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# REMARKS ON FINITE ELEMENT METHODS FOR CORNER SINGULARITIES USING SIF

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**Abstract.** In [15] they introduced a new finite element method for accurate numerical solutions of Poisson equations with corner singularities, which is useful for the problem with known stress intensity factor.

They consider the Poisson equations with homogeneous Dirichlet boundary condition, compute the finite element solution using standard FEM and use the extraction formula to compute the stress intensity factor, then they pose a PDE with a regular solution by imposing the nonhomogeneous boundary condition using the computed stress intensity factor, which converges with optimal speed. From the solution we could get accurate solution just by adding the singular part. This approach works for the case when we have the accurate stress intensity factor.

In this paper we consider Poisson equations with mixed boundary conditions and show the method depends the accrucy of the stress intensity factor by considering two algorithms.

## 1. Introduction

Let $\Omega$ be an open, bounded polygonal domain in $\mathbb{R}^2$ and let $\Gamma_D$ and $\Gamma_N$ be a partition of the boundary of $\Omega$ such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$. For simplicity, assume that $\Gamma_D$ is not empty (i.e., $\mathrm{meas}(\Gamma_\mathrm{D}) \neq 0$). Let $\nu$ denote the outward unit vector normal to the boundary.

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For a given function $f \in L^2(\Omega)$, as a model problem, we consider the following Poisson equation with Mixed boundary conditions:

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma_N, \end{cases} \tag{1}$$

where $\Delta$ stands for the Laplacian operator.

If $\Gamma_N = \emptyset$ ( i.e. Dirichlet boundary condition) and the domain is convex or smooth, the solution belongs to $H^2(\Omega)$ and we expect to have an optimal convergence rate with the standard finite element method. But this is not true for Poisson problems defined on non-convex domains or with mixed boundary condition. In these cases, the solutions of Poisson problems have singular behavior at that concave corner or the point changing boundary conditions and such singular behavior affects the accuracy of numerical solution throughout the whole domain.

Roughly speaking, there were two groups of people who use two different approaches for overcoming this difficulty. One is based on local mesh refinement (see, e.g., [1, 18, 19, 20, 21]). Another is done by augmenting the space of trial/test functions in which one looks for the approximate solution (see, e.g., [16, 13, 4, 5, 9, 17, 7]).

Basically the approaches of [7, 15] and this paper belong to the second one. In [15] they consider Poisson problems with Dirichlet boundary condition defined on a polygonal domain $\Omega$ with one reentrant corner (i.e. $\Gamma_N = \emptyset$.)

We consider the case $\Gamma_N \neq \emptyset$. The solution of (1) has singular behavior at the boundary point where the boundary condition changes as well as its concave corner (even when $f$ is very smooth). For simplicity, we assume there is only one singular point where the boundary conditions changes with the inner angle $w$ : $\frac{\pi}{2} < \omega \leq \frac{3\pi}{2}$. Without the loss of generality, we assume that the singular corner is at the origin. As in [8] we may consider the two cases $D/N$ and $N/D$, where $D/N$ means the boundary condition change from Dirichlet to Neumann countclockwise in the domain, for example, as in **Figure 1** with $\omega = \pi$.

For simplicity again we assume that we have $D/N$ so the singular function $s$ and its dual singular function $s_-$ can be expressed by

$$s = s(r,\theta) = r^{\frac{\pi}{2\omega}} \sin \frac{\pi\theta}{2\omega}, \quad s_- = s_-(r,\theta) = r^{-\frac{\pi}{2\omega}} \sin \frac{\pi\theta}{2\omega} \tag{2}$$

for the model problem (1) and the unique solution $u \in H_D^1(\Omega)$ has the representation (see [13, 8])

$$u = w + \lambda\eta s, \tag{3}$$

where $w \in H^2(\Omega) \cap H^1_D(\Omega)$, and $\eta$ is a smooth cut-off function which equals one identically in a neighborhood of the origin and the support of $\eta$ is small enough so that the function $\eta s$ vanishes identically on $\Gamma_D$. (Here, $(r, \theta)$ is polar coordinate.)

