A CHARACTERIZATION OF THE UNIT GROUP IN $\mathbb{Z}[T \times C_2]$

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Abstract. Describing the group of units $U(\mathbb{Z}G)$ of the integral group ring $\mathbb{Z}G$, for a finite group $G$, is a classical and open problem. In this note, we show that $U_1(\mathbb{Z}[T \times C_2]) \cong [F_{97} \rtimes F_5] \rtimes [T \times C_2]$, where $T = \langle a, b : a^6 = 1, a^3 = b^2, ba = a^5b \rangle$ and $F_{97}, F_5$ are free groups of ranks 97 and 5, respectively.

1. Introduction

Given a finite group $G$ and the ring of integers $\mathbb{Z}$, we denote the integral group ring as $\mathbb{Z}G$. Its elements are all finite formal sums

$$\sum_{g \in G} r_g g, \text{ where } r_g \in \mathbb{Z}.$$ 

There is a surjective ring homomorphism $\epsilon: \mathbb{Z}G \to \mathbb{Z}$, defined by

$$\sum_{g \in G} r_g g \mapsto \sum_{g \in G} r_g.$$ 

The ring homomorphism $\epsilon$ is called the augmentation map and its kernel $\Delta_{\mathbb{Z}}(G) = \langle g - 1 : g \in G \rangle$ is the augmentation ideal. We will denote the group of units of $\mathbb{Z}G$ by $U(\mathbb{Z}G)$, $U_1(\mathbb{Z}G)$ will denote the units of augmentation one in $U(\mathbb{Z}G)$. Thus, $U_1(\mathbb{Z}G)$ is a normal subgroup of $U(\mathbb{Z}G)$ and $\pm U_1(\mathbb{Z}G) = U(\mathbb{Z}G)$. Observe that $\pm G \leq U(\mathbb{Z}G)$. The elements $\pm G$ are called the trivial units of $\mathbb{Z}G$.

Describing the units of the integral group ring is a classical and difficult problem. Over the years, it has drawn the attention of those working in the areas of algebra, number theory, and algebraic topology. Most descriptions of $U(\mathbb{Z}G)$ in the mathematical literature either give an explicit description of the units, the general structure of $U(\mathbb{Z}G)$, or a subgroup of finite index of the unit group $U(\mathbb{Z}G)$. These results were often obtained by using techniques from representation theory and algebraic number theory.

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In 1940, substantial work on the unit problem was done by Graham Higman [5, 6]. He first showed that if $U(ZG) = \pm G$, then $U(G) = \pm G \iff G$ is abelian of exponent $2, 3, 4, 6$ or $G = E \times K$ where $K$ is the quaternion group of order $8$ and $E$ is an elementary abelian $2$-group. Furthermore, Higman gave a general structure theorem for $U(ZA)$, where $A$ is a finite abelian group. Other results include: $A_4$ and $S_4$ by Allen-Hobby [1, 2], $L_2(q)$ by Passman-Smith [21], $G = C_p \times C_q$, where $q$ is a prime dividing $p - 1$ by Galovitch-Reiner-Ullom [4], $|G| = p^3$ by Ritter-Sehgal [23], and $U(ZG^*)$ by Hughes-Pearson [7]. Jespers and Parmenter [10] gave a more explicit description of $U(ZS_3)$. In 1993, Jespers and Parmenter [11] completed the description of $U(ZG)$ for all groups of order 16. Jespers [9], in 1995, gave a description of $U(ZG)$, for the dihedral group of order 12 and for $G = D_5 \times C_2$. More recently, Bilgin [3] gave a characterization of $U_1(ZC_{12})$. Kusmus and Denizler [15] gave a construction of $U(ZG^*)$. Kelebek and Bilgin [14] described the structure of $U_1(ZC_n \times K_4)$. The interested reader is directed to Sehgal’s [24] comprehensive survey on the unit problem in integral group rings.

In [17, 18], a general algebraic framework was developed to study $U(ZG^*)$, where $G^* = G \times C_2$, with $p$ prime. In the following sections of this note, we focus on the case where $p = 2$ and then resolve a conjecture found in [17].

2. $U(ZG^*)$

Here, we obtain a result which helps us to answer the following question: Assuming that we have a good description of $U(ZG)$, can we obtain a description of $U(ZG^*)$, where $G^* = G \times C_2$?

