CERTAIN SEMISYMMETRY PROPERTIES OF 
$(\kappa, \mu)$-CONTACT METRIC MANIFOLDS

UDAY CHAND DE, JAE-BOK JUN, AND SHRIMAYEE SAMUI

Abstract. The object of the present paper is to characterize $(\kappa, \mu)$-contact metric manifolds whose concircular curvature tensor satisfies certain semisymmetry conditions. We also verify that the result holds by a concrete example.

1. Introduction

In [3], Blair, Koufogiorgos and Papantoniou introduced $(\kappa, \mu)$-contact metric manifolds. A class of contact metric manifolds with contact metric structure $(\varphi, \xi, \eta, g)$ in which the curvature tensor $R$ satisfies the condition

$$R(X, Y)\xi = (\kappa I + \mu h)\{\eta(Y)X - \eta(X)Y\}$$

for all $X$ and $Y \in TM$, where $(\kappa, \mu) \in \mathbb{R}^2$ is called $(\kappa, \mu)$-metric manifolds.

A transformation of an $(2n+1)$-dimensional Riemannian manifold $M$, which transforms every geodesic circle of $M$ into a geodesic circle is called a concircular transformation ([8], [12]). Here geodesic circle means a curve in $M$ whose first curvature is constant and whose second curvature is identically zero. A concircular transformation is always a conformal transformation [8].

Concircular curvature tensor is defined by [12]

$$Z(X, Y)W = R(X, Y)W - \frac{r}{2n(2n+1)}\{g(Y, W)X - g(X, W)Y\},$$

where $X, Y, W \in TM$ and $R$ and $r$ is the curvature tensor and the scalar curvature respectively.

Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

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In [2], D. E. Blair et al. started a study of concircular curvature tensor of contact metric manifolds. Also concircular curvature tensor in \((\kappa, \mu)\)-contact metric manifolds has been studied by U. C. De and Sujit Ghosh [6].

A Riemannian manifold is said to be semisymmetric if its curvature tensor \(R\) satisfies 
\[
R(X, Y) \cdot R = 0, \quad X, Y \in TM,
\]
where \(R(X, Y)\) acts on \(R\).

Recently, in [13] Yildiz and De studied \(\varphi\)-projectively semisymmetric and 
\(h\)-projectively semisymmetric \((\kappa, \mu)\)-contact metric manifolds.

Motivated by the above studies, we study in this paper certain semisymmetry properties of the concircular curvature tensor in \((\kappa, \mu)\)-contact metric manifolds.

The paper is organized as follows:
In Section 2, we give necessary details about \((\kappa, \mu)\)-contact metric manifolds. Section 3 deals with \(\varphi\)-concircularity semisymmetric \((\kappa, \mu)\)-contact metric manifolds. In Section 4, \(h\)-concircularity semisymmetric \((\kappa, \mu)\)-contact metric manifolds have been studied. Finally, we construct an example of a \((\kappa, \mu)\)-contact metric manifold which verifies Theorem 5.1.

2. Preliminaries

An \((2n+1)\)-dimensional differentiable manifold \(M\) is called an almost contact manifold if there is an almost contact structure \((\varphi, \xi, \eta)\) consisting of a \((1,1)\)-tensor field \(\varphi\), a vector field \(\xi\), a 1-form \(\eta\) satisfying
\[
\varphi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0.
\]

An almost contact structure is said to be normal if the induced almost complex structure \(J\) on the product manifold \(M^{2n+1} \times \mathbb{R}\) defined by
\[
J(X, f \frac{dt}{\eta}) = (\varphi X - f\xi, \eta(X)\frac{dt}{\eta})
\]
is integrable, where \(X\) is tangent to \(M\), \(t\) is the coordinate of \(R\) and \(f\) is a smooth function on \(M^{2n+1} \times \mathbb{R}\).

The condition for being normal is equivalent to vanishing of the torsion tensor \([\varphi, \varphi] + 2d\eta \otimes \xi\), where \([\varphi, \varphi]\) is the Nijenhuis tensor of \(\varphi\).

