A REFINEMENT OF THE UNIT AND UNITARY CAYLEY GRAPHS OF A FINITE RING

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Abstract. Let $R$ be a finite commutative ring with nonzero identity. We define $\Gamma(R)$ to be the graph with vertex set $R$ in which two distinct vertices $x$ and $y$ are adjacent if and only if there exists a unit element $u$ of $R$ such that $x + uy$ is a unit of $R$. This graph provides a refinement of the unit and unitary Cayley graphs. In this paper, basic properties of $\Gamma(R)$ are obtained and the vertex connectivity and the edge connectivity of $\Gamma(R)$ are given. Finally, by a constructive way, we determine when the graph $\Gamma(R)$ is Hamiltonian. As a consequence, we show that $\Gamma(R)$ has a perfect matching if and only if $|R|$ is an even number.

1. Introduction

Throughout this paper, $R$ is a finite commutative ring with nonzero identity. The group of units and the Jacobson radical of $R$ are denoted by $U(R)$ and $J(R)$, respectively. The unit graph $G(R)$ is the graph with vertex set $R$ in which two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in U(R)$. Unit graphs were introduced in [2] and their properties were investigated in [7], [16], [17] and [19]. The unitary Cayley graph $G_R$ is the graph with vertex set $R$ such that two distinct vertices $x$ and $y$ are adjacent if and only if $x - y \in U(R)$. Unitary Cayley graphs were introduced in [8] and their properties were investigated in [1], [10], [11], [12] and [15]. For example, in [10] the chromatic number, clique number and independence number of $G_R$ are given along with other results. The authors in [15] give a necessary and sufficient condition for $G_R$ to be Ramanujan graph.

In [9], Khashayarmanesh and Khorsandi provide a generalization of the unit and unitary Cayley graphs as follows: Let $G$ be a multiplicative subgroup of $U(R)$ and $S$ be a non-empty subset of $G$ such that $S^{-1} = \{s^{-1} \mid s \in S\} \subseteq S$. Then $\Gamma(R, G, S)$ is the (simple) graph with vertex set $R$ in which two distinct elements $x, y \in R$ are adjacent if and only if there exists $s \in S$ such that $x + sy \in G$. The authors in [3] derive several bounds for the genus of $\Gamma(R, U(R), S)$.
this paper, we use $\Gamma(R)$ to denote the graph $\Gamma(R, U(R), U(R))$. For a subset $C$ of $R$, the induced subgraph of $\Gamma(R)$ over $C$ is denoted by $\Gamma(C)$.

We recall that a ring $R$ is said to have unit 1-stable range if, whenever $Rx + Ry = R$ ($x, y \in R$), there exists $u \in U(R)$ such that $x + uy \in U(R)$. We refer the reader to [6] and [13] for more information about unit 1-stable range rings.

In [18], Sharma and Bhatwadekar defined another graph on $R$, $\Omega(R)$, with vertices the elements of $R$, in which two distinct vertices $x$ and $y$ are adjacent if and only if $Rx + Ry = R$. It is easy to see that $\Gamma(R)$ is a subgraph of $\Omega(R)$. The concepts of $\Gamma(R)$ and $\Omega(R)$ give an interesting graph interpretation of unit 1-stable range rings. In fact, a commutative ring $R$ has unit 1-stable range if and only if $\Gamma(R) \cong \Omega(R)$. This provides a motivation to introduce and study the properties of $\Gamma(R)$.

For a graph $G$, $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively. A graph $G$ is called a refinement of a graph $H$ if $V(G) = V(H)$ and if $x, y$ are adjacent in $H$, then $x, y$ are adjacent in $G$. We mention that “$G$ is a refinement of $H$” has the same meaning as “$H$ is a spanning subgraph of $G$”. We note that $\Gamma(R)$ is a refinement of both $G(R)$ and $G_R$. If we omit the word “distinct”, we obtain the graph $\overline{\Gamma}(R)$; this graph may have loops. Some examples of this kind of graphs are displayed in Figure 1.

![Figure 1](image_url)

**Figure 1.** The graphs $\Gamma(R)$ and $\overline{\Gamma}(R)$ of the specific rings $R$. 
For a local ring $R$, we have the following immediate result about the loops of $\Gamma(R)$.

**Proposition 1.1.** Let $R$ be a local ring with maximal ideal $m$. Then

1. If $|R/m| = 2$, then $\Gamma(R)$ has no loop (i.e., $\Gamma(R) = \Gamma(R)$);
2. If $|R/m| \neq 2$, then only the elements of $U(R)$ have a loop in $\Gamma(R)$.

A graph $G$ in which each pair of distinct vertices is joined by an edge is called a complete graph. We use $K_n$ to denote the complete graph with $n$ vertices. For a graph $G$ and vertex $x \in V(G)$, the degree of $x$, denoted by $\deg(x)$, is the number of edges of $G$ incident with $x$. The minimum degree of $G$ is denoted by $\delta(G)$. For $x \in V(G)$, we denote by $N_G(x)$ the set of all vertices of $G$ adjacent to $x$.

A graph $G$ is called bipartite if $V(G)$ admits a partition into two classes such that vertices in the same partition class must not be adjacent. A simple bipartite graph in which every two vertices from different partition classes are adjacent is called a complete bipartite graph, denoted by $K_{m,n}$, where $m$ and $n$ are the sizes of the partition classes. A clique is a set of pairwise adjacent vertices of $G$ (any complete subgraph). The largest integer $n$ such that $K_n$ is a subgraph of $G$ is the clique number $\omega(G)$ of $G$. An independent set is a set of pairwise non-adjacent vertices of $G$. A walk from $x$ to $y$ is an ordered list of vertices (not necessarily distinct) $x = v_0, v_1, \ldots, v_n = y$ such that $v_i$ is adjacent to $v_{i-1}$ for $i = 1, \ldots, n$. We denote this walk by $x \cdots v_i \cdots v_n$. A path of length $n$ is an ordered list of distinct vertices $v_0, v_1, \ldots, v_n$ such that $v_i$ is adjacent to $v_{i-1}$ for $i = 1, \ldots, n$. We denote this path by $v_0 \cdots v_i \cdots v_n$. A cycle is a path $v_0 \cdots v_i \cdots v_n$ with an extra edge $v_0 - v_n$. The union of two simple graphs $G$ and $H$ is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $V(G)$ and $V(H)$ are disjoint, we refer to $G \cup H$ as a disjoint union, and denote it by $G + H$. The join of simple graphs $G$ and $H$, written $G \vee H$, is the graph obtained from the disjoint union $G + H$ by adding edges joining every vertex of $G$ to every vertex of $H$.