Let $G^* = G \times \langle x \rangle$, $x^2 = 1$, with $|G| = n$. Decomposing $G^*$ into two cosets, we have that $G^* = G \cup xG = \{g, g_1, \ldots, g_n, xg_1, \ldots, xg_n\}$. Thus, $ZG^* = ZG \oplus xZG$, a direct sum of abelian groups. Here, the equal sign denotes equality as sets. Now, consider the surjective group homomorphism $\pi : G^* \rightarrow G$ defined by $g \mapsto g, x \mapsto 1$. This induces a ring homomorphism $\pi : ZG^* \rightarrow ZG$; where $\pi(P_1 + xP_2) = P_1 + P_2$, and $P_1, P_2 \in ZG$. At the ring level, $\text{Ker}(\pi) = K^* = (x - 1)ZG$. So, we have the sequence of maps

$$K^* \xrightarrow{i} ZG^* \xrightarrow{\pi} ZG.$$ 

Restricting $\pi$ to the group of units, we obtain the split exact sequence of groups:

$$K \xrightarrow{i} U(ZG^*) \xrightarrow{\pi} U(ZG),$$

where $K = \text{Ker}(\pi)$. Hence, $U(ZG^*) = K \times U(ZG)$. Note that $K = U(ZG^*) \cap (1 + K^*)$. Thus, a unit in $K$ has the form $1 + (x - 1)P$, where $P \in ZG$, and has an inverse $1 - (x - 1)Q$, where $Q \in ZG$.

Also, let us consider the surjective ring homomorphism $\rho : ZG \rightarrow Z_2G$, where $\rho$ reduces the coefficients modulo 2. The kernel of $\rho$, say $M^*$ (as an ideal), is $M^* = 2ZG$. Thus, we have the following sequence of maps:

$$M^* \xrightarrow{i} ZG \xrightarrow{\rho} Z_2G.$$


Furthermore, $\rho$ induces the following exact sequence of groups, which does not necessarily split:

$$M \xrightarrow{i} U(ZG) \xrightarrow{\rho} U(Z_2G),$$

where $M$ is the kernel of the group homomorphism $\rho$. Observe that $M = U(ZG) \cap (1 + M^\ast)$. Thus, a unit in $M$ has the form $1 + 2P$, where $P \in ZG$ and has an inverse $1 + 2Q$, where $Q \in ZG$. Notice that here at the group level, $\rho$ is not necessarily surjective.

Since $G^* = G \times \langle x \rangle$ and $x^2 = 1$, we have the group homomorphism $\sigma : G^* \to U(ZG)$, where $\sigma(g) = g$ and $\sigma(x) = -1$. This extends to a ring homomorphism $\sigma : ZG^* \to ZG$. So, we have the following diagram of rings:

$$K^* \xrightarrow{i} ZG^* \xrightarrow{\pi} ZG$$

$$\sigma \downarrow \quad \sigma \downarrow \quad \rho \downarrow$$

$$M^* \xrightarrow{i} ZG \xrightarrow{\rho} Z_2G.$$

Observe that $\rho \circ \pi = \rho \circ \sigma$. Hence, $\sigma(K^*) \subseteq M^*$. Note that $\sigma$ maps the element $1 + (x - 1)P$ to the element $1 - 2P$ of $M$. Thus, $\sigma(K) \subseteq M$.

**Lemma 2.1.** Let $G^* = G \times \langle x \rangle$, where $x$ has order 2, $u = 1 + (x - 1)P$, $v = 1 + (x - 1)Q$, where $P, Q \in ZG$. Then $u$ and $v$ are multiplicative inverses of each other in $K \iff 1 - 2P$ and $1 - 2Q$ are multiplicative inverses of each other in $U(ZG)$.

