Equivalence of $\mathbb{Z}_4$-actions on Handlebodies of Genus $g$

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ABSTRACT. In this paper we consider all orientation-preserving $\mathbb{Z}_4$-actions on 3-dimensional handlebodies $V_g$ of genus $g > 0$. We study the graph of groups $(\Gamma(v), G(v))$, which determines a handlebody orbifold $V(\Gamma(v), G(v)) \cong V_g/\mathbb{Z}_4$. This algebraic characterization is used to enumerate the total number of $\mathbb{Z}_4$ group actions on such handlebodies, up to equivalence.

1. Introduction

A $G$-action on a handlebody $V_g$, of genus $g > 0$, is a group monomorphism $\phi : G \rightarrow \text{Homeo}^+(V_g)$, where $\text{Homeo}^+(V_g)$ denotes the group of orientation-preserving homeomorphisms of $V_g$. Two actions $\phi_1$ and $\phi_2$ on $V_g$ are said to be equivalent if and only if there exists an orientation-preserving homeomorphism $h$ of $V_g$ such that $\phi_2(x) = h \circ \phi_1(x) \circ h^{-1}$ for all $x \in G$. From [4], the action of any finite group $G$ on $V_g$ corresponds to a collection of graphs of groups. We may assume these particular graphs of groups are in canonical form and satisfy a set of normalized conditions, which can be found in [2].

Let $v = (r, s, t, m, n)$ be an ordered 5-tuple of nonnegative integers. The graph of groups $(\Gamma(v), G(v))$ in canonical form, shown in Figure 1, determines a handlebody orbifold $V(\Gamma(v), G(v))$. The orbifold $V(\Gamma(v), G(v))$ is constructed in a similar manner as described in [2]. Note that the quotient of any $\mathbb{Z}_4$-action on $V_g$ is an orbifold of this type, up to homeomorphism.

An explicit combinatorial enumeration of orientation-preserving $\mathbb{Z}_p$-actions on $V_g$, up to equivalence, is given in [2]. In this work we will be interested in examining the orientation-preserving geometric group actions on $V_g$ for the group $\mathbb{Z}_4$. The case for $\mathbb{Z}_p$, when $p$ is an odd prime is considered in [5] and gives a different result. As we will see, there is exactly one equivalence class of $\mathbb{Z}_4$-actions on the handlebody of genus 2. This result coincides with [3]. In this paper we will prove the following
main theorem:

**Theorem 1.1** If $\mathbb{Z}_4$ acts on $V_g$, where $g > 0$, then $V_g/\mathbb{Z}_4$ is homeomorphic to $V(\Gamma(v), G(v))$ for some 5-tuple $v = (r, s, t, m, n)$ of nonnegative integers with $r + s + t + m + n > 0$ and $g + 3 = 4(r + s + m) + 3t + 2n$. The number of equivalence classes of $\mathbb{Z}_4$-actions on $V_g$ with this quotient type is $m$ if $r + s + t = 0$, and $m + 1$ if $r + s + t > 0$.

To illustrate the theorem, let $g = 3$. Then the genus equation becomes $6 = 4(r + s + m) + 3t + 2n$ so that $r + s + m$ must equal 0 or 1, and $(r, s, t, m, n)$ is one of $(0, 0, 2, 0, 0), (1, 0, 0, 0, 1), (0, 1, 0, 0, 1), \text{or} (0, 0, 0, 1, 1)$. Applying Theorem 1.1 to these four possibilities shows that there are a total of $1 + 1 + 1 = 4$ equivalence classes of orientation-preserving $\mathbb{Z}_4$-actions on $V_3$. Some results that follow directly from Theorem 1.1:

**Corollary 1.2** Every $\mathbb{Z}_4$-action on a handlebody of even genus must have an interval of fixed points and at least two fixed points on the boundary of the handlebody.

**Corollary 1.3** Every $\mathbb{Z}_4$-action that is free on the boundary of the handlebody will have $t = n = 0$ and $g \equiv 1 \pmod{4}$.

2. The Main Theorem

The orbifold fundamental group of $V(\Gamma(v), G(v))$ is an extension of $\pi_1(V_g)$ by the group $G$. We may view the fundamental group as a free product $G_1 \ast G_2 \ast G_3 \ast \cdots \ast G_{r+s+t+m+n}$, where $G_i$ is isomorphic to either $\mathbb{Z}, \mathbb{Z}_4 \times \mathbb{Z}, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}, \text{or} \mathbb{Z}_2$. We establish notation similar to [2] and denote the generators of the orbifold fundamental group by $\{a_i : 1 \leq i \leq r\} \cup \{b_j : 1 \leq j \leq s\} \cup \{d_k : 1 \leq k \leq t\} \cup \{e_l, f_l : 1 \leq l \leq m\} \cup \{g_q : 1 \leq q \leq n\}$ such that $b_j^4 = d_k^4 = 1, [b_j, c_j] = 1, e_l^2 = g_q^2 = 1, \text{and} [e_l, f_l] = 1$.

