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SHARP HEREDITARY CONVEX RADIUS OF CONVEX HARMONIC MAPPINGS UNDER AN INTEGRAL OPERATOR

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Abstract. In this paper, we study the hereditary convex radius of convex harmonic mapping $f(z) = f_1(z) + \overline{f_2(z)}$ under the integral operator $I_f(z) = \int_0^z \frac{f_1(u)}{u} du + \overline{\int_0^z \frac{f_2(u)}{u}}$ and obtain the sharp constant $\frac{4\sqrt{6}-\sqrt{15}}{9}$, which generalized the result corresponding to the class of analytic functions given by Nash.

1. Introduction

Let Ω be a simply connected domain in the complex plane $\mathbb{C}$. We say that a function $f(z) \in C^2(\Omega)$ is a *harmonic mapping* if it satisfies the Laplacian equation $\Delta f(z) = 4f_{z\bar{z}} = 0$. A harmonic mapping defined on Ω has a canonical expression

$$f(z) = f_1(z) + \overline{f_2(z)},$$

where $f_1(z)$ and $f_2(z)$ are analytic on Ω. For an analytic function g, we write

$$f \tilde{*} g := f_1 * g + \overline{f_2 * g},$$

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where $f * g$ denotes the convolution of the functions f and g ($*$ denotes the Hadamard product).

A domain $M \subset \mathbb{C}$ is said to be *convex in the direction* $e^{i\varphi}, \varphi \in [0, 2\pi)$ if for every $a \in \mathbb{C}$ the set

$$M \cap \{a + te^{i\varphi} : t \in \mathbb{R}\}$$

is either connected or empty. A domain $M \subset \mathbb{C}$ is said to be *convex* if it is convex in every direction. We denote by $K(\varphi)$ the family of univalent analytic functions f in the unit disk $\mathbb{D}$ with $f(\mathbb{D})$ convex in the direction $e^{i\varphi}$. Similarly, $K_H(\varphi)$ is the set of all univalent harmonic functions convex in the direction $e^{i\varphi}$. Denote by K and K_H the set of convex univalent functions and convex harmonic functions, respectively.

It is well-known that if $f(z)$ is a conformal mapping which maps the unit disk $\mathbb{D}$ onto a convex domain, then the convex radius of $f(z)$ has the hereditary property, that is, $f(rz)$ maps $\mathbb{D}$ to a convex domain for any $r \in (0, 1)$. However, when $f(z) \in K(\varphi)$, $f(z)$ doesn't have this hereditary property. In 1979, Goodman and Saff [7] constructed an analytic function $g(z) \in K(\pi/2)$ such that $g(rz)$ maps $\mathbb{D}$ onto a convex domain for $0 < r \leq \sqrt{2} - 1$, but not for $\sqrt{2} - 1 < r < 1$. They conjectured that the radius $\sqrt{2} - 1$ is the best possible. In 1989, Ruscheweyh and Salinas [2] verified the Goodman-Saff conjecture, they obtained the following theorem

Theorem A. *Let $f \in K_H(\varphi)$, $0 < r \leq \sqrt{2} - 1$, then $f(rz) \in K_H(\varphi)$.*

In order to prove theorem A, the class of DCP functions were introduced.

Definition 1.1. An analytic function $g(z), z \in \mathbb{D}$ is said to be a directional convex preservation (DCP) function if for every $\varphi \in [0, 2\pi)$ and every $f \in K(\varphi)$, we have $g * f \in K(\varphi)$.

They also gave a criterion for g to be in DCP.

Lemma 1.1. *Let a non-constant function g be analytic in the unit disk $\mathbb{D}$ which is continuous in $\overline{\mathbb{D}}$ with $u(\theta) = \Re g(e^{i\theta}) \in C^3(\mathbb{D})$. Then $g \in DCP$ if and only if u is periodically monotone and satisfies*

$$u'(\theta)u'''(\theta) \leq (u''(\theta))^2, \theta \in \mathbb{R}.$$

Definition 1.2. Let u be a real, continuous and 2π-periodic function. It is said to be periodically monotone if there exist two numbers $\theta_1 < \theta_2 < \theta_1 + 2\pi$ such that u increases on (θ_1, θ_2) and decreases on $(\theta_2, \theta_1 + 2\pi)$.

