ON FUNCTIONAL EQUATIONS OF THE FERMAT-WARING TYPE FOR NON-ARCHIMEDEAN VECTORIAL ENTIRE FUNCTIONS

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Abstract. We show a class of homogeneous polynomials of Fermat-Waring type such that for a polynomial $P$ of this class, if $P(f_1, \ldots, f_{N+1}) = P(g_1, \ldots, g_{N+1})$, where $f_1, \ldots, f_{N+1}; g_1, \ldots, g_{N+1}$ are two families of linearly independent entire functions, then $f_i = cg_i, i = 1, 2, \ldots, N + 1$, where $c$ is a root of unity. As a consequence, we prove that if $X$ is a hypersurface defined by a homogeneous polynomial in this class, then $X$ is a unique range set for linearly non-degenerate non-Archimedean holomorphic curves.

1. Introduction

The function equation $P(f) = P(g)$, where $P$ is a polynomial, $f, g$ are functions in some classes, has a long history, dating back to Ritt ([24]). In recent years the problem of existence or non-existence of solutions to the equation has investigated by many authors (see [1], [2], [6], [8], [10], [20], [21], [22], [23]). For the case of entire functions of one variable in a non-Archimedean field, many interesting results are obtained ([4], [5], [6], [9], [11], [12], [13], [16], [17]).

In this paper we investigated the case of the Fermat-Waring type for non-Archimedean vectorial entire functions. Namely, we consider the equation:

$$P(f_1, f_2, \ldots, f_{N+1}) = P(g_1, g_2, \ldots, g_{N+1}),$$

where $P$ is a polynomial of Fermat-Waring type, and $f_i, g_i$ are entire functions in a non-Archimedean field. We show if $f_1, \ldots, f_{N+1}; g_1, \ldots, g_{N+1}$ are two families of linearly independent entire functions, then $f_i = cg_i, i = 1, 2, \ldots, N + 1$, where $c$ is a root of unity. As a consequence, we obtained a class of unique range sets for linearly non-degenerate non-Archimedean holomorphic curves.

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Throughout this paper, \( \mathbb{K} \) will denote an algebraically closed field of characteristic zero, complete for a non-trivial non-Archimedean absolute value denoted by \( | \cdot | \). We assume that the reader is familiar with the notations in the non-Archimedean Nevanlinna theory (see [14]).

Let \( f \) be a non-constant meromorphic function on \( \mathbb{K} \). For every \( a \in \mathbb{K} \), define the function \( \mu_f^a : \mathbb{K} \to \mathbb{N} \) by

\[
\mu_f^a(z) = \begin{cases} 
0 & \text{if } f(z) \neq a \\
d & \text{if } f(z) = a \text{ with multiplicity } d.
\end{cases}
\]

A non-Archimedean holomorphic map \( f \) is a map \( f = [f_1, \ldots, f_{N+1}] : \mathbb{K} \to \mathbb{P}^N(\mathbb{K}) \), where \( f_1, \ldots, f_{N+1} \) are non-Archimedean entire functions without common zeros. The map \( \tilde{f} = (f_1, \ldots, f_{N+1}) : \mathbb{K} \to \mathbb{K}^{N+1} \setminus \{0\} \) is called a reduced representation of \( f \) (see [25]).

Let \( H \) be a hypersurface of \( \mathbb{P}^N(\mathbb{K}) \) such that the image of \( f \) is not contained in \( H \), and \( H \) is defined by the equation \( F = 0 \). For every \( z \in \mathbb{K} \) set

\[
\mu_f(H, z) = \mu_{F \circ \tilde{f}}(z), \quad \mu_f(H) = \mu_{F \circ \tilde{f}}.
\]

Let us first describe the class of polynomials of Fermat-Waring type considered in this paper.

A family of \( q \) polynomials of \( N+1 \) variables are said to be in general position if no set of \( N+1 \) polynomials in this family has common zeros in \( \mathbb{K}^{N+1} \setminus \{0\} \).

Now let given \( q \) linear forms of \( N+1 \) variables \( (q > N + 1) \) in general position:

\[
L_i = L_i(z_1, \ldots, z_{N+1}) = \alpha_{i,1}z_1 + \alpha_{i,2}z_2 + \cdots + \alpha_{i,N+1}z_{N+1}, \quad i = 1, 2, \ldots, q.
\]

Let \( n, m \) be positive integers, \( m < n, a, b \in \mathbb{K}, \, a, b \neq 0 \).