A Hamiltonian cycle in a graph $G$ is a cycle containing every vertex of $G$ and $G$ is called a Hamiltonian graph if it contains a Hamiltonian cycle. For other notions not mentioned in this paper, one can refer to [4] and [20].

The plan of this paper is as follows: In Section 2, we give some basic properties of $\Gamma(R)$. In Section 3, we determine the clique number of $\Gamma(R)$. In Section 4, by a constructive way, we determine when the graph $\Gamma(R)$ is Hamiltonian. Finally, we determine when the graph $\Gamma(R)$ has a perfect matching.

### 2. Basic properties of $\Gamma(R)$

In this section, we study some basic properties of $\Gamma(R)$. We begin with the following lemma.
Lemma 2.1. Let $R$ be a ring. Then each element of $U(R)$ is adjacent to all elements of $J(R)$.

Proof. Let $x \in U(R)$ and $y \in J(R)$. Suppose on the contrary that $x$ and $y$ are not adjacent. Then $x + uy \notin U(R)$ for all $u \in U(R)$, and so $x - y \notin U(R)$. Therefore there exists a maximal ideal $m$ of $R$ such that $x - y \in m$. This implies that $x \in m$, which is a contradiction. This completes the proof. □

Let $R$ be a ring with maximal ideal $m$ such that $|R/m| = 2$. Then it is easy to see that $\Gamma(R)$ is a bipartite graph. In the next section, we show that the converse of this result is also true (see Corollary 3.2).

In the following theorem, we determine when $\Gamma(R)$ is a complete bipartite graph.

Theorem 2.2. Let $R$ be a ring with maximal ideal $m$ such that $|R/m| = 2$. Then $\Gamma(R)$ is a complete bipartite graph if and only if $R$ is a local ring.

Proof. Suppose that $\Gamma(R)$ is a complete bipartite graph with bipartition $\{V_1, V_2\}$. First we show that $U(R)$ is an independent set of $\Gamma(R)$. Suppose on the contrary that $U(R)$ is not an independent set of $\Gamma(R)$. Then there exist $x, y \in U(R)$ such that $x$ is adjacent to $y$. So, there exists $u \in U(R)$ such that $x + uy \in U(R)$. Since $|R/m| = 2$, there are $m_1, m_2 \in m$ such that $x = 1 + m_1$ and $y = 1 + m_2$. This implies that $1 + m_1 + u + um_2 \in U(R)$. On the other hand, $1 + u \in m$, because $|R/m| = 2$. Therefore we have $1 + u + m_1 + um_2 \in m$, which is a contradiction. Since $\Gamma(R)$ is a complete bipartite graph and $U(R)$ is an independent set of $\Gamma(R)$, without loss of generality, we may assume that $U(R) \subseteq V_1$. We claim that $V_1 = U(R)$. Suppose on the contrary that there exists $v_1 \in V_1 \setminus U(R)$. Then there exists a maximal ideal $n$ of $R$ such that $v_1 \in n$. Since the distinct elements of a maximal ideal can not be adjacent, $n \subseteq V_1$ and so $J(R) \subseteq n \subseteq V_1$, which is a contradiction, by the above lemma. Therefore, $V_1 = U(R)$. It follows that $m \subseteq V_2$. Now we show that $V_2 = m$. Suppose on the contrary that there exists $v_2 \in V_2 \setminus m$. Then $v_2 = 1 + m$ for some $m \in m$. By the assumption, 1 is adjacent to $v_2$, and hence there exists $u_0 \in U(R)$ such that $(1 + m) + u_0, 1 = 1 + m + u_0 \in U(R)$. Hence $1 + m + u_0 = 1 + m_0$ for some $m_0 \in m$. Therefore $u_0 = m_0 - m$, which is a contradiction. Thus $V_2 = m$. It follows that $R$ is a local ring.

The converse follows easily from [9, Proposition 3.2]. □

If $R$ is a local ring with maximal ideal $m$ such that $|R/m| = 2$, then by the above theorem $\deg(x) = |U(R)|$ for each $x \in R$. In the case where $|R/m| > 2$, the following theorem determines the degree of vertices of $\Gamma(R)$.

Theorem 2.3. Let $R$ be a local ring with maximal ideal $m$ such that $|R/m| > 2$ and let $x \in R$. Then

$$\deg(x) = \begin{cases} |R| - 1 & \text{if } x \in U(R), \\ |U(R)| & \text{otherwise}. \end{cases}$$
Proof. Let \( m, u_1 + m, \ldots, u_t + m \) be the set of all distinct cosets of \( R/m \), where \( u_i \in U(R) \) for \( i = 1, \ldots, t \). Let \( x_i \in u_i + m \) and \( x_j \in u_j + m \), where \( i, j \) are two distinct elements of \( \{1, \ldots, t\} \). We claim that \( x_i \) and \( x_j \) are adjacent. Suppose on the contrary that \( x_i \) and \( x_j \) are not adjacent. Therefore, \( u_i + u_j \in m \) for all \( u \in U(R) \) and so \( u_i - u_j \in m \), which is a contradiction. Now let \( k \in \{1, \ldots, t\} \). We show that every pair of elements of the coset \( u_k + m \) are adjacent. Suppose on the contrary that there exist two distinct elements \( m_1, m_2 \in m \) such that \( u_k + m_1 \) and \( u_k + m_2 \) are not adjacent. Then \( (u_k + m_1) + u(u_k + m_2) \in m \) for all \( u \in U(R) \). We conclude that \( u_k(1 + u) \in m \) for all \( u \in U(R) \) and so \( 1 - u \in m \) for all \( u \in U(R) \). This implies that \( |R/m| = 2 \), which is a contradiction. It is clear that the elements of \( u_i + m \) are adjacent to the elements of \( m \), for all \( i = 1, \ldots, t \) and also no pair of elements of \( m \) are adjacent. These observations complete the proof.

