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SOME PROPERTIES OF FUZZY LATTICES AS FUZZY RELATIONS

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Abstract. We define a fuzzy lattice as a fuzzy relation, prove the distributive inequalities and the modular inequality of fuzzy lattices, and show that the fuzzy totally ordered set is a distributive fuzzy lattice and that the distributive fuzzy lattice is modular.

1. Introduction

The concept of a fuzzy set was first introduced by Zadeh ([7]) and this concept was adapted by Goguen ([4]) and Sanchez ([5]) to define and study fuzzy relations. Ajmal and Thomas ([1]) defined a fuzzy lattice as a fuzzy algebra and studied fuzzy lattices. Chon ([2], [3]) defined a fuzzy lattice as a fuzzy relation and characterized that lattice. However, the antisymmetric condition of the fuzzy lattice in [2] seems to be too strong and upper (or lower) bound conditions of the fuzzy lattice in [3] turn out to be inadequate. In this note, we weaken the antisymmetric condition and strengthen the upper (or lower) bound conditions of the fuzzy lattice, redefine a fuzzy lattice, and develop some properties of the fuzzy lattices.

In section 2, we give some definitions and review some basic properties of fuzzy lattices which will be used in the next section. In section 3,

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we prove the distributive inequalities of fuzzy lattices and the modular inequality of fuzzy lattices, and show that a fuzzy totally ordered set is a distributive fuzzy lattice and that a distributive fuzzy lattice is modular.

2. Preliminaries

In this section, we give some definitions and review some basic properties of fuzzy lattices which will be used in the next section.

Definition 2.1. Let X be a set. A function $A : X \times X \to [0,1]$ is called a *fuzzy relation* in X. The fuzzy relation A in X is *reflexive* iff $A(x,x) = 1$ for all $x \in X$ and A is *transitive* iff $A(x,z) \geq \sup_{y \in X} \min(A(x,y), A(y,z))$.

In [8], a fuzzy relation A in a set X was called antisymmetric iff $A(x,y) > 0$ and $A(y,x) > 0$ imply $x = y$. However, this definition seems to be too strong. Venugopalan ([6]) weakened this definition, that is, he call A antisymmetric iff $A(x,y) + A(y,x) > 1$ imply $x = y$. We redefine it in the following definition.

Definition 2.2. Let X be a set. A fuzzy relation A in X is *antisymmetric* iff $A(x,y) + A(y,x) > 1$ implies $x = y$ and $A(x,y) = A(y,x) > 0$ implies $x = y$. A fuzzy relation A in X is a *fuzzy partial order relation* if A is reflexive, antisymmetric, and transitive. A fuzzy partial order relation A in X is a *fuzzy total order relation* iff $A(x,y) > 0$ and $A(y,x) \leq A(x,y)$ or $A(y,x) > 0$ and $A(x,y) \leq A(y,x)$ for all $x, y \in X$. If A is a fuzzy partial order relation in X, then (X, A) is called a *fuzzy partially ordered set.* If A is a fuzzy total order relation in X, then (X, A) is called a *fuzzy totally ordered set.*

Chon ([3]) defined a fuzzy lattice, however the upper (or lower) bound conditions of the fuzzy lattice turn out to be somewhat inadequate. We strengthen those conditions in Definition 2.3 and define a fuzzy lattice based on Definition 2.2 and Definition 2.3.

Definition 2.3. Let (X, A) be a fuzzy partially ordered set and let $S \subseteq X$. An element $u \in X$ is said to be an *upper bound* for a set S iff $A(u,b) \leq A(b,u)$ and $A(b,u) > 0$ for all $b \in S$. An upper bound u_0 for S is the *least upper bound* of S iff $A(u,u_0) \leq A(u_0,u)$ and $A(u_0,u) > 0$ for every upper bound u for S. An element $v \in X$ is said to be a *lower bound* for S iff $A(b,v) \leq A(v,b)$ and $A(v,b) > 0$ for all $b \in S$. A lower bound v_0 for S is the *greatest lower bound* of S iff $A(v_0, v) \leq A(v, v_0)$ and $A(v, v_0) > 0$ for every lower bound v for S.

We denote the least upper bound of the set $\{x, y\}$ by $x \vee y$ and denote the greatest lower bound of the set $\{x, y\}$ by $x \wedge y$. $\vee$ is called a join and $\wedge$ is called a meet.

DEFINITION 2.4. Let (X, A) be a fuzzy partially ordered set. Then (X, A) is a *fuzzy lattice* iff $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$.