The coefficient, $\lambda$, is called 'stress intensity factor' and can be computed by the following extraction formula (see [8]):

$$\lambda = \frac{2}{\pi}\int_\Omega f\eta s_- dx + \frac{2}{\pi}\int_\Omega u\Delta(\eta s_-)dx. \tag{4}$$

Note that both $s$ and $s_-$ are harmonic functions in $\Omega$.

As observed in [15], some numerical approaches (e.g. [2, 4, 7]) use this extraction formula for $\lambda$ and seek the regular part $w \in H^2(\Omega)$ from new partial differential equation, for example,

$$-\Delta w = f + \lambda\Delta(\eta s) \qquad \text{in } \Omega. \tag{5}$$

Unfortunately, the results were not good enough because the input function $f$ was replaced by $f + \lambda\Delta(\eta s)$, etc., whose $L^2-$ norms are quite large compared to that of $f$ (see Lemma 2.2 in [15] ).

In [15] they introduced new partial differential equation, whose solution is in $H^2(\Omega)$ with the same input function by simple changing of the boundary condition. Using this partial differential equation, they suggested an efficient algorithm to compute the numerical solution for Poisson equation with singular domain.

In this paper we consider a PDE with the mixed boundary condition, which has stronger singularity than one with the Dirichlet condition. We consider two algorithms: the first one that is similar to suggested in [15] and the second one which use the stress intensity factor obtained by the method introduced by Cai and Kim([7]). Note both procedure can be stated as the following solution procedure;

**Step 1)** Find the stress intensity factor $\lambda$ using a suitable method for the partial differential equation (1).

**Step 2)** Pose new partial differential equation which has zero stress intensity factor and find the solution $w$

$$\left\{\begin{array}{rcll} -\Delta w &=& f & \text{in } \Omega, \\ w &=& -\lambda s|_{\Gamma_D} & \text{on } \Gamma_D, \\ \frac{\partial w}{\partial \nu} &=& 0 & \text{on } \Gamma_N, \end{array}\right. \tag{6}$$

**Step 3)** Set $u = w + \lambda s$.

**Remark :** As S. Brenner's comments in the paper [4], the stress intensity factor computed from the extraction formula depends on the regularity of the solution $u$. So, the convergence of the solution depend on the accuracy of the stress intensity factors we use in the algorithm.

In Section 2, we suggest two algorithms by choosing two methods to determine the stress intensity factors. A couple of examples will be given in Section 4 with computational results using FreeFEM++ code.([14])

We will use the standard notation and definitions for the Sobolev spaces $H^t(\Omega)$ for $t \geq 0$; the standard associated inner products are denoted by $(\cdot, \cdot)_{t,\Omega}$, and their respective norms and seminorms are denoted by $\|\cdot\|_{t,\Omega}$ and $|\cdot|_{t,\Omega}$. The space $L^2(\Omega)$ is interpreted as $H^0(\Omega)$, in which case the inner product and norm will be denoted by $(\cdot, \cdot)_\Omega$ and $\|\cdot\|_\Omega$, respectively, although we will omit $\Omega$ if there is no chance of misunderstanding. $H^1_D(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}$.

## 2. Two methods for SIF and corresponding algorithms

We need a cut-off function to derive the singular behavior of the problem. We set

$$B(r_1; r_2) = \{(r, \theta) : r_1 < r < r_2 \text{ and } 0 < \theta < \omega\} \cap \Omega$$

and

$$B(r_1) = B(0; r_1),$$

and define a smooth enough cut-off function of $r$ as follows:

$$\eta_\rho(r) = \begin{cases} 1 & \text{in} \quad B(\tfrac{1}{2}\rho), \\ \frac{1}{16}\{8 - 15p(r) + 10p(r)^3 - 3p(r)^5\} & \text{in} \quad \overline{B}(\tfrac{1}{2}\rho; \rho), \\ 0 & \text{in} \quad \Omega \backslash \overline{B}(\rho), \end{cases} \tag{7}$$

with $p(r) = 4r/\rho - 3$. Here, $\rho$ is a parameter which will be determined so that the singular part $\eta_\rho s$ has the same boundary condition as the solution $u$ of the Model problem, where $s$ is the singular function which is given in (2). Note $\eta_\rho(r)$ is $C^2$.