\( \square \)

**Theorem 2.4.** Let \( R \) be a ring. Suppose that \( \Gamma(R) \) is a complete \( n \)-partite graph. Then the following hold:

1. \( R \) is a local ring;
2. \( n = 2 \) or \( n = |U(R)| + 1 \).

Proof. (1) Suppose that \( V \) is the part containing zero. We show that \( V = R \setminus U(R) \). For any \( x \in V \) and any \( u \in U(R) \), we have \( ux \notin U(R) \). Therefore \( V \subseteq R \setminus U(R) \). Now let \( y \) be an element of \( R \setminus U(R) \) such that \( y \notin V \). So \( y \) is adjacent to zero and hence \( uy \in U(R) \), for some \( u \in U(R) \). This yields \( y \in U(R) \), which is a contradiction. Hence \( V = R \setminus U(R) \). Let \( m_1, m_2 \) be two distinct maximal ideals of \( R \). Then \( m_1 + m_2 = R \) and hence \( x + y = 1 \) for some \( x \in m_1 \) and \( y \in m_2 \). Therefore \( x \) and \( y \) are adjacent elements of \( V \), which is a contradiction. This implies that \( R \) is a local ring.

(2) First suppose that \( |R/m| = 2 \). Then \( n = 2 \), by Theorem 2.2. Now let \( |R/m| > 2 \) and \( U(R) = \{u_1, \ldots, u_t\} \). For any \( 1 \leq i \leq t \), we set \( V_i = \{u_i\} \) and \( V_{i+1} = m \). Therefore \( \Gamma(R) \) is a complete \((t + 1)\)-partite graph by Theorem 2.3. This completes the proof.

\( \square \)

**Theorem 2.5.** Let \( R \) be a ring, with exactly two maximal ideal, say \( m_1 \) and \( m_2 \). Then \( \Gamma(R) \) is connected if and only if \( |R/m_1| \neq 2 \) or \( |R/m_2| \neq 2 \).

Proof. Suppose that \( \Gamma(R) \) is not connected. In view of Lemma 2.1 and the fact that every element of \( (m_1 \setminus m_2) \) is adjacent to every element of \( (m_2 \setminus m_1) \), there are two components \( V_1 \) and \( V_2 \) of \( \Gamma(R) \) such that \( V_1 = J(R) \cup U(R) \) and \( V_2 = (m_1 \setminus m_2) \cup (m_2 \setminus m_1) \). We show that \( |R/m_1| = 2 \). Suppose on the contrary that \( |R/m_1| \neq 2 \). So there exists \( x \in R \setminus m_1 \) such that \( 1 - x \notin m_1 \). Then \( 1 - x \in m_2 \setminus m_1 \) or \( 1 - x \in U(R) \). First suppose that \( 1 - x \in m_2 \setminus m_1 \). So \( x \notin m_2 \). Therefore \( x \in U(R) \subseteq V_1 \) and \( 1 - x \in V_2 \), which is a contradiction.

Now suppose that \( 1 - x \in U(R) \). Then \( x \notin m_2 \setminus m_1 \), for otherwise \( 1 \) is adjacent to \( x \), which is a contradiction. Hence \( x \in U(R) \). Since \( R/m_1 \) is a field, there is \( v \in R \setminus m_1 \) such that \( 1 - vx \in m_1 \). We consider the following four cases:
Case 1: $1 - vx \in m_1 \setminus m_2$ and $v \in U(R)$. In this case, we have $vx + (1 - vx) \in U(R)$, which is a contradiction.

Case 2: $1 - vx \in m_1 \setminus m_2$ and $v \in m_2 \setminus m_1$. It follows that $1 - v \not\in U(R) \cup m_2$, and hence $1 - v \in m_1 \setminus m_2$. Now we conclude that $1 - vx - 1 + v \in m_1$ and therefore $v(1 - x) \in m_1$. Since $1 - x$ is unit, we must have $v \in m_1$, which is a contradiction.

Case 3: $1 - vx \in J(R)$ and $v \in m_2 \setminus m_1$. Then it is clear that $vx \in m_2 \setminus m_1$.

But we have $1 - vx + vx \in U(R)$, which is a contradiction.

Case 4: $1 - vx \in J(R)$ and $v \in U(R)$. Let $a$ be an arbitrary element of $m_1 \setminus m_2$. Then we have $a(1 - x) + vx \not\in U(R)$, since $a(1 - x)$ is not adjacent to $v$. Also if $a(1 - x) + vx \in m_1$, then we conclude that $vx \in m_1$, which is a contradiction, and therefore $a(1 - x) + vx \in m_2 \setminus m_1$. Now according to the assumption that $1 - vx \in J(R)$, we have

$$(2.1) 1 + a - ax \in m_2 \setminus m_1.$$ 

Since $1$ is not adjacent to $a$, we have $1 - ax \not\in U(R)$. Also if $1 - ax \in m_1 \setminus m_2$, we conclude that $1 \in m_1 \setminus m_2$, which is a contradiction. So $1 - ax \not\in m_2 \setminus m_1$.

By (2.1), we obtain $a \in m_2 \setminus m_1$, which is a contradiction. Hence the first assumption is not true and therefore $|R/m_1| = 2$. A similar argument shows that $|R/m_2| = 2$.