The sharp convex radius of some subclasses of analytic functions and harmonic mappings have also been studied (see [3], [4] and [5]). Nagpal and Ravichandran [5] utilized coefficient estimates to obtain the hereditary convex radius of certain harmonic mappings. In 1915, Alexander [1] introduced an integral operator

$$I_f(z) = \int_0^z \frac{f(u)}{u} du,$$

where $f(z)$ is an analytic function. Nash [8] estimated the convex radius of I_f, and gave the following theorem

Theorem B. *Let $f \in K(\varphi)$ satisfy $f(0) = 0$. Then $I_f(rz) \in K(\varphi)$ for $0 < r \leq \frac{(4\sqrt{6}-\sqrt{15})}{9}$, and the constant $\frac{4\sqrt{6}-\sqrt{15}}{9}$ is the best possible.*

Nagpal and Rovichandran [6] defined the integral operator I_f for harmonic mappings as

$$I_f(z) = \int_0^z \frac{f_1(u)}{u} du + \overline{\int_0^z \frac{f_2(u)}{u} du},$$

where $f(z) = f_1(z) + \overline{f_2(z)}$ is a harmonic mapping in $\mathbb{D}$.

In this paper, we will generalize the result given by Nash to the class of harmonic mappings (see Theorem 2.1). Moreover, the sharp example is also given (see Example 2.1). In order to prove our result we also need the following lemma (see [2]).

Lemma A. *Let g be analytic in $\mathbb{D}$, Then $f \tilde{*} g \in K_H(\varphi)$ for all $f \in K_H(\varphi)$ if and only if $g \in DCP$.*

2. Main results

Theorem 2.1. *If $f(z) = f_1(z) + \overline{f_2(z)} \in K_H(\varphi)$, satisfies $f(0) = 0$, then $I_f(rz) \in K_H(\varphi)$ for $0 < r \leq r_0 = \frac{4\sqrt{6}-\sqrt{15}}{9}$.*

Proof. First, we will prove $g_r(z) = -\log(1 - rz)$ is a DCP function for some $0 < r \leq r_0$. Since

$$u_r(\theta) = \Re g_r(z) = -\Re \log(1 - rz) = -\frac{1}{2} \log(1 + r^2 - 2r\cos\theta),$$

the fact that the function $\log x$ is increasing and the function $\cos\theta$ is periodically monotone implies that $u_r(\theta)$ is periodically monotone. After taking derivatives of $u_r(\theta)$ in θ, we get

$$u_r'(\theta) = -r\frac{\sin\theta}{1+r^2-2r\cos\theta},$$

$$u_r''(\theta) = -r\frac{(1+r^2)\cos\theta-2r}{(1+r^2-2r\cos\theta)^2},$$

and

$$u_r'''(\theta) = -r\frac{-(1+r^2)^2\sin\theta-2r(1+r^2)\sin\theta\cos\theta+8r^2\sin\theta}{(1+r^2-2r\cos\theta)^3}.$$

Assume that $u_r'(\theta)u'''(\theta) \leq (u_r''(\theta))^2$, we obtain

$$2r(1+r^2)\cos^3\theta - 8r^2\cos^2\theta + 2r(1+r^2)\cos\theta + 4r^2 - (1+r^2)^2 \leq 0.$$

Write

$$f(x) = 2r(1+r^2)x^3 - 8r^2x^2 + 2r(1+r^2)x + 4r^2 - (1+r^2)^2,$$

where $x = \cos\theta$. Then we just need to find the range of r for which the function $f(x)$ is nonpositive in the whole interval $-1 \leq x \leq 1$. We first find the maximum of $f(x), x \in [-1,1]$. Taking the derivative of $f(x)$ in x, we get

$$f'(x) = 6r(1+r^2)x^2 - 16r^2x + 2r(1+r^2).$$

By direct calculation, we have that $f'(x) \geq 0$ for $r \in (0, \frac{1}{\sqrt{3}})$. Thus, we have that if $r \in (0, \frac{1}{\sqrt{3}})$, then $f(x)$ increases monotonically with $x \in [-1,1]$. So $f_{\max} = f(1)$, and for every $r \in (0, \frac{1}{\sqrt{3}})$, $f(1) = -4r^2 - (1+r^2)^2 + 4r(1+r^2) \leq 0$, which shows that $f \in DCP$ for $r \in (0, \frac{1}{\sqrt{3}})$.