Consider the set of pairs $((\Gamma(v), G(v)), \lambda)$, where $\lambda$ is a finite injective epimor-
phism from $\pi_1^{orb}(V(\Gamma(v),G(v)))$ onto $Z_4$. We say $\lambda$ is finite injective since the kernel of $\lambda$ is a free group of rank $g$. We consider only finite injective epimorphisms such that $\ker(\lambda) = \text{im}(\nu_*)$ for some orbifold covering $\nu : V \to V(\Gamma(v),G(v))$. Since $V$ is a handlebody with torsion free fundamental group, $V$ is homeomorphic to a handlebody $V_g$ of genus $g = 1 - 4\chi(\Gamma(v),G(v))$. Define an equivalence relation on this set of pairs by setting $((\Gamma(v),G(v)),\lambda) \equiv ((\Gamma(v),G(v)),\lambda')$ if and only if there exists an orbifold homeomorphism $h : V(\Gamma(v),G(v)) \to V(\Gamma(v),G(v))$ such that $\lambda' = \lambda \circ h_*$. We define the set $\Delta(Z_4, V_g, V(\Gamma(v),G(v)))$ to be the set of equivalence classes $[((\Gamma(v),G(v)),\lambda)]$ under this relation.

Denote the set of equivalence classes $\mathcal{E}(Z_4, V_g, V(\Gamma(v),G(v)))$ to be the set $\{[\phi] \mid \phi : Z_4 \to \text{Homeo}^+(V_g) \text{ and } V_g/\phi \simeq V(\Gamma(v),G(v))\}$. Note that given any $Z_4$-action $\phi : Z_4 \to \text{Homeo}^+(V_g)$, it must be the case that for some $V(\Gamma(v),G(v))$, $[\phi] \in \mathcal{E}(Z_4, V_g, V(\Gamma(v),G(v)))$. The following proposition has a similar proof technique as found in [2].

**Proposition 2.1** Let $v = (r,s,t,m,n)$. The set $\mathcal{E}(Z_4, V_g, V(\Gamma(v)))$ is in one-to-one correspondence with the set $\Delta(Z_4, V_g, V(\Gamma(v),G(v)))$ for every $g > 0$.

To prove the main theorem, we count the number of elements in the delta set and use the one-to-one correspondence given in Proposition 2.1 to give the total count for the set $\mathcal{E}(Z_4, V_g, V(\Gamma(v),G(v)))$. We resort to the following lemma to help count the number of elements in the delta set. The proof is an adaptation from [2].

**Lemma 2.2** If $\alpha$ is an automorphism of $\pi_1^{orb}(V(\Gamma(v),G(v)))$, then $\alpha$ is realizable $[\alpha = h_*$ for some orientation-preserving homeomorphism $h : V(\Gamma(v),G(v)) \to V(\Gamma(v),G(v))]$ if and only if $\alpha(b_j) = x_j b_{\sigma(j)} x_j^{-1}$, $\alpha(c_j) = x_j b_{\sigma(j)} \epsilon_j \sigma(j) x_j^{-1}$, $\alpha(d_k) = y_k d_{\tau(k)} y_k^{-1}$, $\alpha(e_l) = u w_l e_\gamma \gamma(l) u_l^{-1}$, $\alpha(f_l) = u w_l e_\gamma \gamma(l) f_{\gamma(l)} u_l^{-1}$, and $\alpha(g_q) = z_q g_{\delta(q)} z_q^{-1}$, for some $x_j, y_k, u_l, z_q \in \pi_1^{orb}(V(\Gamma(v),G(v)))$, $\sigma \in \sum_s$, $\tau \in \sum_t$, $\gamma \in \sum_m$, $\xi \in \sum_n$, $\epsilon_j, \delta_k, \epsilon_l, \delta_q \in \{+1,-1\}$, and $0 \leq v_j < 4$, $0 \leq w_l < 2$.

Note that $\Sigma_1$ is the permutation group on $l$ letters.