Let \(g\) be a compatible Riemannian metric with structure \((\varphi, \xi, \eta, g)\), that is,
\[
g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y),
\]
or equivalently,
\[
g(X, \xi) = \eta(X), \quad g(\varphi X, Y) = -g(X, \varphi Y)
\]
for all \(X, Y \in TM\).

An almost contact metric structure becomes a contact metric structure if
\[
g(X, \varphi Y) = d\eta(X, Y)
\]
for all \(X, Y \in TM\).

Given a contact metric manifold \(M^{2n+1}(\varphi, \xi, \eta, g)\), we define a \((1,1)\)-tensor field \(h\) by
\[
h = \frac{1}{4}L_{\xi} \varphi,
\]
where \(L\) denotes the Lie differentiation. Then \(h\) is symmetric and satisfies
\[
h \xi = 0, \quad h \varphi + \varphi h = 0,
\]
(2.6) \[ \nabla \xi = -\varphi - \varphi h, \quad \text{trace}(h) = \text{trace}(\varphi h) = 0, \]

where \( \nabla \) is the Levi-Civita connection.

A contact metric manifold is said to be an \( \eta \)-Einstein manifold if

(2.7) \[ S(X, Y) = a g(X, Y) + b \eta(X) \eta(Y), \]

where \( a, b \) are smooth functions on \( M \) and \( S \) is the Ricci tensor.

A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

(2.8) \[ (\nabla_X \varphi)_Y = g(X, Y) \xi - \eta(Y) X. \]

On a Sasakian manifold, the following relation holds

(2.9) \[ R(X, Y) \xi = \eta(Y) X - \eta(X) Y \]

for all \( X, Y \in TM \).

Blair, Koufogiorgos and Papantoniou [3] considered the \((\kappa, \mu)\)-nullity condition and gave several reasons for studying it. The \((\kappa, \mu)\)-nullity distribution \( N(\kappa, \mu) \) ([3], [10]) of a contact metric manifold \( M \) is defined by

\[ N(\kappa, \mu) : p \mapsto \{ W \in T_p M \mid R(X, Y)W = (\kappa I + \mu h)(g(Y, W)X - g(X, W)Y) \} \]

for all \( X, Y \in TM \), where \( (\kappa, \mu) \in \mathbb{R}^2 \).

A contact metric manifold \( M^{2n+1} \) with \( \xi \in N(\kappa, \mu) \) is called a \((\kappa, \mu)\)-contact metric manifold. Then we have

(2.10) \[ R(X, Y) \xi = \kappa[\eta(Y) X - \eta(X) Y] + \mu[\eta(Y) hX - \eta(X) hY] \]

for all \( X, Y \in TM \).

For \((\kappa, \mu)\)-metric manifolds, it follows that \( h^2 = (\kappa - 1) \varphi^2 \). This class contains Sasakian manifolds for \( \kappa = 1 \) and \( h = 0 \). In fact, for a \((\kappa, \mu)\)-metric manifold, the condition of being Sasakian manifold, \( \kappa \)-contact manifold, \( \kappa = 1 \) and \( h = 0 \) are equivalent. If \( \mu = 0 \), then the \((\kappa, \mu)\)-nullity distribution \( N(\kappa, \mu) \) is reduced to \( \kappa \)-nullity distribution \( N(\kappa) \) [11]. If \( \xi \in N(\kappa) \), then we call contact metric manifold \( M \) an \( N(\kappa) \)-contact metric manifold.

\((\kappa, \mu)\)-contact metric manifolds have been studied by several authors ([1], [4], [5], [6], [7], [9]) and many other authors.