The following polynomial is called a \( Y_i \) \( (m,n) \)-polynomial:

\[
Y_i(m,n)(z_1, z_2) = z_1^n - az_1^{n-m}z_2^m + bz_2^n.
\]

Now consider \( q \) homogeneous polynomials:

\[
P_1 = P_1(z_1, \ldots, z_{N+1}) = Y(m,n)(L_1, L_2) = L_1^n - aL_1^{n-m}L_2^m + bL_2^n,
\]

and for \( q \geq i \geq 2 \), set:

\[
P_i = P_i(z_1, \ldots, z_{N+1}) = Y(m,n)(P_{i-1}, L_{i-1}^{n-1}).
\]

Then we consider the following polynomial of Fermat-Waring type of degree \( n^q \):

\[
P(z_1, z_2, \ldots, z_{N+1}) = P_q(z_1, \ldots, z_{N+1}).
\]

The polynomial \( P(z_1, z_2, \ldots, z_{N+1}) \) is called a \( q \)-iteration of \( Y_i \) \( (m,n) \)-polynomials.

For entire functions \( f_1, \ldots, f_{N+1} \); \( g_1, \ldots, g_{N+1} \) over \( \mathbb{K} \) we consider the following equation:

\[
P(f_1, \ldots, f_{N+1}) = P(g_1, \ldots, g_{N+1}).
\]
Denote by \( X \) the hypersurface of Fermat-Waring type in \( \mathbb{P}^N(\mathbb{K}) \), which is defined by the equation
\[
(1.3) \quad P(z_1, \ldots, z_{N+1}) = 0.
\]

We shall prove the following theorems.

**Theorem 1.1.** Let \( P(z_1, z_2, \ldots, z_{N+1}) \) be a \( q \)-iteration of Yi \((m,n)\)-polynomials \( n \geq 2m + 8, m \geq 3, \) and \( f_1, \ldots, f_{N+1}; g_1, \ldots, g_{N+1} \) be two families of linearly independent entire functions over \( \mathbb{K} \), satisfying the equation \( P(f_1, \ldots, f_{N+1}) = P(g_1, \ldots, g_{N+1}) \). Then \( g_i = c_i f_i, c_i \alpha^i = 1, i = 1, \ldots, N + 1 \).

**Theorem 1.2.** Let \( f \) and \( g \) be two linearly non-degenerate holomorphic mappings from \( \mathbb{K} \) to \( \mathbb{P}^N(\mathbb{K}) \). Let \( X \) be the Fermat-Waring hypersurface defined by the equation \( P(z_1, \ldots, z_{N+1}) = 0 \), where \( P(z_1, \ldots, z_{N+1}) \) is a \( q \)-iteration of Yi \((m,n)\)-polynomials, and \( n \geq 2m + 8, m \geq 3 \). Then \( \mu_f(X) = \mu_g(X) \) implies \( f \equiv g \).

The main tool to be used is the non-Archimedean Nevanlinna theory, so we first recall some basic facts of the theory. More details can be found in [3], [14], [15], [17], [19].

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2. Preliminaries

Let \( f \) be a non-constant meromorphic function on \( \mathbb{K} \).

The following lemma were proved in [3], see also [14].

**Lemma 2.1.** Let \( f \) be a non-constant meromorphic function on \( \mathbb{K} \) and let \( a_1, a_2, \ldots, a_q, q \geq 2 \), be distinct points of \( \mathbb{K} \). Then
\[
(q - 1)T(r, f) \leq \overline{N}(r, f) + \sum_{i=1}^{q} \frac{1}{N(r; f - a_i)} - \log r + O(1).
\]

Let \( f \) be a holomorphic curve from \( \mathbb{K} \) to \( \mathbb{P}^N(\mathbb{K}) \) with reduced representation \( \bar{f} = (f_1, \ldots, f_{N+1}) \). Define the characteristic function of \( f \), by
\[
\overline{T}_f(r) = \log ||f||_r, \quad \text{where} \quad ||f||_r = \max_{1 \leq i \leq N+1} ||f_i||_r,
\]
where for an entire function \( f \), denote by \( ||f||_r \) the maximum of \( |f(z)| \) for \( |z| \leq r \).

Let \( H \) be a hypersurface of \( \mathbb{P}^N(\mathbb{K}) \) such that the image of \( f \) is not contained in \( H \), and \( H \) is defined by the equation \( F = 0 \). Set
\[
N_f(H, r) = N(r; \frac{1}{F(f)}), \quad N_{k,f}(H, r) = N_k(r; \frac{1}{F(f)}).
\]

Let \( f \) be a holomorphic curve from \( \mathbb{K} \) to \( \mathbb{P}^N(\mathbb{K}) \). Then \( f \) is called *linearly non-degenerate* if there is not any linear form \( L \) of variables \( z_1, \ldots, z_{N+1} \) such that \( L(\bar{f}) = 0 \), i.e., the image of \( f \) is not contained in any hyperplane of \( \mathbb{P}^N(\mathbb{K}) \).