### 2.1. Singularity and extraction formula

The solution of the Poisson equation on the polygonal domain is well known as in [2, 4, 13]. Given $f \in L^2(\Omega)$, if we assume there is only one reentrant corner with inner angle $\pi < \omega < 2\pi$, then there exists a unique solution $u$ and in addition there exists a unique number $\lambda$ such that

$$u - \lambda s \in H^2(\Omega). \tag{8}$$

By using the cut-off function $\eta = \eta_\rho$ we may write

$$u = w + \lambda \eta s, \tag{9}$$

with $w \in H^2(\Omega) \cap H_0^1(\Omega)$.

The constant $\lambda$ is referred as stress intensity factor and computed by the following formula ([8]);

**Lemma 2.1.** *The stress intensity factor $\lambda$ can be expressed in terms of $u$ and $f$ by the following extraction formula*

$$\lambda = \frac{2}{\pi} \int_\Omega f \eta s_- dx + \frac{2}{\pi} \int_\Omega u \Delta(\eta s_-) dx. \tag{10}$$

Assume that (1) has a solution $u$ as in (9) and the stress intensity factor $\lambda$ is known, then we introduce the following boundary value problem:.

$$\begin{cases} -\Delta w &= f & \text{in } \Omega, \\ w &= -\lambda s & \text{on } \Gamma_D, \\ \frac{\partial w}{\partial \nu} &= 0 & \text{on } \Gamma_N, \end{cases} \tag{11}$$

Note the input function $f$ is the same as in (1) and $s = s|_{\Gamma_D}$ is the restriction of the singular function $s$ to the boundary $\Gamma_D$.

### 2.2. Regularity of new Partial Differential Equation

The following theorems show (11) has a regular solution.

**Theorem 2.2.** *If* (1) *has a solution $u$ as in* (9) *with the stress intensity factor $\lambda$, then* (11) *has a unique solution $w$ in $H^2(\Omega)$.*

*Proof.* First, we note (1) has a unique solution and its stress intensity factor is $\lambda$. The uniqueness of the solution of Poisson problem also implies the following equation has a unique solution with the stress intensity factor $-\lambda$ :

$$\begin{cases} -\Delta p &= 0 & \text{in } \Omega, \\ p &= -\lambda s & \text{on } \Gamma_D, \\ \frac{\partial p}{\partial \nu} &= 0 & \text{on } \Gamma_N. \end{cases} \tag{12}$$

( Note $p = -\lambda s$ is the unique solution and the coefficient of the singular function $s$ is the stress intensity factor.) By adding two equations, (1) and (12), we have the following equation

$$\left\{ \begin{array}{rcll} -\Delta w & = & f & \text{in } \Omega, \\ w & = & -\lambda s & \text{on } \Gamma_D, \\ \frac{\partial w}{\partial \nu} & = & 0 & \text{on } \Gamma_N, \end{array} \right. \tag{13}$$

whose solution $w = u + p$ belongs to $H^2(\Omega)$. □

**Theorem 2.3.** *If $\lambda$ is the stress intensity factor given by* (10) *with the solution $u$ in* (1) *and $w$ is the solution of* (11), *then $u = w + \lambda s$ is the unique solution of* (1).

*Proof.* We only need to show $u = w + \lambda s$ is the solution to (1) when $w$ is the solution of (11). Since $\Delta s = 0$, we have

$$-\Delta u = -\Delta w - \lambda \Delta s = \Delta w = f.$$

Moreover, we have

$$u|_{\Gamma_D} = w|_{\Gamma_D} + \lambda s|_{\Gamma_D} = -\lambda s + \lambda s = 0,$$

and

$$\frac{\partial u}{\partial \nu}|_{\Gamma_N} = \frac{\partial w}{\partial \nu}|_{\Gamma_N} + \lambda \frac{\partial s}{\partial \nu}|_{\Gamma_N} = 0 + \lambda \cdot 0 = 0.$$

□

### 2.3. Proposed two algorithms

Now we suggest two algorithms in variational form for the solution $u$ of the model problem (1), which use two different methods to compute approximated stress intensity factor, respectively.