Example of a fuzzy lattice. Let $X = \{x, y, z\}$ and let $A : X \times X \to [0, 1]$ be a fuzzy relation such that $A(x, x) = A(y, y) = A(z, z) = 1$, $A(x, y) = 0.2$, $A(x, z) = 0.1$, $A(y, z) = 0.1$, $A(y, x) = 0.5$, $A(z, x) = 0.3$, and $A(z, y) = 0.2$. Then it is easily checked that (X, A) is a fuzzy lattice. In ([3]), a fuzzy relation B is antisymmetric iff $B(x, y) > 0$ and $B(y, x) > 0$ implies $x = y$. Clearly A is not antisymmetric by the definition in [3], and hence (X, A) is not a fuzzy lattice in [3].

From the above example, the fuzzy lattice in this note is not same as that defined in [3].

PROPOSITION 2.5. *Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then*

(1) $A(x, x \vee y) > 0$ *and* $A(x \vee y, x) \leq A(x, x \vee y)$
(2) $A(y, x \vee y) > 0$ *and* $A(x \vee y, y) \leq A(y, x \vee y)$
(3) $A(x \wedge y, x) > 0$ *and* $A(x, x \wedge y) \leq A(x \wedge y, x)$
(4) $A(x \wedge y, y) > 0$ *and* $A(y, x \wedge y) \leq A(x \wedge y, y)$
(5) *If* $A(x, z) > 0$, $A(z, x) \leq A(x, z)$, $A(y, z) > 0$, *and* $A(z, y) \leq A(y, z)$, *then* $A(x \vee y, z) > 0$ *and* $A(z, x \vee y) \leq A(x \vee y, z)$.
(6) *If* $A(z, x) > 0$, $A(x, z) \leq A(z, x)$, $A(z, y) > 0$, *and* $A(y, z) \leq A(z, y)$, *then* $A(z, x \wedge y) > 0$ *and* $A(x \wedge y, z) \leq A(z, x \wedge y)$.
(7) $A(x, y) > 0$ *and* $A(y, x) \leq A(x, y)$ *iff* $x \vee y = y$.
(8) $A(x, y) > 0$ *and* $A(y, x) \leq A(x, y)$ *iff* $x \wedge y = x$.

Proof. The proof is same as that of Proposition 2.5 of [2]. □

PROPOSITION 2.6. *Let (X, A) be a fuzzy lattice and let $x, y \in X$. Then*

(1) $x \vee x = x$, $x \wedge x = x$.
(2) $x \vee y = y \vee x$, $x \wedge y = y \wedge x$.
(3) $(x \vee y) \wedge x = x$, $(x \wedge y) \vee x = x$.

Proof. The proof is same as that of Proposition 2.6 of [2]. □

We may show that the operations of join and meet in fuzzy lattices are isotone and associative by the same way as shown in [2].

3. Some properties of fuzzy lattices

In this section, we prove the distributive inequalities of fuzzy lattices and the modular inequality of fuzzy lattices, and show that the fuzzy totally ordered set is a distributive fuzzy lattice and that the distributive fuzzy lattice is modular.

THEOREM 3.1. *(Distributive inequalities) Let* (X, A) *be a fuzzy lattice and let* $x, y, z \in X$. *Then*

(1) $A(x \wedge (y \vee z), (x \wedge y) \vee (x \wedge z)) \leq A((x \wedge y) \vee (x \wedge z),\ x \wedge (y \vee z))$ *and* $A((x \wedge y) \vee (x \wedge z),\ x \wedge (y \vee z)) > 0$

(2) $A((x \vee y) \wedge (x \vee z), x \vee (y \wedge z)) \leq A(x \vee (y \wedge z),\ (x \vee y) \wedge (x \vee z))$ *and* $A(x \vee (y \wedge z),\ (x \vee y) \wedge (x \vee z)) > 0$.

Proof.

(1) Since $A(x \wedge y,\ y) > 0$ and $A(y,\ y \vee z) > 0$, $A(x \wedge y,\ y \vee z) > 0$. Clearly

$$\begin{aligned} A(x \wedge y, y \vee z) &\geq \min\ [A(x \wedge y, y), A(y, y \vee z)] \\ &\geq \min\ [A(y, x \wedge y), A(y \vee z,\ y)]. \end{aligned}$$

(i) We consider the case of $A(y \vee z, y) > A(y, x \wedge y)$.
Then $A(x \wedge y, y \vee z) \geq A(y, x \wedge y)$. Also

$$\begin{aligned} A(y, x \wedge y) &\geq \min\ [A(y, y \vee z), A(y \vee z,\ x \wedge y)] \\ &\geq \min\ [A(y \vee z, y), A(y \vee z,\ x \wedge y)]. \end{aligned}$$

If $A(y \vee z, y) \leq A(y \vee z, x \wedge y)$, then $A(y, x \wedge y) \geq A(y \vee z, y)$ and this contradicts $A(y \vee z, y) > A(y, x \wedge y)$. That is, $A(y \vee z, y) > A(y \vee z, x \wedge y)$, and hence $A(y, x \wedge y) \geq A(y \vee z, x \wedge y)$. Thus $A(x \wedge y, y \vee z) \geq A(y, x \wedge y) \geq A(y \vee z, x \wedge y)$.