Let \( q, N \) be positive integers with \( q \geq N + 1 \). We say that the hypersurfaces \( H_1, \ldots, H_q \) of \( \mathbb{P}^N(\mathbb{K}) \) are *in general position* if \( \cap_{i=1}^{N+1} H_j = \emptyset \) for every subset.
Lemma 2.2. Let $f$ be a linearly non-degenerate holomorphic curve from $\mathbb{K}$ to $\mathbb{P}^N(\mathbb{K})$ and $H_1, \ldots, H_q$ be hyperplanes of $\mathbb{P}^N(\mathbb{K})$ in general position. Then
\[(q - N - 1)T_f(r) \leq \sum_{i=1}^{q} N_{N,f}(H_i,r) - \frac{N(N+1)}{2} \log r + O(1)\.

Lemma 2.3. Let $f$ be a non-constant meromorphic function on $\mathbb{K}$ and let $a_1, a_2, \ldots, a_q$, $q \geq 3$, be distinct points of $\mathbb{K} \cup \{\infty\}$. Suppose either $f - a_i$ has no zeros, or all the zeros of the functions $f - a_i$ have multiplicity at least $m_i$, $i = 1, \ldots, q$. Then
\[\sum_{i=1}^{q} (1 - \frac{1}{m_i}) < 2.\]

3. Functional equations and unique range sets

We first need the following lemmas:

Lemma 3.1. Let $d, N \in \mathbb{N}^*$, $q_i \in \mathbb{N}$ and $z_i^{d-q_i} D_i(z_1, \ldots, z_{N+1})$ be a family in general position of homogeneous polynomials with coefficients in $\mathbb{K}$ of degree $d$ such that $f_i^{d-q_i} D_i(f_1, \ldots, f_{N+1}) \neq 0$, $1 \leq i \leq N + 1$. Suppose
\[\sum_{i=1}^{N+1} f_i^{d-q_i} D_i(f_1, \ldots, f_{N+1}) = 0, \quad d \geq N^2 - 1 + \sum_{i=1}^{N+1} q_i, \quad N > 1.\]

Then $f_1^{d-q_1} D_1(f_1, \ldots, f_{N+1}), \ldots, f_N^{d-q_N} D_N(f_1, \ldots, f_{N+1})$ are linearly dependent on $\mathbb{K}$.

Proof. We consider the following possible cases:

Case 1: $f_1, \ldots, f_{N+1}$ have no common zeros.

By the hypothesis, $z_i^{d-q_i} D_i(z_1, \ldots, z_{N+1})$ is a family in general position, we then get
\[\bar{F} = (f_1^{d-q_1} D_1(f_1, f_2, \ldots, f_{N+1}), \ldots, f_N^{d-q_N} D_N(f_1, f_2, \ldots, f_{N+1}))\]
which is a reduced representation of the holomorphic curve
\[F = [f_1^{d-q_1} D_1(f_1, f_2, \ldots, f_{N+1}); \ldots; f_N^{d-q_N} D_N(f_1, f_2, \ldots, f_{N+1})]\]
from $\mathbb{K}$ to $\mathbb{P}^{N-1}(\mathbb{K})$. Assume that $F$ is linearly non-degenerate. By the hypothesis we have
\[\sum_{i=1}^{N+1} f_i^{d-q_i} D_i(f_1, f_2, \ldots, f_{N+1}) = 0.\]

We first prove $dT_f(r) = T_F(r) + O(1)$. Set
\[R_i(z_1, \ldots, z_{N+1}) = z_i^{d-q_i} D_i(z_1, z_2, \ldots, z_{N+1}), \quad i = 1, \ldots, N + 1.\]
From the hypothesis of general position and the Hilbert Nullstellensatz [26] it implies that for any integer \( k, 1 \leq k \leq N + 1 \), there is an integer \( m_k \geq d \) such that

\[
z_k^{m_k} = \sum_{i=1}^{N+1} a_{ik}(z_1, \ldots, z_{N+1})R_i(z_1, \ldots, z_{N+1}),
\]

where \( a_{ik}(z_1, \ldots, z_{N+1}), 1 \leq i \leq N + 1 \), are homogeneous polynomials with coefficients in \( \mathbb{K} \) of degree \( m_k - d \). Therefore

\[
f_k^{m_k} = \sum_{i=1}^{N+1} a_{ik}(f_1, \ldots, f_{N+1})R_i(f_1, \ldots, f_{N+1}), \quad k = 1, \ldots, N + 1.
\]

It implies that

\[
T_{f_k}^{m_k}(r) = m_k T_f(r) \leq (m_k - d) T_f(r) + \max_{1 \leq i \leq N+1} T_{R_i(f_1, \ldots, f_{N+1})}(r) + O(1),
\]

(3.2)

\[
dT_f(r) \leq \max_{1 \leq i \leq N+1} T_{R_i(f_1, \ldots, f_{N+1})}(r) + O(1).
\]

On the other hand,

\[
T_{R_i(f_1, \ldots, f_{N+1})}(r) = T_{f_i^{d-a_i D_i(f_1, f_2, \ldots, f_{N+1})}}(r) \leq d T_f(r) + O(1)
\]

(3.3)

for all \( i = 1, \ldots, N + 1 \).