Conversely, let $|R/m_1| = |R/m_2| = 2$. It is enough to show that every element of $U(R)$ is not connected to elements of $(m_1 \setminus m_2) \cup (m_2 \setminus m_1)$. Let $z \in m_1 \setminus m_2$ and $u$ be an arbitrary element of $U(R)$. Suppose on the contrary that $u$ is adjacent to $z$. Then $u + vz \in U(R)$ for some $v \in U(R)$. Since $|R/m_1| = |R/m_2| = 2$, we have $1 - u - vz \in m_1 \cap m_2$. Also, since $|R/m_2| = 2$, we have $1 - u \in m_2$. Hence $vz \in m_2$ and therefore $z \in m_2$, which is a contradiction. A similar argument shows that every element of $U(R)$ is not connected to elements of $m_2 \setminus m_1$. This completes the proof. □

Corollary 2.6. Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a ring such that $R_i$ is a local ring with maximal ideal $m_i$. Then $\Gamma(R)$ is connected if and only if $R/J(R)$ has at most one $\mathbb{Z}_2$ as a summand.

Proof. Suppose that $R/J(R)$ has at least two $\mathbb{Z}_2$ as summands. Without loss of generality, we may assume $|R_1/m_1| = |R_2/m_2| = 2$. Let $S := R_1 \times R_2$. By the above theorem $\Gamma(S)$ is disconnected and therefore it is easy to see that $\Gamma(R)$ is disconnected. Conversely, suppose that $R/J(R)$ has at most one $\mathbb{Z}_2$ as a summand. Let $(u_1, \ldots, u_n) \in U(R)$, $m_1 \in m_1$ and let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be arbitrary vertices of $\Gamma(R)$. Put $M := (m_1, u_2, \ldots, u_n)$ and $U := (u_1, \ldots, u_n)$ such that $U \not\in \{X, Y\}$. We consider the following two cases:

Case 1: $|R_i/m_i| > 2$ for all $1 \leq i \leq n$. Then, by Theorem 2.3, $X - U - Y$ is a path between $X$ and $Y$. So $\Gamma(R)$ is connected in this case.

Case 2: $|R_i/m_i| = 2$ and $|R_i/m_i| > 2$ for all $2 \leq i \leq n$. First suppose that $x_1, y_1 \in m_1$. Then $X - U - Y$ is a path from $X$ to $Y$. If $x_1, y_1 \in U(R_1)$, then we have the path $X - M - Y$ from $X$ to $Y$. Now, suppose that $x_1 \in m_1$ and
with maximal ideal $m$. In this case $X \rightarrow U \rightarrow M \rightarrow Y$ is a path from $X$ to $Y$. If $x_1 \in U(R_1)$ and $y_1 \in m_1$, a similar argument shows that $X$ is connected to $Y$. Therefore $\Gamma(R)$ is connected. \hfill \Box

3. Clique number

The purpose of this section is to determine the clique number of $\Gamma(R)$. 

**Theorem 3.1.** Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a ring, where $R_i$ is a local ring with maximal ideal $m_i$. Then

$$\omega(\Gamma(R)) = \begin{cases} 2 & \text{if } |R_i/m_i| = 2 \text{ for some } 1 \leq i \leq n, \\ |U(R)| + n & \text{otherwise.} \end{cases}$$

**Proof.** Let $|R_i/m_i| = 2$ for some $1 \leq i \leq n$. Then $M := R_1 \times \cdots \times R_{i-1} \times m_i \times R_{i+1} \times \cdots \times R_n$ is a maximal ideal of $R$ such that $|R/M| = 2$. Therefore the remark before Theorem 2.2 implies that $\omega(\Gamma(R)) = 2$.

Now suppose that $|R_i/m_i| > 2$ for all $1 \leq i \leq n$. We set:

$$S_i := U(R_1) \times U(R_2) \times \cdots \times U(R_{i-1}) \times m_i \times R_{i+1} \times \cdots \times R_n, \quad (1 \leq i \leq n),$$

$$S_{n+1} := U(R_1) \times U(R_2) \times \cdots \times U(R_n).$$

It is easy to see that $S_i \cap S_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i=1}^{n+1} S_i = R$. By Theorem 2.3 and Proposition 1.1, $S_{n+1}$ is a clique. Set

$$C := S_{n+1} \cup \{(0,1,\ldots,1), (1,0,\ldots,1), (1,1,\ldots,0)\}.$$ 

It is easy to see that $C$ is a clique of $\Gamma(R)$. Since $S_i(1 \leq i \leq n)$ is an independent set, every clique of $\Gamma(R)$ contains at most one element of $S_i(1 \leq i \leq n)$. Therefore $\omega(\Gamma(R)) = |U(R_1)| \times |U(R_2)| \times \cdots \times |U(R_n)| + n = |U(R)| + n$. \hfill \Box

**Corollary 3.2.** Let $R$ be a ring such that $\Gamma(R)$ is a bipartite graph. Then there is a maximal ideal $m$ of $R$ such that $|R/m| = 2$.

**Proof.** Let $R = R_1 \times R_2 \times \cdots \times R_n$ such that $R_i$ is a local ring with maximal ideal $m_i$ for $1 \leq i \leq n$ (see [4, Theorem 8.7]). Suppose on the contrary that for all ideals of $R$, we have $|R/m| > 2$. Equivalently, $|R_i/m_i| > 2$ for all $1 \leq i \leq n$. In view of Theorem 3.1, we conclude that

$$|U(R_1)| \times |U(R_2)| \times \cdots \times |U(R_n)| + n = 2.$$ 

So we have $n = 1$ (i.e., $R = R_1$) and $|U(R_1)| = 1$. Suppose that $|R| > 2$. Let $x$ be an element of $R$ such that $x \not\in \{0,1\}$. Then $1+x \not\in U(R)$ and $x \not\in U(R)$. So $1 = (1+x) - x \in m$, which is a contradiction. Therefore $|R| = 2$ and hence $R = \mathbb{Z}_2$, which is again a contradiction. This completes the proof. \hfill \Box
4. Connectivity

In the following, we use \( \kappa(G) \) and \( \kappa'(G) \) to denote the vertex-connectivity and edge-connectivity of a graph \( G \), respectively. The local connectivity between distinct vertices \( x \) and \( y \) is the maximum number of pairwise internally disjoint \( xy \)-paths, denoted by \( p(x, y) \) (see [5, Page 206]). We begin with the following notation:

**Notation.** Let \( S = R_1 \times \cdots \times R_n, T = R_{n+1} \times \cdots \times R_m \) and \( R = S \times T \) such that \( R_i \) is ring for all \( 1 \leq i \leq m \). Suppose that \( X = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_m) \in R, \hat{X} = (x_1, x_2, \ldots, x_n) \in S, \hat{Y} = (x_{n+1}, \ldots, x_m) \in T. \) For convenience, we let \( X \) denote one of the following expressions:

\[
\begin{align*}
(\hat{X}, \hat{Y}), \\
(\hat{X}, x_{n+1}, \ldots, x_m), \\
(x_1, x_2, \ldots, x_n, \hat{Y}).
\end{align*}
\]

**Theorem 4.1.** Let \( R = F_1 \times F_2 \times \cdots \times F_n \) be a ring such that \( F_i \) is field. If \( \Gamma(R) \) is connected, then \( \kappa(\Gamma(R)) = \kappa'(\Gamma(R)) = \delta(\Gamma(R)) = |U(R)|. \)

**Proof.** Since \( \Gamma(R) \) is connected, by Corollary 2.6, we have the following cases:

**Case 1:** \(|F_i| > 2\) for all \( 1 \leq i \leq n \). We decompose \( R \) to the subsets \( S_i \), as defined in Theorem 3.1. Set \( S := S_1 \cup S_2 \cup \cdots \cup S_n \). It is easy to see that \( \Gamma(R) \cong \Gamma(S) \), where the vertex \((0,0,\ldots,0) \in S \) is an isolated vertex, and therefore \( \kappa(S) = 0 \). Also we know that \( \Gamma(S) \cong K_{|U(R)|} \), and hence \( \kappa(\Gamma(S)) = |U(R)| - 1 \). On the other hand, it is clear that \( \delta(\Gamma(S)) = \operatorname{deg}(\{0,0,\ldots,0\}) = |U(R)| \). By using [5, Exercises 9.1.2, 9.3.2], we conclude that \( \kappa(\Gamma(R)) = \kappa'(\Gamma(R)) = \delta(\Gamma(R)) = |U(R)|. \) The assertion is proved.

**Case 2:** \(|F_i| = 2 \) and \(|F_i| > 2\) for all \( 2 \leq i \leq n \). Let \( X := (x_2, \ldots, x_n) \) and \( Y := (y_2, \ldots, y_n) \) be arbitrary distinct elements of \( R \). Let \( \hat{X} := (x_2, \ldots, x_n) \in F_2 \times \cdots \times F_n \) and \( \hat{Y} := (y_2, \ldots, y_n) \in F_2 \times \cdots \times F_n \). We consider the following four subcases:

**Subcase 1.** No entries of \( \hat{X} \) and \( \hat{Y} \) are equal to zero. Thus, \( \hat{X} \) and \( \hat{Y} \) are adjacent in \( \Gamma(F_2 \times \cdots \times F_n) \). Also for each \( A \in (F_2 \setminus \{0\}) \times \cdots \times (F_n \setminus \{0\}) \setminus \{\hat{X}, \hat{Y}\} \), \( \hat{X} \rightarrow A \rightarrow \hat{Y} \) is a path of length two between \( \hat{X} \) and \( \hat{Y} \). The number of such distinct \( A \) is \((f_2-1) \cdots (f_n-1) - 2\). Now we consider the following two cases: If \( x = y \), we choose \( t \in \mathbb{Z}_2 \setminus \{x\} \) and construct the following pairwise internally disjoint paths from \( X \) to \( Y \):

\[
\begin{align*}
X = (x, \hat{X}) \rightarrow (t, A) \rightarrow Y = (x, \hat{Y}), \\
X = (x, \hat{X}) \rightarrow (t, \hat{X}) \rightarrow Y = (x, \hat{Y}), \\
X = (x, \hat{X}) \rightarrow (t, \hat{Y}) \rightarrow Y = (x, \hat{Y}),
\end{align*}
\]

where \( A \in (F_2 \setminus \{0\}) \times \cdots \times (F_n \setminus \{0\}) \setminus \{\hat{X}, \hat{Y}\} \).
If $x \neq y$, we have the following pairwise internally disjoint paths:

$$X = (x, \hat{X}) \rightarrow Y = (y, \hat{Y}),$$
$$X = (x, \hat{X}) \rightarrow (y, A) \rightarrow (x, A) \rightarrow Y = (y, \hat{Y}),$$
$$X = (x, \hat{X}) \rightarrow (y, \hat{X}) \rightarrow (x, \hat{Y}) \rightarrow Y = (y, \hat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}) \setminus \{\hat{X}, \hat{Y}\}$.

Hence, in this case, $p(X, Y) \geq (f_2 - 1) \cdots (f_n - 1) - 2 + 2 = |U(R)| = \delta(\Gamma(R))$.

**Subcase 2.** Both $\hat{X}$ and $\hat{Y}$ have at least one entry which is equal to zero. Then for any $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\})$, $\hat{X} \rightarrow A \rightarrow \hat{Y}$ is a path from $\hat{X}$ to $\hat{Y}$ in $\Gamma(F_2 \times \cdots \times F_n)$. The number of such distinct $A$, and therefore such paths, is $(f_2 - 1) \cdots (f_n - 1)$. We consider the following two cases:

If $x = y$, we construct the following paths from $X$ to $Y$:

$$X = (x, \hat{X}) \rightarrow (t, A) \rightarrow Y = (x, \hat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\})$, $t \in \mathbb{Z}_2 \setminus \{x\}$.

If $x \neq y$, we provide the following internally disjoint paths:

$$X = (x, \hat{X}) \rightarrow (y, A) \rightarrow (x, A) \rightarrow Y = (y, \hat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\})$.

In this case we also deduce that $p(X, Y) \geq (f_2 - 1) \cdots (f_n - 1) = |U(R)| = \delta(\Gamma(R))$.