When $r \in [\frac{1}{\sqrt{3}}, 1)$, the equation $f'(x) = 0$ have two roots which are

$$x_1 = \frac{4r - \sqrt{16r^2 - 3(1+r^2)^2}}{3(1+r^2)}, \quad x_2 = \frac{4r + \sqrt{16r^2 - 3(1+r^2)^2}}{3(1+r^2)}.$$

Since $f(-1) = -4r(1+r^2) - 4r^2 - (1+r^2)^2 \leq 0$ and $f(1) \leq 0$, we assume that $f(x)$ has a local maximum at x_1. Now we only need to find the minimum positive root of the equation $f(x_1) = 0$ in r. By direct calculation we have that $f(x_1) \leq 0$ implies that $\frac{1}{\sqrt{3}} \leq r \leq \frac{4\sqrt{6}-\sqrt{15}}{9}$. So, $g_r(z) \in DCP$ for $0 < r \leq \frac{4\sqrt{6}-\sqrt{15}}{9}$.

We know that $I_f(rz) = f(z)\tilde{*}g_r(z)$, then we have from Lemma A that $I_f(rz) \in K_H(\varphi)$ for $0 < r \leq r_0 = \frac{4\sqrt{6}-\sqrt{15}}{9}$. □

The following example shows that the constant $\frac{4\sqrt{6}-\sqrt{15}}{9}$ is sharp.

EXAMPLE 2.1. Under the action of integral operator I_f, the mapping

$$L(z) = \frac{1}{2}\frac{2z - z^2}{(1-z)^2} + \overline{\frac{1}{2}\frac{-z^2}{(1-z)^2}}$$

becomes a new harmonic mapping

$$F(z) = \frac{1}{2}(-\log(1-z) + \frac{z}{1-z}) + \overline{\frac{1}{2}(-\log(1-z) - \frac{z}{1-z})}.$$

Moreover, $F(z)$ maps the subdisk $\{z, |z| < r \leq r_0\}$ onto a convex region, where $r_0 = \frac{4\sqrt{6}-\sqrt{15}}{9}$.

Proof. In order to prove our result, it is necessary to study the argument change of the tangent direction

$$\Psi_r(\theta) = \arg\{\frac{\partial}{\partial_\theta}F(re^{i\theta})\}$$

of the image curve as the point $z = re^{i\theta}$ moves around the circle $|z| = r$. Note that

$$\frac{\partial}{\partial\theta}F(re^{i\theta}) = A(r,\theta) + iB(r,\theta),$$

where

$$A(r,\theta) = \frac{-r\sin\theta}{|1-z|^2}, \quad B(r,\theta) = \frac{r\cos\theta - 2r^2 + r^3\cos^3\theta}{|1-z|^4}.$$

Now we need to find the values of r such that $\tan\Psi_r(\theta)$ is a nondecreasing function in θ for $0 \leq \theta \leq \pi$, and

$$\tan\Psi_r(\theta) = \frac{B(r,\theta)}{A(r,\theta)} = \frac{(1+r^2)\cos\theta - 2r}{-(1+r^2)\sin\theta + r\sin 2\theta}.$$

By direct verification, we have

$$(-(1+r^2)\sin\theta + r\sin 2\theta)^2\frac{\partial}{\partial\theta}\tan\Psi_r(\theta) = p(r,x),$$

where $x = \cos\theta$ and

$$p(r,x) = -2r(1+r^2)x^3 + 8r^2x^2 - 2r(1+r^2)x + (1+r^2)^2 - 4r^2.$$

The problem is now equivalent to find the value of the parameter r for which the polynomial $p(r,x)$ is nonnegative in the whole interval $-1 \leq r \leq 1$. By the proof of Theorem 2.1, we know that the harmonic mapping $F(z)$ sends each disk $\{z, |z| < r \leq r_0\}$ to a convex region, but the image is not convex when $r_0 \leq r \leq 1$.

$\square$

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