By (3.2) and (3.3) we have \( d T_f(r) = \max_{1 \leq i \leq N+1} T_{R_i(f_1, \ldots, f_{N+1})}(r) + O(1) \). Therefore \( d T_f(r) = T_F(r) + O(1) \). Consider the following hyperplanes in general position in \( \mathbb{P}^{N-1} \):

\[
H_1 : x_1 = 0; \quad H_2 : x_2 = 0; \quad \ldots; \quad H_N : x_N = 0; \quad H_{N+1} : x_1 + x_2 + \cdots + x_N = 0.
\]

Using Lemma 2.2, and noting that \( d - q_i \geq N - 1 \),

\[
N_{N-1,F}(H_{N+1}, r) = N_{N-1}(r, \frac{1}{d-q_{N+1} D_{N+1}(f_1, f_2, \ldots, f_{N+1})}),
\]

we have

\[
d T_f(r) = T_F(r) + O(1) \leq \sum_{i=1}^{N+1} N_{N-1,F}(H_i, r) - \frac{N(N-1)}{2} \log r + O(1)
\]

\[
\leq (N-1) \sum_{i=1}^{N+1} N(r, \frac{1}{f_i}) + \sum_{i=1}^{N+1} N(r, \frac{1}{D_i(f_1, f_2, \ldots, f_{N+1})})
\]

\[
- \frac{N(N-1)}{2} \log r + O(1)
\]

\[
\leq (N-1)(N+1) T_f(r) + \sum_{i=1}^{N+1} q_i T_f(r) - \frac{N(N-1)}{2} \log r + O(1)
\]

\[
\leq (N^2 - 1 + \sum_{i=1}^{N+1} q_i) T_f(r) - \frac{N(N-1)}{2} \log r + O(1),
\]
and
\[ (d - (N^2 - 1) - \sum_{i=1}^{N+1} q_i)T_f(r) + \frac{N(N-1)}{2} \log r \leq O(1). \]

Because \( d \geq N^2 - 1 + \sum_{i=1}^{N+1} q_i \), we have a contradiction.

So \( f_1^{d-q_1}D_1(f_1, \ldots, f_{N+1}), \ldots, f_N^{d-q_N}D_N(f_1, \ldots, f_{N+1}) \) are linearly dependent on \( K \).

**Case 2**: \( f_1, \ldots, f_{N+1} \) have common zeros. Let \( l \) be a greatest common divisor of \( f_1, f_2, \ldots, f_{N+1} \). Write \( f_i = lh_i, i = 1, \ldots, N+1 \). Then \( h_1, \ldots, h_{N+1} \) have no common zeros. From (3.1) we obtain
\[ l^{d-q_1}D_1(h_1, \ldots, h_{N+1}) = 0, \]
and
\[ \sum_{i=1}^{N+1} h_i^{d-q_i}D_i(h_1, \ldots, h_{N+1}) = 0. \]

By a similar argument as in the proof of Case 1 for (3.4) we get that
\[ h_1^{d-q_1}D_1(h_1, \ldots, h_{N+1}), \ldots, h_N^{d-q_N}D_N(h_1, \ldots, h_{N+1}) \]
are linearly dependent on \( K \). So
\[ f_1^{d-q_1}D_1(f_1, \ldots, f_{N+1}), \ldots, f_N^{d-q_N}D_N(f_1, \ldots, f_{N+1}) \]
are linearly dependent on \( K \).

Lemma 3.1 is proved. \( \square \)

**Lemma 3.2.** Let \( n, n_1, n_2, \ldots, n_q, q \in \mathbb{N}^*, a_1, \ldots, a_q, c \in K, c \neq 0, \) and \( q \geq 2 + \sum_{i=1}^{q} \frac{n_i}{n} \). Then the functional equation
\[ (f - a_1)^{n_1}(f - a_2)^{n_2} \cdots (f - a_q)^{n_q} = cg^a \]
has no non-constant meromorphic solutions \( (f, g) \).

**Proof.** Suppose that \( (f, g) \) is a non-constant meromorphic solution of the equation:
\[ (f - a_1)^{n_1}(f - a_2)^{n_2} \cdots (f - a_q)^{n_q} = cg^a. \]

From this we see that if \( z_0 \in K \) is a zero of \( f - a_i \) for some \( 1 \leq i \leq q \), then \( z_0 \) is a zero of \( g \) and \( n_i \mu_f^a(z_0) = n_i \mu_g^a(z_0) \). So
\[ \frac{1}{f - a_i} \leq n_i \mu_f^a(z_0) \leq \frac{1}{f - a_i} \leq \frac{n_i}{n} T(r, f) + O(1). \]

From this and by Lemma 2.1,
\[ (q - 2)T(r, f) \leq \sum_{i=1}^{q} \frac{n_i}{n} \mu_f^a(z_0) - \log r + O(1) \]
\[ \leq \sum_{i=1}^{q} \frac{n_i}{n} N(r, f - a_i) - \log r + O(1) \]
Since $q \geq 2 + \sum_{i=1}^{\infty} \frac{n_i}{m_i}$, we obtain a contradiction.