**Proof.** Let $u, v \in K$; with $uv = 1$. It is straightforward to see that $uv = 1 + (x - 1)(P + Q - 2PQ)$.

Hence, $uv = 1 \iff (x - 1)(P + Q - 2PQ) = 0$

$\iff (2PQ - P - Q) + (P + Q - 2PQ)x = 0$

$\iff 2PQ - P - Q = 0$

$\iff 4PQ - 2P - 2Q = 0$

$\iff 1 - 2P - 2Q + 4PQ = 1$

$\iff (1 - 2P)(1 - 2Q) = 1$. \qed

**Lemma 2.2.** The map $\sigma : K \to M$ is an isomorphism of groups.

**Proof.** Note that $\sigma$ maps the element $1 + (x - 1)P$ of $K$ to the element $1 - 2P$ of $M$. It is then easy to show that $\sigma$ is injective. It follows from Lemma 2.1 that $\sigma$ is surjective. \qed

Summarizing, we have the following diagram of groups:

$$K \xrightarrow{i} U(ZG^*) \xrightarrow{\pi} U(ZG)$$

$$\sigma \downarrow \quad \sigma \downarrow \quad \rho \downarrow$$

$$M \xrightarrow{i} U(ZG) \xrightarrow{\rho} U(Z_2G).$$
Theorem 2.3. \(U(\mathbb{Z}G^*) = K \rtimes U(\mathbb{Z}G) \cong M \rtimes U(\mathbb{Z}G)\).

Proof. The elements of the semi-direct product \(M \rtimes U(\mathbb{Z}G)\) should be viewed as ordered pairs \((a, w)\), where \(a \in M\) and \(w \in U(\mathbb{Z}G)\). If \(k \in K\) and \(w \in U(\mathbb{Z}G)\), then the isomorphism maps \(kw\) to \((\sigma(k), w)\) with the action of \(U(\mathbb{Z}G)\) on \(M\) induced by conjugation in \(U(\mathbb{Z}G)\). □

The problem of describing \(U(\mathbb{Z}G^*)\) has been reduced to the problem of describing \(M\). In the next section, we apply Theorem 2.3 and resolve a conjecture involving \(U(\mathbb{Z}G^*)\), where \(G^*\) is a particular non-abelian group of order 24.

3. Resolution of a conjecture

It was shown by Jespers [8] that there are only four finite groups \(G\) with the property that \(G\) has a non-abelian free normal complement in \(U_1(\mathbb{Z}G)\), namely \(G = S_3, D_4\) (the dihedral group of order 8), \(P = \langle a, b : a^4 = 1 = b^5, bab^{-1}a^{-1} = a^2 \rangle\), and the non-abelian group \(T\) (of order 12) described by the presentation \(T = \langle a, b : a^6 = 1, a^3 = b^2, ba = a^5b \rangle\).

In [9, 11, 17], the structure of \(U(\mathbb{Z}[G \times C_2])\) is determined for \(G = S_3, D_4\) and \(P\). In this section, we disprove the following conjecture, first posed in [17]:

Conjecture. Let \(T^* = T \times C_2\), where \(T = \langle a, b : a^6 = 1, a^3 = b^2, ba = a^5b \rangle\). Then, \(U_1(\mathbb{Z}T^*) \cong [F_{33} \times F_{35}] \rtimes T^*\), where \(F_i\) is a free group of rank \(i\).

This is certainly a plausible conjecture. Later, it was shown in [12] that \(U(\mathbb{Z}[T \times C_2])\) is commensurable with a free-by-free group. We will show that if \(F_{33}\) is replaced with \(F_{27}\), then a correct result is obtained.

In 1993, Parmenter [20] showed that \(U_1(\mathbb{Z}T) = V \rtimes T\), where \(V = \langle v_1, v_2, v_3, v_4, v_5 \rangle\) is a free group of rank five. He also gave the generators of \(V\) to be:

\[
\begin{align*}
v_1 &= 1 + (1 + a^3)(-a^2 + b) + (1 - a^2), \\
v_2 &= 1 + (1 + a^3)(-a^2 + b)(1 - a^2), \\
v_3 &= 1 + (1 + a^3)(-a^2 + b)(1 - a^2), \\
v_4 &= 1 + [1 + (1 + a^3)a^2(a^2 + ba)^2](1 - a^2), \\
v_5 &= 1 + [-1 - a^2 + (1 + a^3)a(1 - a - 2ba)](1 - a^2).
\end{align*}
\]

