REGULARITY FOR FRACTIONAL ORDER RETARDED NEUTRAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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Abstract. In this paper, we study the existence of solutions and $L^2$-regularity for fractional order retarded neutral functional differential equations in Hilbert spaces. We no longer require the compactness of structural operators to prove the existence of continuous solutions of the nonlinear differential system, but instead we investigate the relation between the regularity of solutions of fractional order retarded neutral functional differential systems with unbounded principal operators and that of its corresponding linear system excluded by the nonlinear term. Finally, we give a simple example to which our main result can be applied.

1. Introduction

Let $H$ and $V$ be two complex Hilbert spaces such that $V$ is a dense subspace of $H$. In this paper, we study the existence of solutions and $L^2$-regularity for the following fractional order retarded neutral functional differential equation:

\[
\begin{aligned}
\frac{d^\alpha}{dt^\alpha} [x(t) + g(t, x_t)] &= Ax(t) + \int_{-h}^0 a_1(s) A_1 x(t + s) ds + (F x)(t) + k(t), \quad t > 0, \\
x(0) &= \phi^0, \quad x(s) = \phi^1(s), \quad -h \leq s < 0,
\end{aligned}
\]

where $1/2 < \alpha < 1$, $h > 0$, $a_1(\cdot)$ is Hölder continuous, $k$ is a forcing term, and $g$, $f$, are given functions satisfying some assumptions. Moreover, $A : D(A) \subset H \to H$ is unbounded but $A_1$ is bounded. For each $s \in [0, T]$, we define $x_s : [-h, 0] \to H$ as $x_s(r) = x(s + r) \quad r \in [-h, 0]$ and $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$.

This kind of systems arises in many practical mathematical models arising in dynamic systems, economy, physics, biological and engineering problems, etc. (see [5, 6, 17, 18]). There has been a significant development in fractional...
differential equations in recent years, see [9, 14, 16, 20] and the references therein.

In [4, 7, 8], the authors have discussed the existence of solutions for mild solutions for the neutral differential systems with state-dependence delay. Most studies about the neutral initial value problems governed by retarded semilinear parabolic equation have been devoted to the control problems. As for the retarded differential equations, Jeong et al. [12, 13], Sukavanam et al. [22], and Wang [25], have discussed the regularity of solutions and controllability of the semilinear retarded systems, and see [12, 13, 22, 25] and references therein for the linear retarded systems.

Recently, the existence of mild solutions for fractional neutral evolution equations has been studied in [10, 14], the existence of solutions of inhomogeneous fractional diffusion equations with a forcing function in Baeumer et al. [2], and the existence and approximation of solutions to fractional evolution equation in Muslim [19]. In addition, Sukavanam et al. [21] studied approximate controllability of fractional order semilinear delay systems.

In this paper, we propose a different approach of the earlier works used properties of the relative compactness. Our approach is that regularity results of general retarded linear systems of Di Blasio et al. [3] and semilinear systems of [13] remain valid under the above formulation of fractional order retarded neutral differential system (1.1) even though the system (1.1) contains unbounded principal operators, delay term, and local Lipschitz continuity of the nonlinear term. The methods of the functional analysis concerning an analytic semigroup of operators and some fixed point theorems are applied effectively.

The paper is organized as follows. In Section 2, we deal with properties of the analytic semigroup constructing the strict solution of the corresponding linear systems excluded by the nonlinear term and introduce basic properties. In Section 3, by using properties of the strict solutions in dealt in Section 2, we will obtain the $L^2$-regularity of solutions of (1.1), and a variation of constant formula of solutions of (1.1). Finally, we also give an example to illustrate the applications of the abstract results.

2. Preliminaries and lemmas

The inner product and norm in $H$ are denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively. $V$ is another Hilbert space densely and continuously embedded in $H$. The notations $||\cdot||$ and $||\cdot||_*$ denote the norms of $V$ and $V^*$ as usual, respectively. For brevity we may regard that

$\tag{2.1} ||u||_* \leq |u| \leq ||u||, \quad u \in V.$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding’s inequality

$\tag{2.2} \text{Re } a(u, u) \geq c_0 ||u||^2 - c_1 |u|^2, \quad c_0 > 0, \ c_1 \geq 0.$
Let $A$ be the operator associated with the sesquilinear form $-a(\cdot, \cdot)$:

$$(c_1 - A)u, v = -a(u, v), \quad u, v \in V.$$  

It follows from (2.2) that for every $u \in V$

$$\text{Re} \,(Au, u) \geq c_0 \|u\|^2.$$  

Then $A$ is a bounded linear operator from $V$ to $V^*$ according to the Lax-Milgram theorem, and its realization in $H$ which is the restriction of $A$ to

$$D(A) = \{ u \in V; Au \in H \}$$

is also denoted by $A$. Then $A$ generates an analytic semigroup $S(t) = e^{tA}$ in both $H$ and $V^*$ as in Theorem 3.6.1 of [23]. Moreover, there exists a constant $C_0$ such that

$$(2.3) \quad \|u\| \leq C_0 \|u\|_{D(A)}^{1/2} \|u\|^{1/2}$$

for every $u \in D(A)$, where

$$\|u\|_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*,$$

where each space is dense in the next one and continuous injection.

**Lemma 2.1.** With the notations (2.1), (2.3), we have

$$(V, V^*)_{1/2, 2} = H,$$

$$(D(A), H)_{1/2, 2} = V,$$

where $(V, V^*)_{1/2, 2}$ denotes the real interpolation space between $V$ and $V^*$ (see Section 1.3.3 of [24]).

If $X$ is a Banach space and $1 < p < \infty$, $L^p(0, T; X)$ is the collection of all strongly measurable functions from $(0, T)$ into $X$ the $p$-th powers of norms are integrable. $L(X, Y)$ is the collection of all bounded linear operators from $X$ into $Y$, and $L(X, X)$ is simply written as $L(X)$.