**Subcase 3.** No entry of $\hat{X}$ is equal to zero and at least one entry of $\hat{Y}$ is zero. Hence for any $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}) \setminus \{\hat{X}\}$, $\hat{X} \rightarrow A \rightarrow \hat{Y}$ is a path from $\hat{X}$ to $\hat{Y}$. Note that $\hat{X}$ has loop and also $\hat{X}$ is adjacent to $\hat{Y}$. The number of such $A$ is $(f_2 - 1) \cdots (f_n - 1) - 1$. We consider the following two cases:

If $x = y$, we provide the following paths from $X$ to $Y$:

$$X = (x, \hat{X}) \rightarrow (t, A) \rightarrow Y = (x, \hat{Y}),$$
$$X = (x, \hat{X}) \rightarrow (t, \hat{X}) \rightarrow Y = (x, \hat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}) \setminus \{\hat{X}\}$.

If $x \neq y$, we have the following paths from $X$ to $Y$:

$$X = (x, \hat{X}) \rightarrow (y, A) \rightarrow (x, A) \rightarrow Y = (y, \hat{Y}),$$
$$X = (x, \hat{X}) \rightarrow (y, \hat{X}) \rightarrow (x, \hat{Y}) \rightarrow Y = (y, \hat{Y}),$$

where $A \in (F_2 \setminus \{0\} \times \cdots \times F_n \setminus \{0\}) \setminus \{\hat{X}\}$.

Therefore, $p(X, Y) \geq (f_2 - 1) \cdots (f_n - 1) - 1 + 1 = |U(R)| = \delta(\Gamma(R))$.

**Subcase 4.** No entry of $\hat{Y}$ is equal to zero and at least one entry of $\hat{X}$ is zero. This subcase is similar to the previous subcase and so we omit the argument. Hence, for every $X, Y \in R$, we have $p(X, Y) \geq |U(R)| = \delta(\Gamma(R))$. This implies that $\kappa(\Gamma(R)) = \delta(\Gamma(R))$. This completes the proof. \(\square\)

Let $G$ be a connected graph. A non-empty subset $S$ of vertices of $G$ is called a **vertex cut** if $G - S$ (the removal of vertices of $S$ from $G$) is not connected.
Theorem 4.2. Let $R$ be a ring. Then
\[ \kappa(\Gamma(R)) = \kappa(\Gamma(R/J(R)))/J(R). \]

Proof. Let $\kappa(\Gamma(R/J(R)) = t$ and \{b_1 + J(R), b_2 + J(R), \ldots, b_t + J(R)\} be a vertex cut of $\Gamma(R/J(R))$. Then, by [14, Proposition 4.8], it is not hard to see that $\bigcup_{j=1}^t b_j + J(R)$ is a vertex cut of $\Gamma(R)$. Therefore $\kappa(\Gamma(R)) \leq \kappa(\Gamma(R/J(R)))/J(R)$.

Let $\kappa(\Gamma(R)) = n$ and $C$ be a vertex cut of $\Gamma(R)$ such that $|C| = n$. We claim that $C = \bigcup_{j=1}^m a_j + J(R)$ for some $a_i \in R$. Let $a + j \in C$, where $a \in R$ and $j \in J(R)$. We show that $a + J(R) \subseteq C$. Suppose on the contrary that $a + j_0 \not\in C$ for some $j_0 \in J(R)$. Since $C$ is a vertex cut, there are $x, y \in R$ such that $x$ is not connected to $y$ in $\Gamma(R) \setminus C$. On the other hand, $\Gamma(R) \setminus (C \setminus \{a + j\})$ is a connected graph. So we have the following walk in $\Gamma(R) \setminus (C \setminus \{a + j\})$:
\[ x = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_i \rightarrow (a + j) \rightarrow x_i \rightarrow \cdots \rightarrow x_n = y, \]
where $x_i \in G \setminus C$. Since $a + j_0 \not\in C$ and $N_{\Gamma(R)}(a + j) = N_{\Gamma(R)}(a + j_0)$, we have the following walk in $\Gamma(R) \setminus C$:
\[ x = x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_i \rightarrow (a + j_0) \rightarrow x_i \rightarrow \cdots \rightarrow x_n = y, \]
which is a contradiction. Therefore $C = \bigcup_{j=1}^m a_j + J(R)$ for some $a_i \in R$ and hence $n = m|J(R)|$. By [14, Proposition 4.8], it is easy to see that $\{a_1 + J(R), a_2 + J(R), \ldots, a_m + J(R)\}$ is a vertex cut of $\Gamma(R/J(R))$. So
\[ \kappa(\Gamma(R/J(R))) \leq m = n|J(R)| = \kappa(\Gamma(R))/|J(R)|. \]

This completes the proof. \(\Box\)

The following theorem is one of our main results in this paper.

Theorem 4.3. Let $R$ be a ring. Then $\kappa(\Gamma(R)) = \kappa'(\Gamma(R)) = \delta(\Gamma(R)) = |U(R)|$.

Proof. Let $R = R_1 \times \cdots \times R_n$ be a ring such that $R_i$ is a local ring with maximal ideal $m_i$. By Theorems 4.1 and 4.2, we have
\[
\kappa(\Gamma(R)) = \kappa(\Gamma(R/J(R)))/J(R) = \kappa(\Gamma(R/m_1 \times \cdots \times R_n/m_n))/|m_1| \cdots |m_n| = (|R_1/m_1| - 1) \cdots (|R_n/m_n| - 1)|m_1| \cdots |m_n| = (|R_1| - |m_1|) \cdots (|R_n| - |m_n|) = |U(R)|.
\]
This completes the proof. \(\Box\)
5. Hamiltonian cycle and matching

Let \( R \neq \mathbb{Z}_2 \) be a ring. Since \( \Gamma(R) \) is a refinement of the unit graph \( G(R) \), [17, Theorem 2.1] implies that \( \Gamma(R) \) is Hamiltonian. In this section, by a simple and constructive method, we show that \( \Gamma(R) \) is Hamiltonian if and only if it is connected. As a consequence of this result, we show that \( \Gamma(R) \) has a perfect matching if and only if \(|R|\) is an even number. We begin with the following lemma.