(ii) We consider the case of $A(y, x \wedge y) > A(y \vee z, y)$.
Then $A(x \wedge y, y \vee z) \geq A(y \vee z, y)$. Also

$$\begin{aligned} A(y \vee z, y) &\geq \min\ [A(y \vee z, x \wedge y), A(x \wedge y, y)] \\ &\geq \min\ [A(y \vee z, x \wedge y), A(y, x \wedge y)]. \end{aligned}$$

If $A(y, x \wedge y) \leq A(y \vee z, x \wedge y)$, then $A(y \vee z, y) \geq A(y, x \wedge y)$ and this contradicts $A(y, x\wedge y) > A(y\vee z, y)$. That is, $A(y, x\wedge y) > A(y\vee z, x\wedge y)$, and hence $A(y\vee z, y) \geq A(y\vee z, x\wedge y)$. Thus $A(x\wedge y, y\vee z) \geq A(y\vee z, y) \geq A(y \vee z, x \wedge y)$.

(iii) We consider the case of $A(y, x\wedge y) = A(y\vee z, y)$ and $A(y\vee z, x\wedge y) \geq A(x \wedge y, y)$.

Since $A(y \vee z, y) \geq \min\,[A(y \vee z, x \wedge y), A(x \wedge y, y)] \geq A(x \wedge y, y)$,

$$A(y \vee z, y) \geq A(x \wedge y, y) \geq A(y, x \wedge y) = A(y \vee z, y).$$

Thus $A(x \wedge y, y) = A(y, x \wedge y) > 0$. That is, $x \wedge y = y$. Hence $A(x \wedge y, y \vee z) = A(y, y \vee z) \geq A(y \vee z, y) = A(y \vee z, x \wedge y)$.

(iv) We consider the case of $A(y, x\wedge y) = A(y\vee z, y)$ and $A(y\vee z, x\wedge y) < A(x \wedge y, y)$.

Then $A(y \vee z, y) \geq \min\,[A(y \vee z, x \wedge y), A(x \wedge y, y)] = A(y \vee z, x \wedge y)$. Since $A(x \wedge y, y \vee z) \geq A(y \vee z, y) = A(y, x \wedge y)$, $A(x \wedge y, y \vee z) \geq A(y \vee z, y) \geq A(y \vee z, x \wedge y)$.

From (i), (ii), (iii), and (iv), $A(x \wedge y, y \vee z) \geq A(y \vee z, x \wedge y)$. Clearly $A(x \wedge y, y \vee z) > 0$. Also $A(x \wedge y,\ x) > 0$ and $A(x \wedge y, x) \geq A(x, x \wedge y)$. By (6) of Proposition 2.5,

$$A(x\wedge y,\ x\wedge(y\vee z)) \geq A(x\wedge(y\vee z),\ x\wedge y) \text{ and } A(x\wedge y,\ x\wedge(y\vee z)) > 0.$$

That is, $x \wedge (y \vee z)$ is an upper bound of $\{x \wedge y\}$.

Since $A(x \wedge z,\ z) > 0$ and $A(z,\ y \vee z) > 0$, $A(x \wedge z,\ y \vee z) > 0$. Clearly

$$A(x\wedge z, y\vee z) \geq \min\,[A(x\wedge z, z), A(z, y\vee z)] \geq \min\,[A(z, x\wedge z), A(y\vee z,\ z)].$$

$(i)'$ We consider the case of $A(y \vee z, z) > A(z, x \wedge z)$.

Then $A(x \wedge z, y \vee z) \geq A(z, x \wedge z)$. Also

$$\begin{aligned} A(z, x \wedge z) &\geq \min\,[A(z, y \vee z), A(y \vee z,\ x \wedge z)] \\ &\geq \min\,[A(y \vee z, z), A(y \vee z,\ x \wedge z)]. \end{aligned}$$

Since $A(y \vee z, z) > A(z, x \wedge z)$, $A(z, x \wedge z) \geq A(y \vee z, x \wedge z)$. Thus $A(x \wedge z, y \vee z) \geq A(y \vee z, x \wedge z)$.

$(ii)'$ We consider the case of $A(z, x \wedge z) > A(y \vee z, z)$.

Then $A(x \wedge z, y \vee z) \geq A(y \vee z, z)$. Also

$$\begin{aligned} A(y \vee z, z) &\geq \min\,[A(y \vee z, x \wedge z), A(x \wedge z, z)] \\ &\geq \min\,[A(y \vee z, x \wedge z), A(z, x \wedge z)]. \end{aligned}$$

Since $A(z, x \wedge z) > A(y \vee z, z)$, $A(y \vee z, z) \geq A(y \vee z, x \wedge z)$. Thus $A(x \wedge z, y \vee z) \geq A(y \vee z, x \wedge z)$.