In a \((\kappa, \mu)\)-contact metric manifold, the following relations hold [3]:

(2.11) \[ h^2 = (\kappa - 1) \varphi^2, \]

(2.12) \[ (\nabla_X \varphi)_Y = g(X + hX, Y) \xi - \eta(Y)(X + hX), \]

(2.13) \[ R(\xi, X)Y = \kappa g(X, Y) \xi - \eta(Y)X + \mu g(hX, Y) \xi - \eta(Y)hX, \]

(2.14) \[ S(X, \xi) = 2n \kappa \eta(X), \]

(2.15) \[ S(X, Y) = \{2n - 2 - n \mu\} g(X, Y) + \{2n - 2 + \mu\} g(hX, Y) \]

\[ + \{(2 - 2n) + n + (2\kappa + \mu)\} \eta(X) \eta(Y), \]
\[ r = 2n(2n - 2 + \kappa - n\mu), \]

\[ S(X, hY) = \{(2n-2) - n\mu\}g(X, hY) - (\kappa - 1)(2n-2 + \mu)g(X, Y) + (\kappa - 1)(2n-2 + \mu)\eta(X)\eta(Y), \]

\[ Q\varphi - \varphi Q = 2\{2n - 2 + \mu\}h\varphi, \]

where \( Q \) is the Ricci operator defined by \( g(QX, Y) = S(X, Y) \).

From [3] we can state the following results:

**Lemma 2.1.** Let \( M \) be an \((2n + 1)\)-dimensional contact metric manifold with \( \xi \) belonging to the \((\kappa, \mu)\)-nullity distribution. Then we have

\[ R(X, Y)\varphi W - \varphi R(X, Y)W = \{(1 - \kappa)[g(\varphi Y, W)\eta(X) - g(\varphi X, W)\eta(Y)] + (1 - \mu)[g(\varphi h Y, W)\eta(X) - g(\varphi h X, W)\eta(Y)]\}\xi \\
- g(Y + hY, W)(\varphi X + \varphi h X) + g(X + hX, W)(\varphi Y + \varphi h Y) \\
- g(\varphi Y + \varphi h Y, W)(X + hX) + g(\varphi X + \varphi h X, W)(Y + hY) \\
- \eta(W)\{(1 - \kappa)[\eta(X)\varphi Y - \eta(Y)\varphi X] + (1 - \mu)[\eta(X)\varphi h Y - \eta(Y)\varphi h X]\} \]

for any vector fields \( X, Y, W \).

**Lemma 2.2.** Let \( M \) be an \((2n + 1)\)-dimensional contact metric manifold with \( \xi \) belonging to the \((\kappa, \mu)\)-nullity distribution. Then we have

\[ R(X, Y)hW - hR(X, Y)W = \{\kappa[g(hY, W)\eta(X) - g(hX, W)\eta(Y)] + \mu(1 - \kappa)[g(X, W)\eta(Y) - g(Y, W)\eta(X)]\}\xi \\
+ \kappa[g(Y, \varphi W)\varphi h X - g(X, \varphi W)\varphi h Y] + g(W, \varphi h Y)\varphi X - g(W, \varphi h X)\varphi Y \\
+ \eta(W)\{(1 - \kappa)[\eta(X)h Y - \eta(Y)h X]\} \\
- \mu(1 - \kappa)[\eta(Y)h X + \mu(1 - \kappa)\eta(Y)h W] + 2g(X, \varphi Y)\varphi h W \]

for any vector fields \( X, Y, W \).

### 3. \( \eta \)-Einstein \((\kappa, \mu)\)-contact metric manifolds

In general, in a \((\kappa, \mu)\)-contact metric manifold, the Ricci operator \( Q \) does not commute with \( \varphi \). However, Yildiz and De [13] proved the following:
Proposition 3.1. In a non-Sasakian \((\kappa, \mu)\)-contact metric manifold, the following conditions are equivalent:

(a) \(\eta\)-Einstein manifold,
(b) \(Q\varphi = \varphi Q\).

For \(n = 1\), from (2.18) and Proposition 3.1 we can state the following:

Corollary 3.1. A 3-dimensional non-Sasakian \(\eta\)-Einstein \((\kappa, \mu)\)-contact metric manifold is an \(N(\kappa)\)-contact metric manifold.