For the first algorithm we use the approximated stress intensity factor $\lambda_{BD}$ form the formula in (10) with the approximated solution obtained by standard finite element method. For the second algorithm use the stress intensity factor $\lambda_{CK}$ computed by the method introduced by Cai and Kim([7]).

So the followings are two algorithm;

**The first algorithm (V1)**

**V1-1:** To find $u \in H^1_D(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall\, v \in H^1_D(\Omega). \tag{14}$$

**V1-2:** Then compute $\lambda = \lambda_{BD}$ by (10) with $u$.

**V1-3:** To find $w$ such that $w + \lambda_{BD} s \in H^1_D(\Omega)$ and

$$(\nabla w, \nabla v) = (f, v), \quad \forall\, v \in H^1_D(\Omega). \tag{15}$$

**V1-4:** Finally set $u = w + \lambda_{BD} s$.

The existence and uniqueness of the solution $u$ and $w$ in **V1** and **V2** is clear. By Theorem 2.2 and Theorem 2.3 we have the solution $w \in H^2(\Omega)$ and $u$ is the solution of (1).

Now we state the second algorithm:

**The second algorithm (V2)**

**V2-1:** First compute $\lambda = \lambda_{CK}$ by the method introduced by Cai and Kim([7]).

**V2-2:** Then find $w$ such that $w + \lambda_{CK} s \in H^1_D(\Omega)$ and

$$(\nabla w, \nabla v) = (f, v), \quad \forall\, v \in H^1_D(\Omega). \tag{16}$$

**V2-3:** Finally set $u = w + \lambda_{CK} s$.

## 3. Finite Element Approximation

In this section we present standard finite element approximation for $u$ obtained in the algorithm in the $L^2$ and $H^1$ norms. Let $T_h$ be a partition of the domain $\Omega$ into triangular finite elements; i.e., $\Omega = \cup_{K \in T_h} K$ with $h = \max\{\text{diam} K : K \in T_h\}$. Let $V_h$ be continuous piecewise linear finite element space; i.e.,

$$V_h = \{\phi_h \in C^0(\Omega) : \phi_h|_K \in P_1(K)\ \forall K \in T_h, \phi_h = 0 \text{ on } \Gamma_{\rm D}\} \subset {\rm H}^1_{\rm D}(\Omega),$$

where $P_1(K)$ is the space of linear functions on $K$.

Now the error analysis of the method in the standard norms, $\|\cdot\|$ and $|\cdot|_1$, is carried out with a regular triangulation and continuous piecewise linear finite element space $V_h$. ( See [15])

Note we can find approximated solution $u_h$ using the following Algorithm:

Algorithm 1 (**A1**) :

**A1-1:** To find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v) = (f, v) \quad \forall\, v \in V_h. \tag{17}$$

**A1-2:** Then compute $\lambda_{BD,h}$ by

$$\lambda_{BD,h} = \frac{2}{\pi} \int_\Omega f \eta s_- dx + \frac{2}{\pi} \int_\Omega u_h \Delta(\eta s_-) dx. \tag{18}$$

**A1-3:** To find $w_h$ such that $w_h + \lambda_{BD,h} s \in V_h$ and

$$(\nabla w_h, \nabla v) = (f, v) \quad \forall\, v \in V_h. \tag{19}$$

**A1-4:** Then $u_h = w_h + \lambda_{BD,h} s$.

The second one using $\lambda = \lambda_{CK,h}$ from [7] is the following.