For the sake of simplicity we assume that the semigroup $S(t)$ generated by $A$ is uniformly bounded, that is, There exists a constant $M_0$ such that

$$(2.4) \quad \|S(t)\|_{L(H)} \leq M_0, \quad \|AS(t)\|_{L(H)} \leq \frac{M_0}{t}.$$  

The following lemma is from [23, Lemma 3.6.2].

**Lemma 2.2.** There exists a constant $M_0$ such that the following inequalities hold:

$$(2.5) \quad \|S(t)\|_{L(V, H)} \leq t^{-1/2} M_0,$$

$$(2.6) \quad \|S(t)\|_{L(V^*, V)} \leq t^{-1} M_0,$$

$$(2.7) \quad \|AS(t)\|_{L(H, V)} \leq t^{-3/2} M_0.$$
The following initial value problem for the abstract linear parabolic equation
\[ \begin{align*}
\frac{dx(t)}{dt} &= Ax(t) + \int_{-h}^{t} a_1(s) A_1 x(t + s) ds + k(t), \\
x(0) &= \phi^0, \quad x(s) = \phi^1(s) \quad s \in [-h, 0].
\end{align*} \tag{2.8} \]

Then the mild solution \( x(t) \) is represented by
\[
x(t) = S(t)\phi^0 + \int_{0}^{t} S(t-s) \left\{ \int_{-h}^{0} a_1(\tau) A_1 x(s+\tau) d\tau + f(s, x(s)) \right\} ds
+ \int_{0}^{t} S(t-s) k(s) ds,
\]
\[\begin{align*}
x(0) &= \phi^0, \quad x(s) = \phi^1(s) \quad s \in [-h, 0].
\end{align*} \]

By virtue of Theorem 2.1 of [11] or [3], we have the following result on the corresponding linear equation of (2.8).

**Lemma 2.3.** (1) For \((\phi^0, \phi^1) \in V \times L^2(-h, 0; D(A)) \) and \( k \in L^2(0, T; H) \), \( T > 0 \), there exists a unique solution \( x \) of (2.8) belonging to
\[ L^2(0, T; D(A)) \cap W^{1, 2}(0, T; H) \subset C([0, T]; F) \]
and satisfying
\[ \|x\|_{L^2(0, T; D(A)) \cap W^{1, 2}(0, T; H)} \leq C_1 (\|\phi^0\|_F + \|\phi^1\|_{L^2(-h, 0; D(V))} + \|k\|_{L^2(0, T; H)}), \]
where \( C_1 \) is a constant depending on \( T \) and
\[ \|x\|_{L^2(0, T; D(A)) \cap W^{1, 2}(0, T; H)} = \max \left\{ \|x\|_{L^2(0, T; D(A))}, \|x\|_{W^{1, 2}(0, T; H)} \right\}. \]

(2) Let \((\phi^0, \phi^1) \in H \times L^2(-h, 0; V) \) and \( k \in L^2(0, T; V^*) \), \( T > 0 \). Then there exists a unique solution \( x \) of (2.8) belonging to
\[ L^2(0, T; V) \cap W^{1, 2}(0, T; V^*) \subset C([0, T]; H) \]
and satisfying
\[ \|x\|_{L^2(0, T; V) \cap W^{1, 2}(0, T; V^*)} \leq C_1 (\|\phi^0\| + \|\phi^1\|_{L^2(-h, 0; V)} + \|k\|_{L^2(0, T; V^*)}), \]
where \( C_1 \) is a constant depending on \( T \).

Let the solution spaces \( W_0(T) \) and \( W_1(T) \) of strong solutions be defined by
\[ \begin{align*}
W_0(T) &= L^2(0, T; D(A)) \cap W^{1, 2}(0, T; H), \\
W_1(T) &= L^2(0, T; V) \cap W^{1, 2}(0, T; V^*).
\end{align*} \]

Here, we note that by using interpolation theory, we have
\[ W_0(T) \subset C([0, T]; V), \quad W_1(T) \subset C([0, T]; H). \]

Thus, there exists a constant \( c_1 > 0 \) such that
\[ \|x\|_{C([0, T]; V)} \leq c_1 \|x\|_{W_0(T)}, \quad \|x\|_{C([0, T]; H)} \leq c_1 \|x\|_{W_1(T)}. \tag{2.11} \]

In what follows in this section, we assume \( c_1 = 0 \) in (2.2) without any loss of generality. So we have that \( 0 \in \rho(A) \) and the closed half plane \( \{ \lambda : \text{Re} \lambda \geq 0 \} \)
is contained in the resolvent set of $A$. In this case, it is possible to define the fractional power $A^\alpha$ for $\alpha > 0$. The subspace $D(A)\alpha$ is dense in $H$ and the expression

$$||x||\alpha = ||A^\alpha x||, \quad x \in D(A)\alpha$$

defines a norm on $D(A)\alpha$. It is also well known that $A^\alpha$ is a closed operator with its domain dense and $D(A^\alpha) \supset D(A^\beta)$ for $0 < \alpha < \beta$. Due to the well known fact that $A^{-\alpha}$ is a bounded operator, we can assume that there is a constant $C_{-\alpha} > 0$ such that

$$(2.12) \quad ||A^{-\alpha}||_{L(H)} \leq C_{-\alpha}, \quad ||A^{-\alpha}||_{L(V^*, V)} \leq C_{-\alpha}.$$ 

3. Existence of solutions

Consider the following fractional order retarded neutral differential system:

$$\frac{d^\alpha}{dt^\alpha}[x(t) + g(t, x_t)] = Ax(t) + \int_{-h}^{0} a_1(s) A_1 x(t + s) ds + (Fx)(t) + k(t), \quad t > 0,$$

$$x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \leq s < 0,$$

where $0 < \alpha < 1$ and $A$ and $A_1$ are the linear operators defined as in Section 2. For each $s \in [0, T]$, we define $x_s : [-h, 0] \rightarrow H$ as

$$x_s(r) = x(s + r), \quad -h \leq r \leq 0.$$ 

We will set

$$\Pi = L^2(-h, 0; V).$$

**Definition.** The fractional integral of order $\alpha > 0$ with the lower limit 0 from a function $f$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,$$

provided the right hand side is pointwise defined on $[0, \infty)$, $\Gamma$ is the Gamma function.