Lemma 3.3. Let $n, m \in \mathbb{N}^*$, $n \geq 2m + 8$, $a_1, b_1, a_2, b_2, c, e \in \mathbb{K}$, $a_1 \neq 0$, $b_1 \neq 0$, $a_2 \neq 0$, $b_2 \neq 0$, $c \neq 0$, and let $f_1, f_2, g_1, g_2$ be non-zero entire functions.

1. Suppose that $\frac{f_1}{f_2}$ is a non-constant meromorphic function, and

\[
(3.5) \quad f_1^n + a_1 f_1^{n-m} f_2^m + b_1 f_2^n = b_2 g_2^n.
\]

Then there exists $c_1 \neq 0$ such that $c_1 b_2 g_2^n = b_1 f_2^n$, $g_2 = h f_2$ with $b_1 = c_1 b_2 h^n$, $h \in \mathbb{K}$.

2. Suppose that $\frac{f_1}{f_2}$ and $\frac{g_1}{g_2}$ are non-constant meromorphic functions, and

\[
(3.6) \quad f_1^n + a_1 f_1^{n-m} f_2^m + b_1 f_2^n = c(g_1^n + a_2 g_1^{n-m} g_2^{m} + b_2 g_2^n).
\]

i. If $m \geq 2$, then

\[
c_1 b_2 g_2^n = b_1 f_2^n, \quad g_2 = h f_2 \text{ with } b_1 = c_1 b_2 h^n, \quad h \in \mathbb{K}.
\]

ii. If $m \geq 3$, then

\[
g_1 = l f_1, \quad g_2 = h f_2 \text{ with } 1 = c l^n, a_1 = c a_2^{n-m} h^n, b_1 = c b_2 h^n, \quad l, h \in \mathbb{K}.
\]

Proof. 1. From (3.5) we have

\[
(3.7) \quad f_1^n + a_1 f_1^{n-m} f_2^m + b_1 f_2^n = 0.
\]

Note that $x_1^{n-m}(x_1^n + a_1 x_1^m), b_1 x_2^m, -b_2 x_2^n$ are the homogeneous polynomials of degree $n$ in general position. Since $n \geq 2m + 8$ and by Lemma 3.1, there exists $c_1 \neq 0$ such that $c_1 b_2 g_2^n = b_1 f_2^n$. Therefore $g_2 = h f_2$ with $b_1 = c_1 b_2 h^n$, $h \in \mathbb{K}$.

2. We consider the possible cases:

Case 1: $c = 1$. Then

\[
(3.8) \quad f_1^n + a_1 f_1^{n-m} f_2^m + b_1 f_2^n = g_1^n + a_2 g_1^{n-m} g_2^m + b_2 g_2^n.
\]

i.e.,

\[
(3.9) \quad b_1 f_2^n + f_1^{n-m}(f_1^n + a_1 f_2^n) - b_2 g_2^n - g_1^{n-m}(g_1^n + a_2 g_2^m) = 0.
\]

Note that $b_1 x_1^n, x_2^{n-m}(x_2^m + a_1 x_1^n), -b_2 x_2^n, -x_4^{n-m}(x_4^m + a_2 x_3^n)$ are the homogeneous polynomials of degree $n$ in general position. Since $n \geq 2m + 8$ and by Lemma 3.1, there exist constants $C_1, C_2, C_3, (C_1, C_2, C_3) \neq (0, 0, 0), \text{ such that}$

\[
(3.10) \quad C_1 b_1 f_2^n + C_2 f_1^n + C_3 f_2^n = 0.
\]

We consider the following possible subcases:

Subcase 1: $C_3 = 0$. Then from (3.10) we have

\[
C_1 b_1 f_2^n + C_2 f_1^n + C_3 f_2^n = 0.
\]
Since $f_2$ is a non-zero entire function, we have $C_2 \neq 0$. If $C_1 = 0$, then $\frac{f_1}{f_2}$ is a constant, a contradiction. So $C_1, C_2 \neq 0$. Then $\frac{f_1}{f_2}$ is a constant, a contradiction. So $C_3 \neq 0$.