4. \(\varphi\)-concircularly semisymmetric \((\kappa, \mu)\)-contact metric manifolds

Definition 4.1. A \((\kappa, \mu)\)-contact metric manifold is said to be \(\varphi\)-concircularly semisymmetric if 
\[
Z(X, Y) \varphi = \varphi Z(X, Y) W = 0.
\]

Using (1.1) and (2.19) in (4.1) we have

\[
Z(X, Y) \varphi W - \varphi(Z(X, Y) W) = 0.
\]

Putting \(X = \varphi X\) and using (2.1) we have

\[
g(X, hW) = ag(X, W) + b\eta(X)\eta(W),
\]

where

\[
a = -\frac{\kappa - 3 - \frac{r(2n-1)}{2n(2n+1)} + (2n+1)}{2 - \mu - 2n}
\]

and

\[
b = \frac{\kappa - 3 - \frac{r(2n-1)}{2n(2n+1)} + (2n+1)}{2 - \mu - 2n}.
\]

Putting (4.4) in (2.15) we have

\[
S(X, W) = a_4 g(X, W) + b_4 \eta(X)\eta(W),
\]
where
\[ a_1 = \{(2n - 2) - n\mu\} - \{(2n - 2) + \mu\} \frac{\kappa - 3 - \frac{r(2n-1)}{2n(2n+1)} + (2n + 1)}{2 - \mu - 2n} \]
and
\[ b_1 = \{(2 - 2n) + n(2\kappa + \mu)\} + \{(2n - 2) + \mu\} \frac{\kappa - 3 - \frac{r(2n-1)}{2n(2n+1)} + (2n + 1)}{2 - \mu - 2n} \].

From (4.5) we can conclude the following:

**Theorem 4.1.** An \((2n+1)\)-dimensional \(\varphi\)-concircularly semisymmetric \((\kappa, \mu)\)-contact metric manifold reduces to an \(\eta\)-Einstein manifold.

From Proposition 3.1 and Theorem 4.1 we can state that:

**Corollary 4.1.** Let \(M\) be an \((2n+1)\)-dimensional \(\varphi\)-concircularly semisymmetric \((\kappa, \mu)\)-contact metric manifold. Then the Ricci operator \(Q\) commutes with \(\varphi\). That is, \(Q\varphi = \varphi Q\).

5. 3-dimensional \(\varphi\)-concircularly semisymmetric \((\kappa, \mu)\)-contact metric manifolds

Suppose \(M\) is a 3-dimensional \(\varphi\)-concircularly semisymmetric \((\kappa, \mu)\)-contact metric manifold.

Putting \(n = 1\) in equation (4.3) we have
\[ g(\varphi X, W)(\kappa - \frac{r}{6}) + g(\varphi X, hW)\mu = 0. \]  

(5.1)

Substituting \(W = hW\) in (5.1) we obtain
\[ (\kappa - \frac{r}{6})hW + \mu h^2 W = 0. \]  

(5.2)

Applying trace in both side of the equation (5.2) and using \(\text{trace} h = 0\), we get
\[ \mu = 0. \]  

(5.3)

From (5.3) we can state the following:

**Theorem 5.1.** A 3-dimensional \(\varphi\)-concircularly semisymmetric \((\kappa, \mu)\)-contact metric manifold reduces to an \(N(\kappa)\)-contact metric manifold.

6. \(h\)-concircularly semisymmetric \((\kappa, \mu)\)-contact metric manifolds

**Definition 6.1.** A \((\kappa, \mu)\)-contact metric manifold is said to be \(h\)-concircularly semisymmetric if \(Z(X, Y) \cdot h = 0\) for all \(X, Y \in TM\).
Suppose \( M \) is an \((2n + 1)\)-dimensional \( h \)-concircularly semisymmetric \((\kappa, \mu)\)-contact metric manifold. Then we get

\[
Z(X, Y)hW - h(Z(X, Y)W) = 0.
\]

Using (1.1) and (2.20) in (6.1) we have

\[
\{\kappa [g(Y, hW)\eta(X) - g(hX, W)\eta(Y)] + \mu(1 - \kappa)[g(X, W)\eta(Y) - g(Y, W)\eta(X)]\} \xi + \mu \{\eta(Y)[(1 - \kappa)\eta(W)X + \mu\eta(X)hW] - \eta(X)[(1 - \kappa)\eta(W)Y + \mu\eta(Y)hW] + 2g(X, \varphi Y)\varphi hW\}
\]

\[
- \frac{r}{2n(2n + 1)} [g(Y, hW)X - g(X, hW)Y - g(Y, W)hX + g(X, W)hY]
\]

\[
= 0.
\]