The fractional derivative of order $\alpha > 0$ in the Caputo sense with the lower limit 0 from a function $f \in C^n[0, \infty)$ is defined as

$$\frac{d^n}{dt^n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \quad n-1 < \alpha < n.$$ 

For the basic results about fractional integrals and fractional derivative, one can refer to [20].

The mild solution of the system (3.1) is represented as (see [10, 26]):

$$(3.2) \quad x(t) = S(t)[g^0 + g(0, \phi^1)] - g(t, x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} AS(t-s)g(s, x_s) ds$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} S(t-s)$$
\[
\times \left\{ \int_{-\infty}^{0} a_1(\tau) A_1(x(s + \tau) d\tau + (Fx)(s) + k(s) \right\} ds.
\]

To establish our results, we introduce the following assumptions on system (3.1).

**Assumption (A).** We assume that \( a_1(\cdot) \) is Hölder continuous of order \( \rho \):
\[
|a_1(0)| \leq H_1, \quad |a_1(s) - a_1(\tau)| \leq H_1(s - \tau)^\rho.
\]

**Assumption (F).** \( F \) is a nonlinear mapping of \( L^2(0, T; V) \) into \( L^2(0, T; H) \) satisfying following:
\[
\begin{align*}
&\text{(i)} \quad \text{There exists a function } L_f : \mathbb{R}_+ \to \mathbb{R} \text{ such that} \\
&\quad \|Fx - Fy\|_{L^2(0, T; H)} \leq L_f(r) \|x - y\|_{L^2(0, T; V)}, \quad t \in [0, T] \\
&\quad \text{hold for } \|x\|_{L^2(0, T; V)} \leq r \text{ and } \|y\|_{L^2(0, T; V)} \leq r. \\
&\text{(ii)} \quad \text{The inequality} \\
&\quad \|Fx\|_{L^2(0, T; H)} \leq L_f(r)(\|x\|_{L^2(0, T; V)} + 1) \\
&\quad \text{holds for every } t \in [0, T] \text{ and } \|x\|_{L^2(0, T; V)} \leq r.
\end{align*}
\]

**Assumption (G).** Let \( g : [0, T] \times \Pi \to H \) be a nonlinear mapping such that there exists a constant \( L_g \) satisfying the following conditions hold:
\[
\begin{align*}
&\text{(i)} \quad \text{For any } x \in \Pi, \text{ the mapping } g(\cdot, x) \text{ is strongly measurable;} \\
&\text{(ii)} \quad \text{There exists a positive constant } \beta > 1 - 2\alpha/3 \text{ such that} \\
&\quad \|A^3g(t, 0)\| \leq L_g, \quad \|A^3g(t, x) - A^3g(t, \dot{x})\| \leq L_g \|x - \dot{x}\|_H \\
&\quad \text{for all } t \in [0, T], \text{ and } x, \dot{x} \in \Pi.
\end{align*}
\]

**Lemma 3.1.** Let \( x \in L^2(-h, T; V) \). Then the mapping \( s \mapsto x_s \) belongs to \( C([0, T]; \Pi) \), and
\[
\begin{align*}
&\|x_t\|_H \leq \|x\|_{L^2(-h, T; V)}(t > 0), \\
&\|x\|_{L^2(0, T; H)} \leq \sqrt{T}\|x\|_{L^2(-h, T; V)}.
\end{align*}
\]

**Proof.** The first paragraph is easy to verify. Moreover, we have
\[
\|x_t\|_H = \left[ \int_{-h}^{0} (x(s + \tau) d\tau \right]^{1/2} \leq \left[ \int_{-h}^{0} \|x(\tau)\|_V^2 d\tau \right]^{1/2} \leq \|x\|_{L^2(-h, t; V)}, \quad t > 0,
\]
and
\[
\|x\|_{L^2(0, T; H)} \leq \int_{0}^{T} \|x_s\|_H^2 ds \leq \int_{0}^{T} \int_{-h}^{0} \|x(s + \tau)\|_V^2 d\tau ds \leq \int_{0}^{T} ds \int_{-h}^{0} \|x(\tau)\|_V^2 d\tau \leq T\|x\|_{L^2(-h, T; V)}. \quad \square
\]

One of the main useful tools in the proof of existence theorems for nonlinear functional equations is the following fixed point theorem.
Lemma 3.2 (Krasnoselski [15]). Suppose that \( \Sigma \) is a closed convex subset of a Banach space \( X \). Assume that \( K_1 \) and \( K_2 \) are mappings from \( \Sigma \) into \( X \) such that the following conditions are satisfied:

(i) \( (K_1 + K_2)(\Sigma) \subset \Sigma \),

(ii) \( K_1 \) is a completely continuous mapping,

(iii) \( K_2 \) is a contraction mapping.

Then the operator \( K_1 + K_2 \) has a fixed point in \( \Sigma \).

From now on, we establish the following results on the solvability of the equation (3.1).