Let us determine \(\rho(V)\). It is straightforward to verify the following facts. First, \(\rho(v_i)\rho(v_j) = \rho(v_j)\rho(v_i)\), where \(1 \leq i, j \leq 3\). Also, \(\rho(v_1)^2 = \rho(v_2)^2 = \rho(v_3)^2 = 1\) and thus, \(E = \langle \rho(v_1), \rho(v_2), \rho(v_3) \rangle \cong C_2 \times C_2 \times C_2\). Now, calculations show that \(a^2\rho(v_1)a^4 = \rho(v_2), a^2\rho(v_2)a^4 = \rho(v_3), a^2\rho(v_3)a^4 = \rho(v_1)\), and \([\rho(v_1)]^3a^4 = \rho(v_5)\). Thus, \(\langle a^2, \rho(v_1) \rangle = \langle a^2, \rho(v_2) \rangle = \langle a^2, \rho(v_3) \rangle = \rho(V)\).

Lemma 3.1. \(\rho(V) = E \rtimes \langle a^2 \rangle, \) a group of order 24.

Proof. Since \(E = \langle \rho(v_1), \rho(v_2), \rho(v_3) \rangle\) is normalized by \(\rho(v_1), \rho(v_2), \rho(v_3)\), and \(a^2\), we have that \(E \unlhd \rho(V)\). So, \(E \cdot \langle a^2 \rangle \leq \rho(V)\). In fact, \(E \cdot \langle a^2 \rangle = \rho(V)\) and
Lemma 3.2. \( \rho[U_1(ZT)] = E \times T \).

Proof. Clearly, \( \rho[U_1(ZT)] = \rho(V \times T) = \rho(V) \cdot T \). Since \( E \leq \rho(V) \), we have that \( \rho[U_1(ZT)] = E \cdot T \). Since \( ap(v_1)a^5 = \rho(v_3), ap(v_2)a^5 = \rho(v_2), ap(v_2)a^7 = \rho(v_1), b\rho(v_1)b^3 = \rho(v_1), b\rho(v_2)b^3 = \rho(v_3), b\rho(v_3)b^3 = \rho(v_2) \), we see that \( E \) is normalized by \( T \). Note that \( E \cap T = 1 \). Hence, the lemma is established.

A remark should be made at this point. Since \( \rho[U_1(ZT)] \) has order 96, \( |\rho(V)| = 24 \), and \( |T| = 12 \), this implies that \( |\rho(V) \cap T| = 3 \). But \( \langle a^2 \rangle \leq \rho(V) \cap T \), where the order of \( a^2 \) is 3. Hence, \( \rho(V) \cap T = \langle a^2 \rangle \). Now, we have the diagram:

\[
\begin{array}{cccc}
K & \xrightarrow{\iota} & U(ZT^*) & \xrightarrow{\pi} & U(ZT) \\
\cong & \downarrow & \sigma & \downarrow & \rho \\
M & \xrightarrow{\iota} & U(ZT) & \xrightarrow{\rho} & U(Z_2T) \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
M^+ & \xrightarrow{\iota} & U_1(ZT) &\xrightarrow{\text{onto}} & E \times T \\
\downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
M^+ \cap V & \xrightarrow{\iota} & V & \xrightarrow{\text{onto}} & E \times \langle a^2 \rangle.
\end{array}
\]

Lemma 3.3. \( \rho(V) = \langle \rho(v_1), \rho(v_2), \rho(v_3), \rho(v_4) \rangle = \{ [\rho(v_1)]^{i_1} : [\rho(v_2)]^{i_2}, [\rho(v_3)]^{i_3}, [\rho(v_4)]^{i_4} : 0 \leq i_1, i_2, i_3, i_4 \leq 1; 0 \leq i_4 \leq 2 \} \). Furthermore, this canonical representation is unique.