Algorithm 2 (**A2**) :

**A2-1:** Compute $\lambda_{CK,h}$ using the method by Cai and Kim([7]).
**A2-2:** Find $w_h$ such that $w_h + \lambda_{CK,h} s \in V_h$ and

$$(\nabla w_h, \nabla v) = (f, v) \quad \forall\, v \in V_h. \tag{20}$$

**A2-3:** Then $u_h = w_h + \lambda_{CK,h} s$.

## 4. Numerical results and conclusions

In this section we consider two examples with mixed boundary condition, with inner angles $\omega = \pi$ and $\omega = \frac{3\pi}{2}$. We note that the later one is more singular than the first.

**Example 1.** Consider the Poisson equation in (1) with mixed boundary conditions on the rectangular domain $\Omega_1 = \{(x, y) \in R^2 : -1 < x < 1, 0 < y < 1\}$ with $\Gamma_N = \{(x, 0) \in R^2 : -1 < x < 0\}$ and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ (see **Figure 1**). This problem has a singularity at the origin (0, 0), where the boundary conditions change from Dirichlet to Neumann with an internal angle $\omega = \pi$. More specifically, the corresponding singular function has the form

$$s = r^{\frac{1}{2}} \sin(\frac{\theta}{2}).$$

Let $\eta_{ex} = \eta_{3/4}$ be the cut-off function in (7) with $\rho = 3/4$ and choose the right-hand side function in (1) to be

$$f = -\Delta(\eta_{ex} s).$$

Then the exact solution of the underlying problem is

$$u = \eta_{ex} s.$$

The exact stress intensity factor is 1 and the errors of the computed stress intensity factors, $\lambda_{BD}$ and $\lambda_{CK}$, are given in Table 1, The errors and rates of approximated solutions by two algorithms, (**A1**) and (**A2**), are presented in Table 2 and 3, respectively.

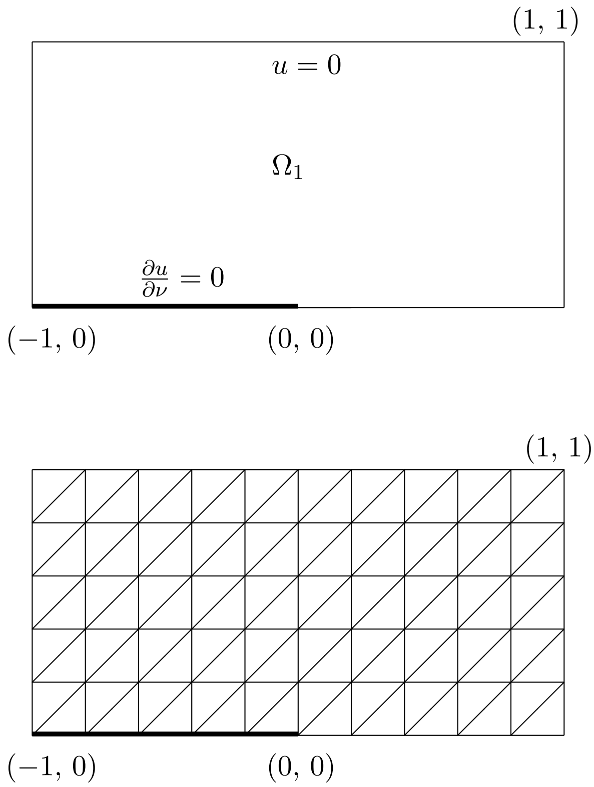


FIGURE 1. Rectangular domain with Mixed boundary condition and its regular mesh

| Mesh Size | ERROR OF $\lambda_{BD}$ | RATE | ERROR OF $\lambda_{CK}$ | RATE |
|---|---|---|---|---|
| $h = \frac{1}{4}$ | 1.1329e-01 | | 1.5081e-01 | |
| $h = \frac{1}{8}$ | 4.4056e-03 | 4.6845 | 5.5008e-03 | 4.7769 |
| $h = \frac{1}{16}$ | 3.8815e-03 | 0.1827 | 8.6641e-03 | -0.6554 |
| $h = \frac{1}{32}$ | 3.9057e-03 | -0.0089 | 7.9650e-04 | 3.4433 |
| $h = \frac{1}{64}$ | 2.1765e-03 | 0.8435 | 4.6521e-04 | 0.7757 |
| $h = \frac{1}{128}$ | 1.1943e-03 | 0.8658 | 6.4045e-05 | 2.8607 |
| $h = \frac{1}{256}$ | 6.1818e-04 | 0.9501 | 1.9587e-05 | 1.7092 |