Taking inner product with \( Z \) of (6.2) and contracting \( Y, Z \) we obtain

\[
\{\kappa + 2\mu + \frac{r}{2n}\} g(hW, X) + \mu(\kappa - 1)g(X, W)
\]

\[
-(2n + 1)\mu(\kappa - 1)\eta(X)\eta(W) = 0,
\]

which implies that

\[
g(X, hW) = ag(X, W) + b\eta(X)\eta(W),
\]

where

\[
a = -\frac{\mu(\kappa - 1)}{\kappa + 2\mu + \frac{r}{2n}}
\]

and

\[
b = \frac{(2n + 1)\mu(\kappa - 1)}{\kappa + 2\mu + \frac{r}{2n}}.
\]

Putting (6.4) in (2.15) we have

\[
S(X, W) = a_1 g(X, W) + b_1 \eta(X)\eta(W),
\]

where

\[
a_1 = \{(2n - 2) - n\mu\} - \{(2n - 2) + \mu\} \frac{\mu(\kappa - 1)}{\kappa + 2\mu + \frac{r}{2n}}
\]

and

\[
b_1 = \{(2 - 2n) + n(2\kappa + \mu)\} + \{(2n - 2) + \mu\} \frac{(2n + 1)(\kappa - 1)\mu}{\kappa + 2\mu + \frac{r}{2n}}.
\]

From (6.5) we can conclude the following:

**Theorem 6.1.** An \((2n+1)\)-dimensional \( h \)-concircularly semisymmetric \((\kappa, \mu)\)-contact metric manifold is an \( \eta \)-Einstein manifold.

From Proposition 3.1 and Theorem 6.1 we can state that:
Corollary 6.1. Let $M$ be an $(2n+1)$-dimensional $h$-concircularly semisymmetric $(\kappa, \mu)$-contact metric manifold. Then the Ricci operator $Q$ commutes with $\varphi$. That is, $Q\varphi = \varphi Q$.

7. An example

Let us consider a 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 : (x,y,z) \neq (0,0,0)\}$, where $(x,y,z)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields $e_1 = e^z-x \frac{\partial}{\partial x}$, $e_2 = e^z-y \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$ are linearly independent at each point of $M$. Let $g$ be the metric defined by

$$g(e_i, e_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

(7.1)

Here $i$ and $j$ runs from 1 to 3.

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_1)$ for any vector field $Z$ tangent to $M$. Let $\varphi$ be the $(1,1)$-tensor field defined by $\varphi e_2 = -e_3$, $\varphi e_3 = e_2$, $\varphi e_1 = 0$. From the properties of $\varphi$ and $\eta$ we can state the following:

$$g(\varphi Z, \varphi W) = g(Z, W) - \eta(Z)\eta(W)$$

for any vector field $Z, W$.

Then for $e_1 = \xi$, the structure $(\varphi, \xi, \eta, g)$ defines a contact metric structure on $M$.

Let $\nabla$ be the Levi-Civita connection on $M$ with respect to the metric $g$. Then we have

$$[e_1, e_2] = e^z-x \frac{\partial}{\partial x} (e^z-y \frac{\partial}{\partial y}) - e^z-y \frac{\partial}{\partial y} (e^z-x \frac{\partial}{\partial x})$$

$$= e^z-x e^z-y \frac{\partial^2}{\partial x \partial y} - e^z-y e^z-x \frac{\partial^2}{\partial x \partial y}$$

$$= 0.$$

Similarly,

$$[e_1, e_3] = -e_1, \quad [e_2, e_3] = -e_2, \quad [e_2, e_1] = 0,$$


From Koszul’s formula, the Riemannian connection $\nabla$ of the metric $g$ is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$

$$- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

(7.2)
Using (7.2) we have
\[ \nabla_{e_1} e_1 = e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_1, \]
\[ \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_2 = e_3, \quad \nabla_{e_3} e_3 = -e_2, \]
\[ \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \]