Theorem 3.3. Let Assumptions (A), (F) and (G) be satisfied. Assume that \( (\phi^0, \phi^1) \in H \times \Pi \) and \( k \in L^2(0,T;V^*) \) for \( T > 0 \). Then, there exists a solution \( x \) of the system (3.1) such that

\[
x \in W_1(T) \Rightarrow L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \to C([0,T];H).
\]

Moreover, there is a constant \( C_2 \) independent of the initial data \( (\phi^0, \phi^1) \) and the forcing term \( k \) such that

\[
(3.5) \quad ||x||_{L^2(-h,T;V^*)} \leq C_2(1 + ||\phi^0|| + ||\phi^1|| + ||k||_{L^2(0,T;V^*)}).
\]

Proof. Let

\[
r := 2[C_1||\phi^0|| + C_1C_{-\alpha}L_g(||\phi^1|| + 1)],
\]

and

\[
N := C_{-\alpha}L_g(||\phi^1|| + ||x||_{L^2(0,T_1;V)} + 1)
\]

\[
+ \frac{C_1(2\alpha - 1)^{-1} - 1}{\Gamma(\alpha)}
\]

\[
\times \left(||\phi^0|| + ||\phi^1||_{L^2(-h,0,V)} + L_f(r)(||x||_{L^2(0,T_1;V)} + 1) + ||k||_{L^2(0,T_1;V)}\right)
\]

\[
+ \frac{C_{1-\beta}L_g}{(\alpha - 3(1 - \beta)/2)(2\alpha + 3\beta - 2)^{1/2}} ||\phi^1|| + ||x||_{L^2(0,T_1;V)} + 1),
\]

where \( C_1 \) is the constants in Lemma 2.3 and \( \beta > 1 - 2\alpha/3 \) in Assumption (G). Let

\[
T_1^* := \max\{T_1^{1/2}, T_1^{(2\alpha + 3\beta - 2)/2}\}
\]

and choose \( 0 < T_1 < T \) such that

\[
(3.6) \quad T_1^*N \leq \frac{T}{2} = [C_1||\phi^0|| + C_1C_{-\alpha}L_g(||\phi^1|| + 1)],
\]

and

\[
(3.7) \quad \tilde{N} := T_1^* \left\{ C_{-\alpha}L_g + \frac{C_{1-\beta}L_g}{(\alpha - 3(1 - \beta)/2)(2\alpha + 3\beta - 2)^{1/2}} \Gamma(\alpha)
\]

\[
+ \frac{C_1(2\alpha - 1)^{-1} - 1}{\Gamma(\alpha)}L_f(r)\right\}
\]

\[
< 1.
\]
Let $J$ be the operator on $L^2(0,T_1;V)$ defined by

$$(Jx)(t) = S(t)[\phi^0 + g(0, \phi^1)] - g(t,x_t) + \frac{1}{I(\alpha)} \int_0^t (t-s)^{(\alpha-1)} A S(t-s) g(s,x_s) ds$$

$$+ \frac{1}{I(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s) \left\{ \int_{-h}^0 a_1(\tau) A_1 x(s+\tau) d\tau + (F x)(s) + k(s) \right\} ds.$$

Let

$$\Sigma = \{ x \in L^2(-h,T;V) : x(0) = \phi^0, \text{ and } x(s) = \phi^1(s) (s \in [-h,0)) \}$$

and

$$\Sigma_r = \{ x \in \Sigma : ||x||_{L^2(0,T_1;V)} \leq r \},$$

which is a bounded closed subset of $L^2(0,T_1;V)$.

Now, in order to show that the operator $J$ has a fixed point in $\Sigma_r \subset L^2(0,T_1;V)$, we take the following steps according to the process of Lemma 3.2.

Step 1. $J$ maps $\Sigma_r$ into $\Sigma_r$.

By (2.10), (2.14) and Assumption (G), and noting $x_0 = \phi^0$, we know

$$(3.8) \quad ||S(\cdot)g(0,x_0)||_{L^2(0,T_1;V)} = C_1|g(0,\phi^1)|$$

$$= C_1||A^{-\beta}||_{L^2(0,T_1;V)} \left(|A^3 g(0,\phi^1) - A^3 g(0,0)| + |A^3 g(0,0)| \right)$$

$$\leq C_1 C_{-\alpha} L_g ||\phi^1|| + 1.$$

From (2.10) of Lemma 2.3 it follows

$$(3.9) \quad ||S(t)\phi^0||_{L^2(0,T_1;V)} \leq C_1|\phi^0|,$$

and by using Hölder inequality

$$(3.10) \quad \int_0^t (t-s)^{\alpha-1} ||S(t-s) \left\{ \int_{-h}^0 a_1(\tau) A_1 x(s+\tau) d\tau + (F x)(s) + k(s) \right\} || ds$$

$$\leq (2\alpha - 1)^{-1/2} (2\alpha - 1)^{1/2} C_1 (||\phi^0|| + ||\phi^1||_{L^2(-h,0;V)} + ||F x||_{L^2(0,T;V)})$$

$$+ ||k||_{L^2(0,T;V)}.$$

Define the operator $I_1$ from $L^2(0,T_1;V)$ to itself by

$$(I_1 x)(t) = \frac{1}{I(\alpha)} \int_0^t (t-s)^{\alpha-1} S(t-s) \left\{ \int_{-h}^0 a_1(\tau) A_1 x(s+\tau) d\tau + (F x)(s) + k(s) \right\} ds.$$

Then according to (3.10) we obtain the following inequality

$$(3.11) \quad ||I_1||_{L^2(0,T_1;V)} \leq \frac{C_1 (2\alpha)^{-1/2} (2\alpha - 1)^{-1/2} T^\alpha}{I(\alpha)} (||\phi^0|| + ||\phi^1||_{L^2(-h,0;V)})$$

$$+ L_f(r)(||x||_{L^2(0,T_1;V)} + 1) + ||k||_{L^2(0,T_1;V)}.$$
By using Assumption (G) and Lemma 3.1, we have

\[
(3.12) \quad \|g(\cdot, x)\|_{L^2(0, T_1; V)} = \left( \int_0^{T_1} \|A^{-\beta}A^\beta g(t, x_t)\|^2 \, dt \right)^{1/2}
\leq C_\alpha \left( \int_0^{T_1} \|A^\beta g(t, x_t)\|^2 \, dt \right)^{1/2}
\leq C_{-\alpha} L_g \sqrt{T_1} (\|x_t\|_{\Pi} + 1)
\leq C_{-\alpha} L_g \sqrt{T_1} (\|\phi^I\|_{\Pi} + \|x\|_{L^2(0, T_1; V)} + 1).
\]