TABLE 1. Errors of the $\lambda_{BD}$ and $\lambda_{CK}$ and convergence rates

| Mesh Size | $L^2$-NORM | RATE | $H^1$-NORM | RATE |
|---|---|---|---|---|
| $h = \frac{1}{4}$ | 6.1711e-02 | | 8.0260e-01 | |
| $h = \frac{1}{8}$ | 1.7006e-02 | 1.8594 | 4.2647e-01 | 0.9122 |
| $h = \frac{1}{16}$ | 4.5586e-03 | 1.8993 | 2.1866e-01 | 0.9637 |
| $h = \frac{1}{32}$ | 1.1557e-03 | 1.9798 | 1.1024e-01 | 0.9880 |
| $h = \frac{1}{64}$ | 2.9189e-04 | 1.9852 | 5.5529e-02 | 0.9893 |
| $h = \frac{1}{128}$ | 7.3039e-05 | 1.9987 | 2.7738e-02 | 1.0013 |
| $h = \frac{1}{256}$ | 1.8216e-05 | 2.0034 | 1.3872e-02 | 0.9996 |

TABLE 2. Errors and convergence rates with **A1** when $\omega = \pi$

| Mesh Size | $L^2$-NORM | RATE | $H^1$-NORM | RATE |
|---|---|---|---|---|
| $h = \frac{1}{4}$ | 5.7964e-02 | | 7.9108e-01 | |
| $h = \frac{1}{8}$ | 1.6219e-02 | 1.2219 | 4.1912e-01 | 0.9165 |
| $h = \frac{1}{16}$ | 4.5317e-03 | 1.8849 | 2.1810e-01 | 0.9424 |
| $h = \frac{1}{32}$ | 1.1528e-03 | 1.8943 | 1.1005e-01 | 0.9868 |
| $h = \frac{1}{64}$ | 2.9215e-04 | 1.9858 | 5.5557e-02 | 0.9861 |
| $h = \frac{1}{128}$ | 7.3149e-05 | 1.9843 | 2.7764e-02 | 1.0007 |
| $h = \frac{1}{256}$ | 1.8258e-05 | 2.0011 | 1.3892e-02 | 0.9990 |

TABLE 3. Errors and convergence rates with **A2** when $\omega = \pi$

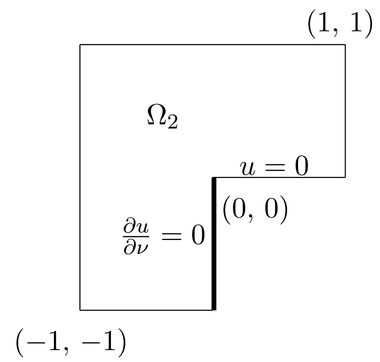


FIGURE 2. L-shape domain with mixed boundary conditions

**Example 2.** Consider the Poisson equation in (1) with mixed boundary conditions on a $\Gamma$-shaped domain $\Omega_2 = (-1,1) \times (-1,1) \setminus ([0,1] \times [-1,0])$ with $\Gamma_N = \{(0,y) \in R^2 : -1 < y < 0\}$ and $\Gamma_D = \partial\Omega \setminus \Gamma_N$ (see **Figure 2**). This problem also has a singularity at the origin (0, 0), where the boundary conditions change from Dirichlet to Neumann with an internal angle $\omega = \frac{3\pi}{2}$. More specifically, the corresponding singular function has the form

$$s = r^{\frac{1}{3}} \sin(\frac{\theta}{3}).$$

Let $\eta_{ex} = \eta_{3/4}$ be the cut-off function in (7) with $\rho = 3/4$ and choose the right-hand side function in (1) to be

$$f = -\Delta(\eta_{ex} s).$$

Then the exact solution of the underlying problem is

$$u = \eta_{ex} s.$$

The exact stress intensity factor is 1 and the errors of the computed stress intensity factors, $\lambda_{BD}$ and $\lambda_{CK}$, are given in Table 4, The errors and rates of approximated solutions by two algorithms, (**A1**) and (**A2**), are presented in Table 5 and 6, respectively.