**Subcase 2:** $C_2 = 0$. Then from (3.10) we have $C_1b_1f_2^n + C_3b_2g_2^n = 0$. Because $f_2, g_2$ are non-zero entire functions, we have $C_1 \neq 0, C_3 \neq 0$. From this and (3.9) it follows that $g_2^n = -\frac{C_3}{C_2}f_2^n$, $g_2^n = h, h \in \mathbb{K}, h \neq 0$, and

$$b_1 \left(1 + \frac{C_1}{C_2}\right)f_2^n + f_1^{n-m}(f_1^{m} + a_1f_2^m) - g_1^{n-m}(g_1^{m} + a_2g_2^m) = 0,$$

(3.11) $-g_1^n + f_1^{n-m}(f_1^{m} + a_1f_2^m) + \left(b_1(1 + \frac{C_1}{C_2})f_2^{n-m} - a_2h^m g_1^{n-m}\right)f_2^m = 0.$

Suppose that $1 + \frac{C_1}{C_2} \neq 0$. Then, from the similarity of (3.11) and (3.9), by a similar argument as in (3.9), there exist constants $C'_1, C'_2, (C'_1, C'_2) \neq (0, 0)$, such that

$$C'_2g_1^n + C'_1f_1^{n-m}(f_1^{m} + a_1f_2^m) = 0.$$  

(3.12) $C'_1 f_1^{n-m}(f_1^{m} + a_1f_2^m) = -C'_2g_1^n, C'_1 \left(\frac{f_1}{f_2}\right)^n + C'_1d_1 \left(\frac{f_1}{f_2}\right)^{n-m} = -C'_2(\frac{g_1}{f_2})^n,$

(3.13) $C'_2 \left(\frac{f_1}{f_2}\right)^{n-m} \left(\frac{f_1}{f_2}\right)^m + a_1 = -C'_2(\frac{g_1}{f_2})^n.$

Note that the equation $z^m + a_1 = 0$ has $m$ distinct roots $d_1, d_2, \ldots, d_m$. Set $f = \frac{f_1}{f_2}$, $g = \frac{g_1}{f_2}$. Consequently, by (3.13) we have

$$f^{n-m}(f - d_1) \cdots (f - d_m) = Cg^n, C \neq 0.$$  

(3.14) $f^{n-m}(f - d_1) \cdots (f - d_m) = Cg^n, C \neq 0.$

Since $\frac{f_1}{f_2}$ is not a constant, neither is $\frac{g_1}{f_2}$. By $m \geq 2, n \geq 2m + 8$ we have $m + 1 \geq 2 + \frac{n-m}{n} + \sum_{i=1}^{m-1} \frac{1}{m}$. Then applying Lemma 3.2 to (3.14) with $q = m + 1$, $n = n, n_1 = n - m, n_2 = 1 = n_3 = \cdots = n_m$, we have a contradiction. So $1 + \frac{C_1}{C_2} = 0$. Therefore $cb_2g_2^n = b_1f_2^n$, and $g_2 = hf_2$ with $b_1 = cb_2h^n$.

**Subcase 3:** $C_1 = 0$. From (3.10) we have $C_2 f_1^{n-m}(f_1^{m} + a_1f_2^m) + C_3b_2g_2^n = 0$. Then, from the similarity of this equation and (3.12), by a similar argument as in (3.12) we have a contradiction.

**Subcase 4:** $C_1 \neq 0, C_2 \neq 0, C_3 \neq 0$.

By a similar argument as in (3.7) we obtain a contradiction. So $b_2g_2^n = b_1f_2^n$, $g_2 = hf_2, h \in \mathbb{K}, h \neq 0$, with $b_1 = 2b_2h^n$.

**Case 2.** $c \neq 0$. Set $b^n = c_1, c_2 = bg_2$. From this and (3.6) we get

$$f_1^{n} + a_1f_1^{n-m}f_2^n + b_1f_2^n = c_1^n + a_2c_1^{n-m}e_2^m + b_2e_2^n.$$

Applying the case with $c = 1$ here we obtain $b_2e_2^n = b_2b_1^n g_2^n = b_2g_2^n = b_1f_2^n$, $g_2 = hf_2$ with $b_1 = cb_2h^n$.
2.ii. \( m \geq 3 \). From (3.6) we have
\[
b_1 f_1^n \left( \frac{1}{b_1} f_1^n + \frac{a_1}{b_1} f_1^{n-m} + 1 \right) = cb_2 g_2^n \left( \frac{1}{b_2} g_2^n + \frac{a_2}{b_2} g_2^{n-m} + 1 \right),
\]
where \( f = \frac{f_1}{f_1^n} \), \( g = \frac{g_2}{g_2^n} \). Set \( \frac{1}{b_1} = a_3 \), \( \frac{a_1}{b_1} = b_3 \), \( \frac{1}{b_2} = a_4 \), \( \frac{a_2}{b_2} = b_4 \). Since \( cb_2 g_2^n = b_1 f_1^n \)
\[
a_3 f_1^n + b_3 f_1^{n-m} = a_4 g_2^n + b_4 g_2^{n-m}.
\]
Set \( h_1 = \frac{g}{f} \). From this we obtain
\[
a_3 f_1^n + b_3 = a_4 \left( \frac{g}{f} \right)^n f_1^n + b_4 \left( \frac{g}{f} \right)^{n-m}, \quad a_3 f_1^n + b_3 = a_4 h_1^n + b_4 h_1^{n-m},
\]
(3.15) \( f_1^n (a_3 - a_4 h_1^n) = b_4 h_1^{n-m} - b_3 - \frac{a_4 (h_1^n - \frac{a_3}{h_1^n})}{b_4 (h_1^{n-m} - \frac{a_3}{h_1^n})} = \left( \frac{1}{f} \right)^m \).
Assume that \( h_1 \) is not a constant. Note that the equation \( z^n - \frac{a_3}{h_1^n} = 0 \) has \( n \)
simple roots, the equation \( z^{n-m} - \frac{a_4}{b_4} = 0 \) has \( n - m \) simple roots. Then the
equations \( z^n - \frac{a_3}{h_1^n} = 0, z^{n-m} - \frac{a_4}{b_4} = 0 \) have at most \( n - m \) common simple roots.
Therefore the equation \( z^n - \frac{a_3}{h_1^n} = 0 \) has at least \( m \) distinct roots, which are
not roots of the equation \( z^{n-m} - \frac{a_4}{b_4} = 0 \). Let \( r_1, r_2, \ldots, r_m \) be all these roots.
Then, from (3.15) we see that all the simple zeros of the equations \( h_1 - r_j, \)
\( j = 1, \ldots, m \), have multiplicities \( \geq m \). By Lemma 2.3 we have \( m(1 - \frac{1}{m}) < 2 \).
Therefore \( 0 < m < 3 \). From \( m \geq 3 \), we obtain a contradiction. Thus \( h_1 \)
is constant and so is \( g_1 = f_1 \). Consequently, \( g_1 = f_1, g_2 = h f_2 \). From that and
since \( \frac{f_2}{f_1} \) is not a constant we obtain \( 1 = d^m, a_1 = co \) \( d^{n-m} h^m, b_1 = cb h^n \). \( \Box \)