Proof. Note that \( \rho(V) = \langle \rho(v_1), \rho(v_2), \rho(v_3), \rho(v_4) \rangle = \langle \rho(v_1), \rho(v_2), \rho(v_3), \rho(v_4) \rangle \). Also, calculations show the following:

\[
\begin{align*}
\rho(v_4)\rho(v_1) &= \rho(v_2)a^2\rho(v_1) = \rho(v_2)a^2 = \rho(v_2), \\
\rho(v_4)\rho(v_2) &= \rho(v_2)a^2\rho(v_2) = \rho(v_2)\rho(v_3)a^2 = \rho(v_2)\rho(v_3)\rho(v_2) = \rho(v_3)\rho(v_4), \\
\rho(v_4)\rho(v_3) &= \rho(v_2)a^2\rho(v_3) = \rho(v_2)\rho(v_1)a^2 = \rho(v_2)\rho(v_1)\rho(v_2) = \rho(v_1)\rho(v_4), \\
\rho(v_4)^2\rho(v_1) &= \rho(v_3)\rho(v_4)^2, \\
\rho(v_4)^2\rho(v_2) &= \rho(v_1)\rho(v_4)^2, \\
\rho(v_4)^2\rho(v_3) &= \rho(v_2)\rho(v_4)^2, \\
\rho(v_4)^2\rho(v_4) &= \rho(v_1)\rho(v_4)^2, \\
\rho(v_4)^3\rho(v_1) &= \rho(v_2)\rho(v_4)^3, \\
\rho(v_4)^3\rho(v_2) &= \rho(v_1)\rho(v_4)^3, \\
\rho(v_4)^3\rho(v_3) &= \rho(v_4)^3, \\
\rho(v_4)^3\rho(v_4) &= \rho(v_2)\rho(v_4)^3, \\
\rho(v_4)^4\rho(v_1) &= \rho(v_4)\rho(v_4)^3\rho(v_1) = \rho(v_1)\rho(v_3)\rho(v_4), \\
\rho(v_4)^4\rho(v_2) &= \rho(v_4)\rho(v_4)^3\rho(v_2) = \rho(v_1)\rho(v_2)\rho(v_4), \\
\rho(v_4)^4\rho(v_3) &= \rho(v_4)\rho(v_4)^3\rho(v_3) = \rho(v_4)\rho(v_4)^3\rho(v_3) = \rho(v_1)\rho(v_2)\rho(v_4).
\end{align*}
\]
\[ \rho(v_4)^4 \rho(v_3) = \rho(v_4) \rho(v_3)^3 \rho(v_3) = \rho(v_2) \rho(v_3) \rho(v_2), \]
\[ \rho(v_4)^3 \rho(v_1) = \rho(v_4) \rho(v_2) \rho(v_3)^2 \rho(v_3) = \rho(v_1) \rho(v_2) \rho(v_3)^2, \]
\[ \rho(v_4)^3 \rho(v_2) = \rho(v_1) \rho(v_2) \rho(v_3)^2 \rho(v_3) = \rho(v_1) \rho(v_2) \rho(v_3)^2, \]
\[ \rho(v_4)^3 \rho(v_3) = \rho(v_1) \rho(v_2) \rho(v_3)^2 \rho(v_3) = \rho(v_1) \rho(v_2) \rho(v_3)^2. \]

Thus, every word in \( \rho \) can be put into the canonical form \( [\rho(v_1)]^{i_1} \cdot [\rho(v_2)]^{i_2} \cdot [\rho(v_3)]^{i_3} \cdot [\rho(v_4)]^{i_4} \), where \( 0 \leq i_1, i_2, i_3 \leq 1 \) and \( 0 \leq i_4 \leq 2 \). This representation is unique, since \( |\rho(V)| = 24 \). □

**Lemma 3.4.** Let \( w(\rho(v_1), \rho(v_2), \rho(v_3)) \in E, t \in T \), with \( w(\rho(v_1), \rho(v_2), \rho(v_3)) \cdot t = 1 \). Then \( t = 1 \).

**Proof.** Suppose that \( w(\rho(v_1), \rho(v_2), \rho(v_3)) \cdot t = 1 \). Then, we have \( w(\rho(v_1), \rho(v_2), \rho(v_3)) = t^{-1} \in T \) and \( w(\rho(v_1), \rho(v_2), \rho(v_3)) \in E \). This implies that \( w(\rho(v_1), \rho(v_2), \rho(v_3)) \in E \). Thus, \( w(\rho(v_1), \rho(v_2), \rho(v_3)) = 1 \), which implies that \( t^{-1} = 1 \). Hence, \( t = 1 \). □