Here, we note

\[
(3.13) \quad \|x_t\|_{\Pi} \leq \|x\|_{L^2(-\infty, T_1; V)} \leq \|\phi^I\|_{\Pi} + \|x\|_{L^2(0, T_1; V)}.
\]

Again we define the operator \( I_2 \) from \( L^2(0, T_1; V) \) to itself by

\[
(I_2 x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} AS(t-s)g(s, x_s) \, ds.
\]

From Lemma 2.6 and Assumption (G) we have

\[
\frac{|(t-s)^{(\alpha-1)} AS(t-s)g(s, x_s)|}{(t-s)^{(\alpha-1)} |A^{1-\beta} S(t-s)|_{L(H, V)}} \leq \frac{C_{1-\beta}}{\Gamma(\alpha)} L_g (\|\phi^I\|_{\Pi} + \|x\|_{L^2(0, T_1; V)} + 1),
\]

and hence, by using Hölder inequality and Assumption (G),

\[
(3.14) \quad \|I_2 x\|_{L^2(0, T_1; V)}^2 \leq \left[ \int_0^{T_1} \left( \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} AS(t-s)g(s, x_s) \, ds \right)^2 \, dt \right]^{1/2}
\leq \frac{1}{\Gamma(\alpha)} C_{1-\beta} L_g (\|\phi^I\|_{\Pi} + \|x\|_{L^2(0, T_1; V)} + 1)
\left[ \int_0^{T_1} \left( \int_0^t (t-s)^{1-\alpha+3(1-\beta)/2} \, ds \right)^2 \, dt \right]^{1/2}
\leq \frac{C_{1-\beta} L_g T_1^{1/2+3(1-\beta)/2}}{(\alpha-3(1-\beta)/2)(2\alpha+3\beta-2)^{1/2} \Gamma(\alpha)} (\|\phi^I\|_{\Pi} + \|x\|_{L^2(0, T_1; V)} + 1).
\]

Thus, from (3.8)-(3.14) it follows that

\[
\|J x\|_{L^2(0, T_1; V)}^2 \leq C_1 |\phi^0| + C_1 C_{-\alpha} L_g (\|\phi^I\|_{\Pi} + 1) + C_{-\alpha} L_g \sqrt{T_1} (\|\phi^I\|_{\Pi} + \|x\|_{L^2(0, T_1; V)} + 1)
+ \frac{C_1 (2\alpha-1/2)(2\alpha-1)^{-1/2} T_1^\alpha}{\Gamma(\alpha)}.
\]
Then reflexive. For a fixed \( \lim \) which is equivalent to the completely continuity of \( K \)

\[
\frac{C_{1-\beta}LgT_1(2\alpha + 3\beta - 2)^{1/2} }{(\alpha - 3(1 - \beta)/2)(2\alpha + 3\beta - 2)^{1/2} \Gamma(\alpha)} \left( |\phi| + |x|_{L^2(0, T_1; V)} + 1 \right),
\]

\[
\leq C_1|\phi| + C_1C_2Lg(|\phi| + 1) + T_1\rho N \leq \frac{r}{2} + \frac{r}{2} \leq r.
\]

Therefore, \( J \) maps \( \Sigma_r \) into \( \Sigma_r \).

Define mapping \( K_1 + K_2 \) on \( L^2(0, T_1; V) \) by the formula

\[
(Jx)(t) = (K_1x)(t) + (K_2x)(t),
\]

where

\[
(K_1x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s) \int_0^s a_1(t-s) A_1x(\tau)d\tau ds,
\]

and

\[
(K_2x)(t) = S(t)[\phi + g(0, x_0)] - g(t, x_t)
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} AS(t-s)g(s, x_s)ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s)
\]

\[
\times \left\{ \int_{s-h}^s a_1(t-s) A_1\phi(\tau)d\tau + F(x)(s) + k(s) \right\} ds.
\]

**Step 2.** \( K_1 \) is a completely continuous mapping.

We can now employ Lemma 3.2 with \( \Sigma_r \). Assume that a sequence \( \{x_n\} \)

of \( L^2(0, T_1; V) \) converges weakly to an element \( x_\infty \in L^2(0, T_1; V) \), i.e., \( w - \lim_{n \to \infty} x_n = x_\infty \). Then we will show that

\[
\lim_{n \to \infty} \|K_1x_n - K_1x_\infty\|_{L^2(0, T_1; V)} = 0,
\]

which is equivalent to the completely continuity of \( K_1 \) since \( L^2(0, T_1; V) \) is reflexive. For a fixed \( t \in [0, T_1] \), let \( x^*_n(x) = (K_1x)(t) \) for every \( x \in L^2(0, T_1; V) \).

Then \( x^*_n \in L^2(0, T_1; V^*) \) and we have \( \lim_{n \to \infty} x^*_n(x_n) = x^*_n(x_\infty) \) since \( w - \lim_{n \to \infty} x_n = x_\infty \). Hence,

\[
\lim_{n \to \infty} (K_1x_n)(t) = (K_1x_\infty)(t), \quad t \in [0, T_1].
\]

By using Hölder inequality, we obtain easily the following inequality:

\[
\left( \int_0^s a_1(t-s) A_1x(\tau)d\tau \right)
\]

\[
= \left( \int_0^s (a_1(t-s) - a_1(0)) A_1x(\tau)d\tau \right)
\]