Note that from [1], a generating set for the automorphisms of the handlebody orbifold fundamental group $\pi_1^{orb}(V(\Gamma(v),G(v)))$ is the set of mappings $\{\rho_{ji}(x),\lambda_{ji}(x),\mu_{ji}(x),\omega_{ji}(x),\sigma_i,\phi_i\}$ whose definitions may be found in [1]. The first five maps are realizable. The realizable $\phi_i$'s are of the form found in Lemma 2.2 and will be used in the remaining arguments of this paper.
Lemma 2.3 Let \( v = (r, s, t, m, n) \) with \( m > 0 \) and let \( \lambda_1, \lambda_2 : \pi_1^{orb}(V(\Gamma(v), G(v))) \rightarrow \mathbb{Z}_4 \) be two finite injective epimorphisms such that there exists a \( j \) with \( \lambda_1(f_j) \) being a generator of \( \mathbb{Z}_4 \) and \( \lambda_2(f_i) \) is not a generator of \( \mathbb{Z}_4 \) for all \( i \). Then \( \lambda_1 \) and \( \lambda_2 \) are not equivalent.

Proof. Let \( \lambda_1, \lambda_2 : \pi_1^{orb}(V(\Gamma(v), G(v))) \rightarrow \mathbb{Z}_4 \) be two finite injective epimorphisms such that \( \lambda_1 \) sends \( f_j \) to a generator of \( \mathbb{Z}_4 \) for some \( j \) and \( \lambda_2 \) does not send \( f_i \) to a generator of \( \mathbb{Z}_4 \) for all \( i \). We may assume that \( \lambda_2(f_i) = 0 \) for all \( i \) by composing \( \lambda_2 \) with the realizable automorphism \( \prod \phi_i \), where \( \phi_i \) sends the generator \( f_i \) to the element \( e_i \) and leaves all other generators fixed. Note that \( w_i = 1 \) if \( \lambda_2(f_i) = 0 \) and \( w_i = 1 \) if \( \lambda_2(f_i) = 2 \). To show that \( \lambda_1 \) and \( \lambda_2 \) are not equivalent we will consider the element \( f_j \) such that \( \lambda_1(f_j) \) generates \( \mathbb{Z}_4 \). For contradiction, assume that \( \lambda_1 \) is equivalent to \( \lambda_2 \). Then by Lemma 2.4, there exists a realizable automorphism \( \alpha \) such that \( \alpha(f_j) = w_m f_m \theta_1 u_1 \), where \( u \in \pi_1^{orb}(V(\Gamma(v), G(v))) \) and \( 0 \leq w < 2 \). Hence \( \lambda_1(f_j) = w \lambda_2(e_m) \), where \( \lambda_2(e_m) \) is a multiple of 2, and hence \( \lambda_1(f_j) \) is a multiple of 2. This is impossible since \( \lambda_1(f_j) \) is a generator of \( \mathbb{Z}_4 \). Therefore \( \lambda_1 \) and \( \lambda_2 \) cannot be equivalent, proving the lemma.

Lemma 2.4 Let \( v = (r, s, t, m, n) \) and let \( \lambda : \pi_1^{orb}(V(\Gamma(v), G(v))) \rightarrow \mathbb{Z}_4 \) be a finite injective epimorphism. There exists a finite injective epimorphism \( \hat{\lambda} : \pi_1^{orb}(V(\Gamma(v), G(v))) \rightarrow \mathbb{Z}_4 \) equivalent to \( \lambda \) such that the following hold:

\begin{enumerate}
  \item \( \hat{\lambda}(a_1) = \cdots = \hat{\lambda}(a_r) = 1 \).
  \item \( \hat{\lambda}(b_1) = \cdots = \hat{\lambda}(b_s) = 1 \).
  \item \( \hat{\lambda}(c_i) = 0 \) for all \( 1 \leq i \leq s \).
  \item \( \hat{\lambda}(d_i) = \cdots = \hat{\lambda}(d_t) = 1 \).
  \item \( \hat{\lambda}(e_i) = \cdots = \hat{\lambda}(e_m) = 2 \).
  \item \( \hat{\lambda}(f_i) = 1 \) for all \( i \leq k \) some \( 0 \leq k \leq m \).
  \item \( \hat{\lambda}(f_i) = 0 \) for all \( k < i \leq m \).
  \item \( \hat{\lambda}(g_1) = \cdots = \hat{\lambda}(g_n) = 2 \).
\end{enumerate}

Proof. Let \( \lambda : \pi_1^{orb}(V(\Gamma(v), G(v))) \rightarrow \mathbb{Z}_4 \) be a finite injective epimorphism.

Properties (5) and (8) must occur since \( \lambda \) is finite injective.