**Lemma 3.5.** \( M^+ \leq V \rtimes \langle a^2 \rangle \).

**Proof.** Suppose that \( w(v_1, v_2, v_3, v_4, v_5) \cdot t \in M^+ \), where \( t \in T \). This implies that \( \rho(w(v_1, v_2, v_3, v_4, v_5) \cdot t) = 1 \). By Lemma 3.3, we have that \( \langle \rho(v_1) \rangle^{i_1} \cdot [\rho(v_2)]^{i_2} \cdot [\rho(v_3)]^{i_3} \cdot [\rho(v_4)]^{i_4} \cdot t = 1 \), where \( 0 \leq i_1, i_2, i_3 \leq 1, 0 \leq i_4 \leq 2; t \in T \). Now, \( \langle \rho(v_1) \rangle^{i_1} \cdot [\rho(v_2)]^{i_2} \cdot [\rho(v_3)]^{i_3} \cdot [\rho(v_4)]^{i_4} \) has three possible forms:

\[
\begin{align*}
\rho(v_1)^{i_1} \cdot \rho(v_2)^{i_2} \cdot \rho(v_3)^{i_3} \cdot \rho(v_4)^{i_4}, & \quad \text{if } i_4 = 1; \\
\rho(v_1)^{i_1} \cdot \rho(v_2)^{i_2} \cdot \rho(v_3)^{i_3}, & \quad \text{if } i_4 = 0.
\end{align*}
\]

Using Lemma 3.4, we have \( \langle \rho(v_1) \rangle^{i_1} \cdot [\rho(v_2)]^{i_2} \cdot [\rho(v_3)]^{i_3} \cdot [\rho(v_4)]^{i_4} \cdot t = 1 \) implies that \( t \in \langle a^2 \rangle \). □

**Lemma 3.6.** \( M^+ \) is a free group of rank 97.

**Proof.** Since \( M^+ \leq \rho^{-1}[E \rtimes T] \), where \( E = \langle \rho(v_1), \rho(v_2), \rho(v_3) \rangle \cong C_2 \times C_2 \times C_2 \), we see that \( M^+ \) consists of the elements of the form

\[
\rho^{-1} \rho(v_1)^{j_1} \rho(v_2)^{j_2} \rho(v_3)^{j_3} \cdot t,
\]

where \( 0 \leq j_1, j_2, j_3 \leq 1 \) and \( t \in T \). Since \( M^+ \) is an appropriate kernel of \( \rho \), then \( \rho(M^+) = 1 \). If we consider an element in \( M^+ \) as

\[
\alpha = \rho^{-1} \rho(v_1)^{j_1} \rho(v_2)^{j_2} \rho(v_3)^{j_3} \cdot t,
\]

we see that \( \rho(\alpha) = \rho(v_1)^{j_1} \rho(v_2)^{j_2} \rho(v_3)^{j_3} \cdot t = 1 \). By Lemma 3.4, \( t = 1 \). This implies that \( M^+ \) consists of elements of the form

\[
\rho^{-1} \rho(v_1)^{j_1} \rho(v_2)^{j_2} \rho(v_3)^{j_3} \in V.
\]

Thus, \( M^+ \leq V \). Since \( V \) is a free group, the Nielsen-Schreier Theorem states that \( M^+ \) is a free group. Note that \( M^+ \) is a free group. Now, consider the induced isomorphism \( \tilde{\rho} : M^+ \rtimes V \to E \rtimes \langle a^2 \rangle \), which implies that \( [V : M^+ \cap V] = [V : \)
$M^+ = 24$. Since $V$ is a free group of rank 5, this implies that $M^+$ is a free group of rank $(24)(5) - 24 + 1 = 97$.

**Theorem 3.7.** Let $T^* = T \times C_2$, where $T = \langle a, b : a^6 = 1, a^3 = b^2, ba = a^5b \rangle$. Then, $U_1(ZT^*) \cong [F_97 \rtimes F_5] \rtimes T^*$, where $F_i$ is a free group of rank $i$.

**Proof.** Invoking Theorem 2.3, we obtain

$$U_1(Z(T \times C_2)) = K \rtimes (V \times T) \cong M \rtimes (V \times T) = [M^+ \rtimes C_2] \rtimes (V \times T) = [F_97 \rtimes F_5] \rtimes (T \times C_2),$$

where $F_i$ is a free group of rank $i$.

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