\[
\leq \{(2\rho + 1)^{-1}s^{2\rho+1} + \sqrt{s}\} H_1 ||A_1||_{L^2(H)} \left( \int_0^s ||x(\tau)||^2 d\tau \right)^{1/2}.
\]
Thus, by (2.5) and (3.16) it holds
\[ \|(K_1 x)(t)\| \]
\[ = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} S(t-s) \int_s^t a_1(\tau-s) A_1 x(\tau) d\tau ds \]
\[ \leq \frac{H_1 \|A_1\|_2 \|x\|_2 \|S(t)\|_2}{\Gamma(\alpha)} \int_0^t \frac{1}{(t-s)^{1-\alpha}} \left\{(2\rho + 1)^{-1} s^{(2\rho+1)/2} + \sqrt{s}\right\} ds \]
\[ \leq \frac{H_1 \|A_1\|_2 \|x\|_2 \|S(0,T_1)\|_2}{\Gamma(\alpha)} \left\{(2\rho + 1)^{-1} B(1/2 + \alpha, (2\rho + 3)/2) t^{\rho+1} + B(1/2 + \alpha, 3/2) t\right\} \]
\[ := c_2 \|x\|_2 \|S(0,T_1)\|_2, \]
where \(c_2\) is a constant and \(B(\cdot, \cdot)\) is the Beta function, that is,
\[ B(1/2 + \alpha, (2\rho + 3)/2) = \int_0^t (t-s)^{\alpha-1/2} s^{(2\rho+1)/2} ds. \]
And we know
\[ \sup_{0 \leq t \leq T_1} \|(K_1 x)(t)\| \leq c_2 \|x\|_{L^2(0,T_1; V)} \leq \infty. \]
Therefore, by Lebesgue's dominated convergence theorem it holds
\[ \lim_{n \to \infty} \left( \int_0^{T_1} \|(K_1 x_n)(t)\|^2 dt \right) = \left( \int_0^{T_1} \|(K_1 x_\infty)(t)\|^2 dt \right), \]
i.e., \(\lim_{n \to \infty} \|K_1 x_n\|_{L^2(0,T_1; V)} = \|K_1 x_\infty\|_{L^2(0,T_1; V)}\). Since \(L^2(0,T_1; V)\) is a reflexive space, it holds (3.15).

**Step 3.** \(K_2\) is a contraction mapping.
For every \(x_1\) and \(x_2 \in \Sigma_r\), we have
\[ (K_2 x_1)(t) - (K_2 x_2)(t) = g(t, x_2) - g(t, x_1) \]
\[ - \int_0^t AS(t-s)(g(t, x_1) - g(t, x_2)) ds \]
\[ + \int_0^t S(t-s)\{F(x_1)(s) - F(x_2)(s)\} dW. \]
By the similar way to (3.8)-(3.14), we have
\[ \|K_2 x_1 - K_2 x_2\|_{L^2(0,T_1; V)} \]
\[ \leq \left\{ C_{-\alpha} L g \sqrt{T_1} + \frac{C_{1-\beta} L g T_1^{(2\alpha+3\beta-2)/2}}{(\alpha - 3(1 - \beta)/2)\left(2\alpha + 3\beta - 2\right)^{1/2} \Gamma(\alpha)} \right\} \]
\[ + \frac{C_1 (2\alpha-1)^{-1/2} (2\alpha - 1)^{-1/2} L f(r) T_1^{\beta}}{\Gamma(\alpha)} \]
\[ \leq N \|x_1 - x_2\|_{M^2(0,T_1; V)}. \]
So by virtue of the condition (3.7) the contraction mapping principle gives that
the solution of (3.1) exists uniquely in $L^2(0, T_1; V)$. This has proved the local existence and
uniqueness of the solution of (3.1).

**Step 4.** We drive a priori estimate of the solution.

To prove the global existence, we establish a variation of constant formula
(3.5) of solution of (3.1). Let $x$ be a solution of (3.1) and $\phi^0 \in H$. Then we have that from (3.8)-(3.14) it follows that

$$
||x||_{L^2(0, T_1; V)} 
\leq C_1|\phi^0| + C_1C_{\alpha}L_{\phi}(||\phi^1|| + 1) + C_{\alpha}L_{\phi}\sqrt{T_1}||\phi^1||_H + ||x||_{L^2(0, T_1; V)} + 1
+ \frac{C_1(2\alpha)^{-1/2}(2\alpha - 1)^{-1/2}T_1^{\alpha}}{\Gamma(\alpha)}
\times (|\phi^0| + ||\phi^1||_{L^2(-h, 0; V)} + L_f(||x||_{L^2(0, T_1; V)} + 1) + ||k||_{L^2(0, T_1; V)})
+ \frac{C_{\alpha}L_{\phi}T_1^{(2\alpha + \beta - 2)/2}}{(\alpha - 3(1 - \beta)/2)(2\alpha + \beta - 2)^{1/2}\Gamma(\alpha)}(||\phi^1||_H + ||x||_{L^2(0, T_1; V)} + 1),
$$

Taking into account (3.7) there exists a constant $C_2$ such that

**Step 4.** We drive a priori estimate of the solution.

To prove the global existence, we establish a variation of constant formula
(3.5) of solution of (3.1). Let $x$ be a solution of (3.1) and $\phi^0 \in H$. Then we have that from (3.8)-(3.14) it follows that

$$
||x||_{L^2(0, T_1; V)} \leq (1 - \hat{N})^{-1}\hat{N}_1
\leq C_2(1 + E(||\phi^0||^2) + ||\phi^1||_H + ||k||_{M^2(0, T_1; V^*)}),
$$

which obtain the inequality (3.5).

Now we will prove that $|x(T_1)| < \infty$ in order that the solution can be extended to the interval $[T_1, 2T_1]$. From (2.11) and Lemma 2.3 it follows that

$$
|S(T_1)|\phi^0 + g(0, x_0)| \leq c_1||S(\cdot)||\phi^0 + g(0, x_0)||_{W_1(T_1)}
\leq c_1C_1|\phi^0 + g(0, \phi^1)|
\leq c_1C_1\{||\phi^0|| + C_{\alpha}L_{\phi}(||\phi^1||_H + 1)\} := I,
$$

and by using Assumption (G) we have

$$
g(T_1, x_T) \leq \|A^{\beta}B^\beta g(t, x_T)\|,
\leq C_{\alpha}L_{\phi}(||x_T||_H + 1)
$$
By (3.10), we have

\[(3.20)\]

\[
\| (I_1x)(T_1) \| \leq \frac{1}{\Gamma(\alpha)} \int_0^{T_1} (T_1 - s)^{\alpha - 1} S(T_1 - s) \times \left\{ \int_{-\tau}^{0} a_1(\tau) A_1 x(s + \tau) d\tau + (Fx)(s) + k(s) \right\} ds \leq (2\alpha - 1)^{-1/2} \Gamma(\alpha)^{-1} T_1^{2(\alpha - 1)/2} C_1(\| \phi^0 \| + \| \phi^1 \|_{L^2(-\tau, 0, V)} + L_f(\| x \|_{L^2(0, T_1; V)} + 1) + \| k \|_{L^2(0, T_1; V')} ) \]

\[ := III. \]

