y$ is a unique additive mapping satisfying (3.4). □

**Theorem 3.3.** Let $r < 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (3.3). Then there exists a unique additive mapping $h : X \to Y$ such that
\[
\|f(x) - h(x)\| \leq \frac{2^r \theta}{2^r - 2^r} \|x\|^{2r}
\]
for all $x \in X$.

**Proof.** It follows from (3.5) that
\[
\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{2^r \theta}{2^r - 2^r} \|x\|^{2r}
\]
for all $x \in X$. Hence
\[
\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|
\leq \frac{2^r \theta}{2^r} \sum_{j=l}^{m-1} \frac{2^{rj}}{2^j} \|x\|^{2r}
\]
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{ \frac{1}{2^m} f(2^m x) \}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence $\{ \frac{1}{2^m} f(2^m x) \}$ converges. So one can define the mapping $h : X \to Y$ by
\[
h(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]
for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proof of Theorem 3.2. □

**Remark 3.4.** If $\rho$ is a real number such that $-1 < \rho < 1$ and $Y$ is a real Banach space, then all the assertions in this section remain valid.
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