**Lemma 5.1.** Let \( R \) be a ring. If \( \Gamma(R/J(R)) \) is Hamiltonian, then \( \Gamma(R) \) is also Hamiltonian.

**Proof.** Let \( J(R) = \{ j_1, \ldots, j_n \} \) and \( a_1 + J(R) \cdots - a_k + J(R) \) be a Hamiltonian cycle in \( \Gamma(R/J(R)) \). By [14, Proposition 4.8], we have the following path in \( \Gamma(R) \):

\[
P_i := j_i + a_1 - j_i + a_2 - \cdots - j_i + a_k, \quad (1 \leq i \leq n).
\]

Now we construct the following Hamiltonian cycle in \( \Gamma(R) \):

\[
P_1 - P_2 - \cdots - P_n.
\]

This completes the proof. \( \square \)

**Remark 5.2.** We note that the converse of the above lemma is false. For example, let \( R \neq \mathbb{Z}_2 \) be a ring such that \( R/J(R) = \mathbb{Z}_2 \). Then \( \Gamma(R/J(R)) \) is not Hamiltonian. But it is easy to see that \( R \) is a local ring with maximal ideal \( m \) such that \(|R/m| = 2 \). Therefore \( \Gamma(R) \) is a complete bipartite graph, by Theorem 2.2. Hence \( \Gamma(R) \) is Hamiltonian.

**Theorem 5.3.** Let \( R \) be a ring such that \( R \neq \mathbb{Z}_2 \). Then \( \Gamma(R) \) is a connected graph if and only if \( \Gamma(R) \) is Hamiltonian.

**Proof.** Suppose \( \Gamma(R) \) is a connected graph. In view of [14, Theorem 3.5], we may assume that \( R/J(R) = F_1 \times F_2 \times \cdots \times F_n \), where \( F_i \) is a field. Since \( \Gamma(R) \) is connected, by Corollary 2.6, we have the following cases:

**Case 1:** \(|F_i| > 2\), for all \( 1 \leq i \leq n \). In this case, we claim that \( \Gamma(R) \) is a Hamiltonian graph. More generally, we show that there is a Hamiltonian cycle \( \hat{X}_1 - \hat{X}_2 - \cdots - \hat{X}_n \) such that no entries of \( \hat{X}_1 \) and \( \hat{X}_n \) are zero. We use induction on \( n \). Suppose that \( n = 1 \) and \( F_1 = \{ a_1 = 0, a_2, \ldots, a_{|F_1|} \} \). Then it is easy to see that \( a_2 - 0 - a_3 - a_4 - \cdots - a_{|F_1|} \) is a Hamiltonian cycle in \( \Gamma(F_1) \).

Now suppose that \( n > 1 \). By the induction hypothesis there is a Hamiltonian cycle \( \hat{X}_1 - \hat{X}_2 - \cdots - \hat{X}_{n-1} \) in \( \Gamma(F_1 \times F_2 \times \cdots \times F_{n-1}) \) such that no entries of \( \hat{X}_1 \) and \( \hat{X}_{n-1} \) are zero. Let \( F_n = \{ c_1 = 0, c_2, \ldots, c_{|F_n|} \} \). In view of Proposition 1.1, we define the following path:

\[
P_{i,i+1} := (\hat{X}_i, c_2) - (\hat{X}_{i+1}, 0) - (\hat{X}_i, c_3) - (\hat{X}_{i+1}, c_2) - (\hat{X}_i, 0) - (\hat{X}_{i+1}, c_3) - (\hat{X}_i, c_4) - (\hat{X}_{i+1}, c_3) - \cdots - (\hat{X}_i, c_{|F_n|}) - (\hat{X}_{i+1}, c_{|F_n|}).
\]

Now we have the following two cases:
If $s$ is an even number we construct the following Hamiltonian cycle in $\Gamma(R/J(R))$:

$$P_{1,2} - P_{3,4} - \cdots - P_{s-1,s}.$$  

If $s$ is an odd number we construct the following Hamiltonian cycle in $\Gamma(R/J(R))$:

$$P_{1,2} - P_{3,4} - \cdots - P_{s-2,s-1} - (\tilde{X}_s,0) - (\tilde{X}_s,c_2) - (\tilde{X}_s,c_3) - \cdots - (\tilde{X}_s,c_{|F_s|}).$$  

**Case 2:** $R/J(R) = \mathbb{Z}_2$. In this case $\Gamma(R)$ is Hamiltonian, by Remark 5.2. 

**Case 3:** $n > 1$ and $F_1 = \mathbb{Z}_2$ and $F_i \neq \mathbb{Z}_2$ for all $2 \leq i \leq n$. By Case 1, $\Gamma(F_2 \times F_3 \times \cdots \times F_n)$ has a Hamiltonian cycle, say $\tilde{Y}_1 - \tilde{Y}_2 - \cdots - \tilde{Y}_h$, such that no entries of $\tilde{Y}_1$ and $\tilde{Y}_h$ are zero. We have the following two cases: If $h$ is an even number, we construct the following Hamiltonian cycle in $\Gamma(R/J(R))$:

$$(1, \tilde{Y}_1) - (0, \tilde{Y}_2) - (1, \tilde{Y}_3) - (0, \tilde{Y}_4) - \cdots - (1, \tilde{Y}_{h-1}) - (0, \tilde{Y}_h)$$

$$- (1, \tilde{Y}_h) - (0, \tilde{Y}_{h-1}) - \cdots - (1, \tilde{Y}_2) - (0, \tilde{Y}_1).$$

If $h$ is an odd number, we have the following Hamiltonian cycle in $\Gamma(R/J(R))$:

$$(1, \tilde{Y}_1) - (0, \tilde{Y}_2) - (1, \tilde{Y}_3) - (0, \tilde{Y}_4) - \cdots - (0, \tilde{Y}_{h-1}) - (1, \tilde{Y}_h)$$

$$- (0, \tilde{Y}_h) - (1, \tilde{Y}_{h-1}) - \cdots - (1, \tilde{Y}_2) - (0, \tilde{Y}_1).$$

Now Lemma 5.1 implies that $\Gamma(R)$ is a Hamiltonian graph. The converse is trivial. \(\square\)

A matching in a graph $G$ is a set of edges no two of which share an endpoint. The vertices incident to the edges of a matching $M$ are saturated by $M$. A perfect matching in a graph is a matching that saturates every vertex.