From Lemma 2.4 and Assumption (G) we have

\[
\| (T_1 - s)^{(\alpha - 1)} A S(T_1 - s) g(s, x_s) \| \leq (T_1 - s)^{(\alpha - 1)} |A|^{1-\beta} S(T_1 - s) |\xi(T)| |A^\beta g(s, x_s) | \leq \frac{C_1 - \beta}{(T_1 - s)^{1-\alpha + (1-\beta)}} |A^\beta g(s, x_s) | \leq \frac{C_1 - \beta}{(T_1 - s)^{2-\alpha - \beta}} L_g(\| \phi^1 \|_{II} + \| x \|_{L^2(0, T_1; V)} + 1), \]

and so

\[(3.21)\]

\[
\| (I_2x)(T_1) \| = \left| \frac{1}{\Gamma(\alpha)} \int_0^{T_1} (T_1 - s)^{(\alpha - 1)} A S(T_1 - s) g(s, x_s) ds \right| \leq C_1 - \alpha + 1 \cdot T_1^{-\alpha + \beta - 1} L_g(\| \phi^1 \|_{II} + \| x \|_{L^2(0, T_1; V)} + 1) \]

\[ := IV. \]

Thus, by (3.17)-(3.21) we have

\[
\| x(T_1) \| = |S(T_1)[\phi^0 + g(0, x_0)] - g(T_1, x_{T_1}) + (I_1x)(T_1) + (I_2x)(T_1) | \leq I + II + III + IV < \infty. \]

Hence we can solve the equation in \([T_1, 2T_1]\) with the initial \((x(T_1), x_{T_1})\) and an analogous estimate to (3.4). Since the condition (3.6) is independent of initial values, the solution can be extended to the interval \([0, nT_1]\) for any natural number \(n\), and so the proof is complete. \[\square\]

Remark 3.4. Thanks for Lemma 2.3, we note that the solution of (3.1) under conditions of Theorem 3.1 with \((\phi^0, \phi^1) \in V \times L^2(0, T; D(A))\) and \(k \in L^2(0, T; H)\) for \(T > 0\) belongs to

\[
W_0(T) = L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \hookrightarrow C([0, T]; V). \]
Moreover, there is a constant $C_2$ independent of the initial data $(\phi^0, \phi^1)$ and the forcing term $k$ such that
\[
||x||_{L^2(-h,T;D(A))} \leq C_2(1 + ||\phi^0|| + ||\phi^1||_{L^2(0,T;D(A))} + ||k||_{L^2(0,T;H)}).
\]

Now, we obtain that the solution mapping is Lipschitz continuous in the following result, which is useful for the control problem and physical applications of the given equation.

**Theorem 3.5.** Let Assumptions (A), (F) and (G) be satisfied. Assuming that the initial data $(\phi^0, \phi^1) \in H \times \Pi$ and the forcing term $k \in M^2(0,T;V^*)$. Then the solution $x$ of the equation (3.1) belongs to $L^2(0,T;V)$ and the mapping
\[
(3.22) \quad H \times \Pi \times L^2(0,T;V^*) \ni (\phi^0, \phi^1, k) \mapsto x \in L^2(0,T;V)
\]
is Lipschitz continuous.

**Proof.** From Theorem 3.1, it follows that if $(\phi^0, \phi^1, k) \in L^2(\Omega, H) \times \Pi \times M^2(0,T;V^*)$, then $x$ belongs to $M^2(0,T;V)$. Let $(\phi^0_i, \phi^1_i, k_i)$ and $x^i$ be the solution of (3.1) with $(\phi^0_i, \phi^1_i, k_i)$ in place of $(\phi^0, \phi^1, k)$ for $i = 1, 2$. Let $x_i (i = 1, 2) \in \Sigma_r$. Then it holds
\[
x^1(t) - x^2(t) = S(t)[(\phi^0_1 - \phi^0_2) + (g(0,x^1_0) - g(0,x^2_0))] - (g(t,x^1_t) - g(t,x^2_t)) + \int_0^t (t - s)^{(\alpha - 1)}A S(t - s)(g(s,x^1_s) - g(t,x^2_s))ds
\]
\[
+ \frac{1}{T(\alpha)} \int_0^t (t - s)^{(\alpha - 1)}S(t - s)\{\int_{-h}^0 a_1(\tau)A_1(x^1(s + \tau) - x^2(s + \tau))d\tau\}ds
\]
\[
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{(\alpha - 1)}S(t - s)\{(F x^1)(s) - (F x^2)(s)) + (k_1(s) - k_2(s))\}ds.
\]

Hence, by applying the same argument as in the proof of Theorem 3.1, we have
\[
||x^1 - x^2||_{L^2(0,T;V)} \leq \tilde{N}_1||x^1 - x^2||_{L^2(0,T;V)} + \tilde{N}_2,
\]
where
\[
\tilde{N}_2 = C_1|\phi^0_1 - \phi^0_2| + C_1C_{-a}L_g(||\phi^1_1 - \phi^1_2||_\Pi) + C_{-a}L_g \sqrt{T_1}||\phi^1_1 - \phi^1_2||_\Pi
\]
\[
+ \frac{C_1(2\alpha)^{-1/2}(2\alpha - 1)^{-1/2}T_1}{\Gamma(\alpha)}\times (||\phi^0_1 - \phi^0_2| + ||\phi^1_1 - \phi^1_2||_{L^2(-h,0,V)} + ||k_1 - k_2||_{L^2(0,T;V^*)})
\]
\[
+ \frac{C_{1-\beta}L_g T_1(2\alpha + 3\beta - 2)^{-1/2}}{(\alpha - 3(1 - \beta)/2)(2\alpha + 3\beta - 2)^{1/2} \Gamma(\alpha)}||\phi^1_1 - \phi^1_2||_\Pi
\]
which implies
\[ ||x||_{\mathcal{M}^2(0,T,V)} \leq \hat{N}_2(1 - \hat{N})^{-1}. \]
Therefore, it implies the inequality (3.22).