Now we use the above lemmas to prove the main result of the paper.

Proof of Theorem 1.1. Set \( L_i(f) = L_i(f_1, \ldots, f_{N+1}), L_i(g) = L_i(g_1, \ldots, g_{N+1}), \)
\( i = 1, \ldots, q \). We first prove \( P_i(f) \neq 0, i = 1, 2, \ldots, q \). With \( q > N \), by induction on \( i \). With \( i = 1 \) assume that
\[
P_1(f) = L_1^0(f) - aL_1^{n-m}(f)L_1^0(f) + bL_1^0(f) \equiv 0.
\]
It follows from this and \( L_2^0(f) \neq 0 \) that \( \frac{L_1(f)}{L_2^0(f)} \) is a constant, and we have a
contradiction to the linearly independence of \( f_1, \ldots, f_{N+1} \). With \( i = 2 \), assume that
\[
P_2(f) = P_2^1(f) - aP_1^{n-m}(f)L_2^0(f) + bL_2^0(f) \equiv 0.
\]
Since \( P_1(f) \neq 0, L_3^0(f) \neq 0 \) we see that \( \frac{P_2(f)}{L_2^0(f)} \) is a constant. Hence
\[
L_1^0(f) - aL_1^{n-m}(f)L_1^0(f) + bL_2^0(f) = AL_3^0(f) \equiv 0, A \neq 0.
\]
Since \( L_1(f) \neq 0, L_2(f) \neq 0, L_3(f) \neq 0 \) and \( n \geq 2m + 8, m \geq 3 \), we deduce
from Lemma 3.3 that \( \frac{L_1(f)}{L_3(f)} \) is a constant, and we have a contradiction to the
linearly independence of \( f_1, \ldots, f_{N+1} \).
Now we consider $P_i(\bar{f}) \equiv 0$. Then

\[(3.16) \quad P^n_{i-1}(\bar{f}) - aP^{n-m}_{i-1}(\bar{f})L_i^{n-1-m}(\bar{f}) + bl_i^{n+1}(\bar{f}) \equiv 0.\]

Applying the induction hypothesis and by a similar argument as above we have a contradiction.

Next we consider

\[(3.17) \quad P_i(\bar{f}) = A_iP_i(\bar{g}). \quad A_i \neq 0, \quad i = 1, 2, \ldots, q.\]

We will show that $L_j(\bar{g}) = c_jL_j(\bar{f})$, $c_j \neq 0$, $j = 1, \ldots, i + 1$, by induction on $i$. With $i = 1$ we get $P_1(\bar{f}) = A_1P_1(\bar{g})$.