**Lemma 5.4.** Let $R$ be a ring. If $\Gamma(R/J(R))$ has a perfect matching, then $\Gamma(R)$ also has a perfect matching.

**Proof.** Suppose that $J(R) = \{j_1, \ldots, j_m\}$ and let $a_1 + J(R), \ldots, a_k + J(R)$ be all distinct elements of $R/J(R)$. Let $\{e_1, \ldots, e_{k/2}\}$ be a perfect matching for $\Gamma(R/J(R))$. Without loss of generality, we may assume that $e_i$ is the edge between vertices $a_{2i-1} + J(R)$ and $a_{2i} + J(R)$, for all $1 \leq i \leq k/2$. According to this assumption and [14, Proposition 4.8], we conclude that $a_{2i-1} + j_i$ is adjacent to $a_{2i} + j_i$ in $\Gamma(R)$ by some edge, say $e_{i,t}$, for all $1 \leq i \leq k/2$ and all $1 \leq t \leq m$. Now it is easy to see that $\{e_{i,t} | 1 \leq i \leq k/2, 1 \leq t \leq m\}$ is a perfect matching for $\Gamma(R)$. \(\square\)

**Remark 5.5.** The converse of the above lemma is also true (see Corollary 5.7).

**Theorem 5.6.** Let $R$ be a ring. Then $\Gamma(R)$ has a perfect matching if and only if $|R|$ is an even number.
Proof. Suppose that \(|R|\) is an even number. First assume that \(\Gamma(R)\) is connected. If \(R = \mathbb{Z}_2\), obviously \(R\) has a perfect matching. So let \(R \neq \mathbb{Z}_2\). By Theorem 5.3, \(\Gamma(R)\) has the following Hamiltonian cycle:

\[ v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n. \]

Let \(e_i\) be the edge between the vertices \(v_i\) and \(v_{i+1}\) for all \(1 \leq i \leq n - 1\). Set \(M := \{e_1, e_3, \ldots, e_{n-1}\}\). Then \(M\) is a perfect matching.

Now let \(\Gamma(R)\) be a disconnected graph. By Corollary 2.6, we may assume that \(R/J) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times F_1 \times F_2 \times \cdots \times F_t\), such that \(n \geq 2\), where \(F_i\) is a field and \(F_i \neq \mathbb{Z}_2\), for all \(1 \leq i \leq t\). First consider the ring \(S = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2\). For \(x \in \{0, 1\}\), we define:

\[ x^c := \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x = 1. \end{cases} \]

If \(\widehat{X} = (x_1, x_2, \ldots, x_n)\) is an arbitrary element of \(S\), we define \(\widehat{X}^c := (x_1^c, x_2^c, \ldots, x_n^c)\). It is clear that \(\widehat{X}^c\) is the unique neighborhood of \(\widehat{X}\) and hence every element of \(\Gamma(S)\) has degree 1. Therefore \(\Gamma(S)\) has \(2^n/2\) connected components that are isomorphic to \(K_2\). Now we consider the ring \(R/J(R)\). We have \(R/J(R) = \{\langle \widehat{X}, \widehat{Y} \rangle \mid \widehat{X} \in S\) and \(\widehat{Y} \in F_1 \times \cdots \times F_t\}\). Suppose that \(\widehat{X}\) is an arbitrarily fixed element of \(S\) and set

\[ C := \{\langle \widehat{X}, \widehat{Y} \rangle \mid \widehat{Y} \in F_1 \times \cdots \times F_t\} \cup \{\langle \widehat{X}^c, \widehat{Y} \rangle \mid \widehat{Y} \in F_1 \times \cdots \times F_t\}. \]

Clearly, if \(\widehat{Z} \in S\) and \(\widehat{Z} \notin \{\widehat{X}, \widehat{X}^c\}\), then \(\langle \widehat{Z}, \widehat{Y} \rangle\) is not adjacent to any element of \(C\). We claim that \(C\) is a connected component of \(\Gamma(R/J(R))\) and has a perfect matching. Define the following map:

\[ h : \Gamma(C) \rightarrow \Gamma(\mathbb{Z}_2 \times F_1 \times \cdots \times F_t), \]

where \(h(\widehat{X}, \widehat{Y}) = (0, \widehat{Y})\) and \(h(\widehat{X}^c, \widehat{Y}) = (1, \widehat{Y})\). It is easy to see that any two vertices of \(\Gamma(C)\), say \(c_1, c_2\), are adjacent if and only if \(h(c_1)\) is adjacent to \(h(c_2)\). So \(\Gamma(C)\) is isomorphic to \(\Gamma(\mathbb{Z}_2 \times F_1 \times \cdots \times F_t)\). The graph \(\Gamma(\mathbb{Z}_2 \times F_1 \times \cdots \times F_t)\) has a Hamiltonian cycle, by Theorem 5.3, and has even vertices. Therefore it has a perfect matching. This implies that \(\Gamma(C)\) also has a perfect matching. On the other hand, all connected components of \(\Gamma(R/J(R))\) are isomorphic to \(\Gamma(C)\) and hence \(\Gamma(R/J(R))\) has a perfect matching. Now Lemma 5.4 implies that \(\Gamma(R)\) has a perfect matching.

The converse is trivial. \(\square\)

**Corollary 5.7.** Let \(R\) be a ring. Then \(\Gamma(R)\) has a perfect matching if and only if \(\Gamma(R/J(R))\) has a perfect matching.

**Proof.** Suppose that \(R = R_1 \times \cdots \times R_n\), where \(R_i\) is a local ring with maximal ideal \(m_i\). Suppose \(\Gamma(R)\) has a perfect matching. By Theorem 5.6, \(|R|\) is an even number. Therefore there is \(1 \leq i \leq n\), such that \(|R_i|\) is an even number.
Hence, by [1, Proposition 2.1], $|R_i/m_i|$ is even. So we deduce that $|R/J(R)| = |R_1/m_1| \times \cdots \times |R_n/m_n|$ is an even number. By the above Theorem, we conclude that $\Gamma(R/J(R))$ has a perfect matching.

The converse follows easily from Lemma 5.4. □

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