**Corollary 3.6.** For a forcing term \( k \in L^2(0,T;V^*) \) let \( x_k \) be the solution of equation (3.1). Let us assume that the embedding \( V \subset H \) is compact. Then the mapping \( k \mapsto x_k \) is compact from \( L^2(0,T;V^*) \) to \( L^2(0,T;H) \).

**Proof.** If \( k \in L^2(0,T;V^*) \), then in view of Theorem 3.1
\[ ||x_k||_{W^1(T)} \leq C_3(1 + ||g^0|| + ||g^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}). \]
Hence if \( k \) is bounded in \( L^2(0,T;V^*) \), then so is \( x_k \) in \( L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \). Since \( V \) is compactly embedded in \( H \) by assumption, the embedding
\[ L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \hookrightarrow L^2(0,T;H) \]
is compact in view of Theorem 2 of J. P. Aubin [1].

4. Example

Let
\[ H = L^2(0,\pi), \quad V = H^1_0(0,\pi), \quad V^* = H^{-1}(0,\pi). \]
Consider the following retarded neutral stochastic differential system in Hilbert space \( H \):

\[
\begin{cases}
\frac{dx}{dt} + a_1(s)A_1x(t,s,y)ds + f(|x(t,y)|^2)x(t,y) + k(t,y), & (t,y) \in [0,T] \times [0,\pi], \\
x(0,y) = \phi^0(y), & (s,y) \in [-h,0] \times [0,\pi],
\end{cases}
\]

where \( h > 0, a_1(\cdot) \) is Hölder continuous, and \( A_1 \in \mathcal{L}(H) \). Let
\[ a(u,v) = \int_0^\pi \frac{du(y)}{dy} \frac{dv(y)}{dy} dy. \]
Then
\[ A = \partial^2/\partial y^2 \quad \text{with} \quad D(A) = \{ x \in H^2(0,\pi) : x(0) = x(\pi) = 0 \}. \]
The eigenvalue and the eigenfunction of \( A \) are \( \lambda_n = -n^2 \) and \( z_n(y) = (2/\pi)^{1/2} \sin ny \), respectively. Moreover,

1. \( \{ z_n : n \in N \} \) is an orthogonal basis of \( H \) and
\[ S(t)x = \sum_{n=1}^\infty e^{\lambda_n t} z_n(x, z_n), \quad \forall x \in H, \quad t > 0. \]

Moreover, there exists a constant \( M_0 \) such that \( ||S(t)||_{\mathcal{L}(H)} \leq M_0. \)
(a2) Let $0 < \alpha < 1$. Then the fractional power $A^\alpha : D(A^\alpha) \subset H \rightarrow H$ of $A$ is given by

$$A^\alpha x = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} (x, z_n) z_n, \quad D(A^\alpha) := \{ x : A^\alpha x \in H \}.$$ 

In particular,

$$A^{-1/2} x = \sum_{n=1}^{\infty} \frac{1}{n} (x, z_n) z_n, \quad \|A^{-1/2}\| = 1.$$ 

The nonlinear mapping $f$ is a real valued function belong to $C^2([0, \infty))$ which satisfies the conditions

\begin{enumerate}
\item[(f1)] $f(0) = 0$, $f(r) \geq 0$ for $r > 0$, \\
\item[(f2)] $|f'(r)| \leq c(r + 1)$ and $|qf''(r)| \leq c$ for $r \geq 0$ and $c > 0$.
\end{enumerate}

If we present $F(t, x(t, y)) = f'(|x(t, y)|^2) x(t, y)$, then it is well known that $F$ is a locally Lipschitz continuous mapping from the whole $V$ into $H$ by Sobolev’s imbedding theorem (see [23, Theorem 6.1.6]).

As an example of $q$ in the above, we can choose $q(r) = \mu^2 r + \eta^2 r^2/2$ ($\mu$ and $\eta$ is constants).

Define $g : [0, T] \times \Pi \rightarrow H$ as

$$g(t, x_t) = \sum_{n=1}^{\infty} \int_{-h}^{T} e^{nt} (\int_{-h}^{0} a_2(s) x(t+s,y) ds) dt, \quad t > 0.$$ 

Then it can be checked that Assumption (G) is satisfied. Indeed, for $x \in \Pi$, we know

$$Ag(t, x_t) = (S(t) - I) \int_{-h}^{0} a_2(s) x(t+s) ds,$$

where $I$ is the identity operator form $H$ to itself and, we assume that

$$|a_2(0)| \leq \rho, \quad |a_2(s) - a_2(\tau)| \leq \rho(s - \tau)^\kappa, \quad s, \tau \in [-h, 0]$$

for a constant $\kappa > 0$. Hence we have

$$|Ag(t, x_t)| \leq (M_0 + 1) \{ \int_{-h}^{0} (a_2(0) x(t+s) d\tau) + \int_{-h}^{0} a_2(0) x(t+s) d\tau \}.$$ 

It is immediately seen that Assumption (G) has been satisfied. Thus, all the conditions stated in Theorem 3.1 have been satisfied for the equation (4.1), and so there exists a solution of (4.1) belongs to $W_1(T) = L^2(0, T; V)) \cap W^{1,2}(0, T; V^*) \hookrightarrow C([0, T]; H)$.
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