$L^n_i(\bar{f}) - aL_1^{n-m}(\bar{f})L_2^m(\bar{f}) + bl_2^i(\bar{f}) = A_1(L^n_i(\bar{g}) - aL_1^{n-m}(\bar{g})L_2^1(\bar{g}) + bl_2^i(\bar{g})).$

Since $L_1(\bar{f}) \neq 0$, $L_2(\bar{f}) \neq 0$, $L_1(\bar{g}) \neq 0$, $L_2(\bar{g}) \neq 0$ and $n \geq 2m + 8$, $m \geq 3$, we deduce from Lemma 3.3 and the above equation that $L_j(\bar{g}) = c_jL_j(\bar{f})$, $c_j \neq 0$, $j = 1, 2$. Now we consider (3.17). Then

\[(3.18) \quad P^n_{i-1}(\bar{f}) - aP^{n-m}_{i-1}(\bar{f})L_i^{n-1-m}(\bar{f}) + bl_i^{n+1}(\bar{f}) = A_i(P^n_{i-1}(\bar{g}) - aP^{n-m}_{i-1}(\bar{g})L_i^{n-1-m}(\bar{g}) + bl_i^{n+1}(\bar{g})).\]

Since $P_{i-1}(\bar{f}) \neq 0$, $L_{i+1}(\bar{f}) \neq 0$, $P_{i-1}(\bar{g}) \neq 0$, $L_{i+1}(\bar{g}) \neq 0$ and $n \geq 2m + 8$, $m \geq 3$, we deduce from Lemma 3.3 and (3.18) that

\[P_{i-1}(\bar{f}) = B_{i-1}P_{i-1}(\bar{g}), \quad L_i^{n-1}(\bar{g}) = C_{i+1}L_i^{n-1}(\bar{f}).\]

Applying the induction hypothesis here we have $L_j(\bar{g}) = c_jL_j(\bar{f})$, $c_j \neq 0$, $j = 1, 2, \ldots, i + 1$.

Now we can return to the proof of Theorem 1.1. Consider

\[(3.19) \quad P(\bar{f}) = P(\bar{g}), \quad q > N.\]

From (3.17) we get $L_i(\bar{g}) = c_iL_i(\bar{f})$, $c_i \neq 0$, $i = 1, \ldots, q + 1$. Since $L_i, i = 1, \ldots, N + 1$, are linearly independent and $L_1, \ldots, L_{N+1}, L_j, j \in \{N + 2, \ldots, q + 1\}$ are linearly dependent we get

\[L_j = b_{1j}L_1 + b_{2j}L_2 + \cdots + b_{N+1j}L_{N+1}, b_{kj} \neq 0, \quad k = 1, \ldots, N + 1, \quad j = N + 2, \ldots, q + 1;\]

\[L_j(\bar{f}) = b_{1j}L_1(\bar{f}) + b_{2j}L_2(\bar{f}) + \cdots + b_{N+1j}L_{N+1}(\bar{f}), \quad j = N + 2, \ldots, q + 1;\]

\[L_j(\bar{g}) = b_{1j}L_1(\bar{g}) + b_{2j}L_2(\bar{g}) + \cdots + b_{N+1j}L_{N+1}(\bar{g}), \quad j = N + 2, \ldots, q + 1.\]

From this and $L_i(\bar{g}) = c_iL_i(\bar{f})$, $c_i \neq 0$, $i = 1, 2, \ldots, N + 1$; $L_j(\bar{g}) = c_jL_j(\bar{f})$, we obtain

\[L_j(\bar{g}) = c_1b_{1j}L_1(\bar{f}) + c_2b_{2j}L_2(\bar{f}) + \cdots + c_{N+1}b_{N+1j}L_{N+1}(\bar{f});\]

\[c_1b_{1j}L_1(\bar{f}) + c_2b_{2j}L_2(\bar{f}) + \cdots + c_{N+1}b_{N+1j}L_{N+1}(\bar{f})\]

\[= c_jb_{1j}L_1(\bar{f}) + c_jb_{2j}L_2(\bar{f}) + \cdots + c_jb_{N+1j}L_{N+1}(\bar{f}), \quad j = N + 2, \ldots, q + 1.\]
By the linear independence of $f_1, \ldots, f_{N+1}$ we obtain $c_j = c_1 = c_j = c_2 = \cdots = c_{N+1}, j = N + 2, \ldots, q + 1$. Set $c = c_i, i = 1, \ldots, q + 1$. Then $L_j(\tilde{g}) = cL_j(\tilde{f}), j = 1, \ldots, q + 1$. Then $g_i = cf_i, i = 1, \ldots, N + 1, c^{\alpha_1} = 1$.

Now we are going to complete the proof of Theorem 1.2

Proof of Theorem 1.2. Let $\tilde{f} = (f_1, \ldots, f_{N+1})$ and $\tilde{g} = (g_1, \ldots, g_{N+1})$ be reduced representations of $f$ and $g$, respectively.

Since $\mu_f(X) = \mu_g(X)$, it is easy to see that there exists a non-zero constant $c$ such that $P(\tilde{f}) = cP(\tilde{g})$. Set $l^\alpha = c$ and $\tilde{h} = (lg_1, \ldots, lg_{N+1})$. Then $\tilde{h}$ is a reduced representation of $g$ and $P_q(\tilde{f}) = P_q(\tilde{h})$. By Theorem 1.1, $f \equiv g$. □

References

[22] F. Pakovich, On the equation $P(f) = Q(g)$, where $P, Q$ are polynomials and $f, g$ are entire functions, Amer. J. Math. 132 (2010), no. 6, 1591–1607.

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