| Mesh Size | Error of $\lambda_{BD}$ | Rate | Error of $\lambda_{CK}$ | Rate |
|---|---|---|---|---|
| $h = \frac{1}{4}$ | 1.3385e-01 | | 1.6043e-01 | |
| $h = \frac{1}{8}$ | 3.1545e-02 | 1.51099 | 3.0878e-02 | 1.7169 |
| $h = \frac{1}{16}$ | 3.5361e-02 | 0.08521 | 1.8853e-02 | 1.2371 |
| $h = \frac{1}{32}$ | 2.8238e-02 | 0.35194 | 3.5436e-03 | 2.4239 |
| $h = \frac{1}{64}$ | 1.8457e-02 | 0.57058 | 8.5586e-04 | 1.7402 |
| $h = \frac{1}{128}$ | 1.2015e-02 | 0.61930 | 2.4227e-04 | 1.8208 |
| $h = \frac{1}{256}$ | 7.7300e-03 | 0.63665 | 5.9000e-05 | 2.0387 |

Table 4. Errors of the $\lambda_{BD}$ and $\lambda_{CK}$ and convergence rates

| Mesh Size | $L^2$-norm | Rate | $H^1$-norm | Rate |
|---|---|---|---|---|
| $h = \frac{1}{4}$ | 8.8499e-02 | | 1.1591e-00 | |
| $h = \frac{1}{8}$ | 2.4090e-02 | 1.8772 | 6.1202e-01 | 0.9213 |
| $h = \frac{1}{16}$ | 5.6925e-03 | 2.0813 | 2.9541e-01 | 1.0508 |
| $h = \frac{1}{32}$ | 1.6361e-03 | 1.7988 | 1.5648e-01 | 0.9167 |
| $h = \frac{1}{64}$ | 4.4645e-04 | 1.8736 | 7.8155e-02 | 1.0015 |
| $h = \frac{1}{128}$ | 1.2626e-04 | 1.8220 | 3.7692e-02 | 1.0521 |
| $h = \frac{1}{256}$ | 3.9579e-05 | 1.6736 | 1.7390e-02 | 1.1602 |

Table 5. Errors and convergence rates with **A1** when $\omega = 3\pi/2$

| Mesh Size | $L^2$-NORM | RATE | $H^1$-NORM | RATE |
|---|---|---|---|---|
| $h = \frac{1}{4}$ | 8.9167e-02 | | 1.1614e-00 | |
| $h = \frac{1}{8}$ | 2.4493e-02 | 1.8641 | 6.1904e-01 | 0.9078 |
| $h = \frac{1}{16}$ | 5.9124e-03 | 2.0506 | 3.0321e-01 | 1.0297 |
| $h = \frac{1}{32}$ | 1.6595e-03 | 1.8329 | 1.6170e-01 | 0.9070 |
| $h = \frac{1}{64}$ | 4.2585e-04 | 1.9624 | 8.2044e-02 | 0.9789 |
| $h = \frac{1}{128}$ | 1.0572e-04 | 2.0101 | 4.0842e-02 | 1.0064 |
| $h = \frac{1}{256}$ | 2.5382e-05 | 2.0583 | 2.0030e-02 | 1.0279 |

TABLE 6. Errors and convergence rates with **A2** when $\omega = 3\pi/2$

Now we have the following conclusions from the theorems together with the examples;

**Conclusion 1 :** We may use the method given in [15] for the Poisson problem with mixed boundary condition.

**Conclusion 2 :** As we see in TABLE 2-3, the algorithm **A1** may give almost the same results as the algorithms and **A2**, when $\omega = \pi$.

**Conclusion 3 :** In the case with stronger singularity as in example 2, the method given in [7] gives better stress intensity factor, so the algorithm 2 gives better results than the algorithm 1 as we see in TABLE 5-6.

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