We also know that
\[ \nabla_{e_2} e_1 = -\varphi e_2 - \varphi he_2. \]

Comparing the above two relations for \( \nabla_{e_2} e_1 \) and using \( \varphi e_1 = 0, \varphi e_3 = e_2 \) and \( \varphi e_2 = -e_3 \), we have
\[ he_2 = -e_2. \]

Similarly, we obtain
\[ he_3 = -e_3 \text{ and } he_1 = 0. \]

It is known that Riemannian curvature tensor
\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \]

Using (7.3) we obtain
\[ R(e_2, e_1)e_1 = -e_2, \]
\[ R(e_3, e_1)e_1 = -e_3, \]
\[ R(e_2, e_3)e_1 = 0. \]

We conclude that \( e_1 \) belongs to the \((\kappa, \mu)\)-nullity distribution, where \( \kappa = -1, \mu = 0 \). Hence the manifold reduces to an \( N(\kappa) \)-contact metric manifold.

All nonzero components of the curvature tensor can be written as follows:
\[ R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_3 = -e_1, \]
\[ R(e_2, e_3)e_3 = -e_2, \quad R(e_2, e_3)e_2 = e_3. \]
\[ R(e_1, e_3)e_3 = -e_1, \quad R(e_3, e_1)e_1 = -e_3, \quad R(e_2, e_1)e_1 = -e_2. \]

From the above results, we have the Ricci tensor
\[ S(e_1, e_1) = g(R(e_2, e_1)e_1, e_2) + g(R(e_3, e_1)e_1, e_3) \]
\[ = -2. \]

Similarly, we obtain \( S(e_2, e_2) = -2, S(e_3, e_3) = -2 \) and the scalar curvature \( r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6. \)

From the above calculation we can conclude that \( S(X, Y) = -2g(X, Y) \) for \( X = a_1 e_1 + a_2 e_2 + a_3 e_3 \) and \( Y = b_1 e_1 + b_2 e_2 + b_3 e_3 \).

For 3-dimensional \((\kappa, \mu)\)-contact metric manifolds, Riemannian curvature tensor can be written as follows:
\[ R(X, Y)W = [S(Y, W)X - S(X, W)Y + g(X, W)QX - g(X, W)QY] \]
\[ - \frac{r}{2}[g(Y, W)X - g(X, W)Y]. \]

Using the values of Ricci tensors and the scalar curvature we obtain
\[ R(X, Y)W = -[g(Y, W)X - g(X, W)Y]. \]
From the definition of \( \phi \)-concircularly semisymmetric manifold we obtain

\[
(Z(X, Y) \cdot \phi)W = Z(X, Y)\phi W - \phi Z(X, Y)W \\
= R(X, Y) \phi W - \phi R(X, Y)W \\
- \frac{r}{6} [g(Y, \phi W)X - g(X, \phi W)Y] \\
- g(Y, W) \phi X + g(X, W) \phi Y].
\]

Using (7.5) and the value of Ricci tensor, (7.6) yields

\[
(Z(X, Y) \cdot \phi)W = - [g(Y, \phi W)X - g(X, \phi W)Y] \\
+ [g(Y, W) \phi X - g(X, W) \phi Y] \\
+ [g(Y, \phi W)X - g(X, \phi W)Y] \\
- g(Y, W) \phi X + g(X, W) \phi Y] \\
= 0.
\]

Thus Theorem 5.1 is verified.

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Uday Chand De  
Department of Pure Mathematics  
Calcutta University  
35 Ballygunge Circular Road  
Kol 700019, West Bengal, India  
E-mail address: uc.de@yahoo.com

Jae-Bok Jun  
Department of Mathematics  
College of Natural Science  
Kookmin University  
Seoul 136-702, Korea  
E-mail address: jbjun@kookmin.ac.kr

Srimayee Samui  
Umeschandra college  
13, Surya Sen street  
Kol 700012, West Bengal, India  
E-mail address: srimayee.samui@gmail.com