Property (4) follows by composing \( \lambda \) with the realizable automorphism \( \prod \phi_i \), where \( \phi_i \) sends the generator \( d_i \) to the element \( d_i^{\epsilon_i} \) and leaves all other generators fixed. Note that \( \epsilon_i = 1 \) if \( \lambda(d_i) = 1 \) and \( \epsilon_i = -1 \) if \( \lambda(d_i) = 3 \).

Property (2) follows by a similar technique. Assuming property (2) holds, property (3) follows by composing \( \lambda \) with the realizable automorphism \( \prod \phi_i \), where \( \phi_i \) sends the generator \( c_i \) to the element \( b_i c_i \) and leaves all other generators fixed.
To show properties (6) and (7) hold, we may compose \( \lambda \) with the realizable automorphism \( \prod \phi_i \), where \( \phi_i \) sends the generator \( f_i \) to the element \( e_i^2 f_i \) and leaves all other generators fixed. Note that \( z_i = 1 \) if \( \lambda(f_i) = 2, z_i = 2 \) if \( \lambda(f_i) = 0 \) or \( \lambda(f_i) = 1 \), and \( z_i = -1 \) if \( \lambda(f_i) = 3 \). Furthermore, composing \( \lambda \) with the realizable automorphisms \( \omega_{ij} \) we may interchange \( f_i \) as needed so that the first \( k \) generators map to 1 and the last \( m - k \) generators map to 0.

Finally, to prove property (1) we may assume that there exists an element \( x \in G_j \) (where \( G_j \) is either \( \mathbb{Z}, \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}, \) or \( \mathbb{Z}_2 \times \mathbb{Z} \)) such that \( \lambda(x) = 1 \). Note that we may compose \( \lambda \) with a realizable automorphism that sends \( x \) to \( x^{-1} \) if needed. Now compose \( \lambda \) with the realizable automorphism \( \prod \rho_{ji}(x^{-\lambda(a_i)}+1) \). It may be shown that \( (\lambda \circ \alpha)(a_i) = 1 \) for all \( i \).

**Proposition 2.5** Let \( v = (r, s, t, m, n) \) with \( m > 0 \) and let \( \lambda, \lambda': \pi_1^{orb}(V(\Gamma(v), G(v))) \rightarrow \mathbb{Z}_4 \) be two finite injective epimorphisms that satisfy the conclusion of Lemma 2.4, where \( \lambda(f_i) = 1 \) for all \( 1 \leq i \leq k \) and \( \lambda'(f_i) = 1 \) for all \( 1 \leq i \leq k' \). Then \( \lambda \) is equivalent to \( \lambda' \) if and only if \( k = k' \).

**Proof.** For a contradiction, assume that \( \lambda \) is equivalent to \( \lambda' \) and \( k \neq k' \). Without loss of generality we may assume that \( k > k' \). Hence, \( \lambda \) maps at least one more generator \( f_i \) to 1 as does \( \lambda' \). This would mean that there must exist a realizable automorphism \( \alpha \) such that \( (\lambda \circ \alpha)(f_{k'+1}) = 0 \). By Lemma 2.2, this is impossible. Thus, \( k = k' \). For the reverse implication suppose that \( k = k' \). Then \( \lambda = \lambda' \), proving the proposition.

We will now prove the main theorem.

**Proof.** Define \( \Delta_0(Z_4, V_g, V(\Gamma(v), G(v))) \) to be the set of equivalence classes \( \left[ (\Gamma(v), G(v)), \lambda \right] \) such that \( \lambda(f_i) = 0 \) for all \( 1 \leq i \leq m \). We now define \( \Delta_1(Z_4, V_g, V(\Gamma(v), G(v))) \) to be the set of equivalence classes \( \left[ (\Gamma(v), G(v)), \lambda \right] \) such that \( \lambda(f_i) = 1 \) for at least one \( i \) such that \( 1 \leq i \leq m \). By Lemma 2.3, the delta set \( \Delta(Z_4, V_g, V(\Gamma(v), G(v))) \) may be viewed as the disjoint union \( \Delta_0(Z_4, V_g, V(\Gamma(v), G(v))) \cup \Delta_1(Z_4, V_g, V(\Gamma(v), G(v))) \). Hence, the order of the delta set is the sum of the orders of the two sets \( \Delta_0(Z_4, V_g, V(\Gamma(v), G(v))) \) and \( \Delta_1(Z_4, V_g, V(\Gamma(v), G(v))) \). Applying Lemma 2.4 and Proposition 2.5, we see that \( |\Delta_0(Z_4, V_g, V(\Gamma(v), G(v)))| = m \) and \( |\Delta_0(Z_4, V_g, V(\Gamma(v), G(v))))| = 1 \). Hence by Proposition 2.1, the theorem follows.

**References**