$(iii)'$ We consider the case of $A(z, x\wedge z) = A(y\vee z, z)$.
Then $A(x\wedge z, y\vee z) \geq A(y\vee z, z) = A(z, x\wedge z)$. Also $A(y\vee z, z) \geq \min\,[A(y\vee z, x\wedge z), A(x\wedge z, z)]$. If $A(x\wedge z, z) \geq A(y\vee z, x\wedge z)$, then $A(x\wedge z, y\vee z) \geq A(y\vee z, z) \geq A(y\vee z, x\wedge z)$. If $A(x\wedge z, z) < A(y\vee z, x\wedge z)$, then $A(y\vee z, z) \geq A(x\wedge z, z) \geq A(z, x\wedge z) = A(y\vee z, z)$, that is, $A(x\wedge z, z) = A(z, x\wedge z) > 0$, thus $x\wedge z = z$, and hence $A(x\wedge z, y\vee z) = A(z, y\vee z) \geq A(y\vee z, z) = A(y\vee z, x\wedge z)$.

From $(i)'$, $(ii)'$, and $(iii)'$, $A(x\wedge z, y\vee z) \geq A(y\vee z, x\wedge z)$. Clearly $A(x\wedge z, y\vee z) > 0$. Also $A(x\wedge z,\ x) > 0$ and $A(x\wedge z, x) \geq A(x, x\wedge z)$. By (6) of Proposition 2.5,

$$A(x\wedge z,\ x\wedge(y\vee z)) \geq A(x\wedge(y\vee z), x\wedge z) \text{ and } A(x\wedge z,\ x\wedge(y\vee z)) > 0.$$

Thus $x\wedge(y\vee z)$ is an upper bound of $\{x\wedge y,\ x\wedge z\}$. Since $(x\wedge y)\vee(x\wedge z)$ is the least upper bound of $\{x\wedge y,\ x\wedge z\}$, $A((x\wedge y)\vee(x\wedge z),\ x\wedge(y\vee z)) \geq A(x\wedge(y\vee z), (x\wedge y)\vee(x\wedge z))$ and $A((x\wedge y)\vee(x\wedge z),\ x\wedge(y\vee z)) > 0$.

(2) Since $A(y\wedge z,\ y) > 0$ and $A(y,\ x\vee y) > 0$, $A(y\wedge z,\ x\vee y) > 0$. Clearly

$$\begin{aligned} A(y\wedge z, x\vee y) &\geq \min\,[A(y\wedge z, y), A(y, x\vee y)] \\ &\geq \min\,[A(y, y\wedge z), A(x\vee y,\ y)]. \end{aligned}$$

(i) We consider the case of $A(y, y\wedge z) > A(x\vee y, y)$.
Then $A(y\wedge z, x\vee y) \geq A(x\vee y, y)$. Also $A(x\vee y, y) \geq \min\,[A(x\vee y, y\wedge z), A(y\wedge z,\ y)] \geq \min\,[A(x\vee y, y\wedge z), A(y,\ y\wedge z)]$. Since $A(y, y\wedge z) > A(x\vee y, y)$, $A(x\vee y, y) \geq A(x\vee y, y\wedge z)$. Thus $A(y\wedge z, x\vee y) \geq A(x\vee y, y\wedge z)$.
(ii) We consider the case of $A(x\vee y, y) > A(y, y\wedge z)$.
Then $A(y\wedge z, x\vee y) \geq A(y, y\wedge z)$. Also

$$\begin{aligned} A(y, y\wedge z) &\geq \min\,[A(y, x\vee y), A(x\vee y, y\wedge z)] \\ &\geq \min\,[A(x\vee y, y), A(x\vee y, y\wedge z)]. \end{aligned}$$

Since $A(x\vee y, y) > A(y, y\wedge z)$, $A(y, y\wedge z) \geq A(x\vee y, y\wedge z)$. Thus $A(y\wedge z, x\vee y) \geq A(x\vee y, y\wedge z)$.
(iii) We consider the case of $A(x\vee y, y) = A(y, y\wedge z)$.
Then $A(y\wedge z, x\vee y) \geq A(y, y\wedge z) = A(x\vee y, y)$. Also $A(x\vee y, y) \geq \min\,[A(x\vee y, y\wedge z), A(y\wedge z, y)]$. If $A(y\wedge z, y) \geq A(x\vee y, y\wedge z)$, then $A(y\wedge z, x\vee y) \geq A(x\vee y, y) \geq A(x\vee y, y\wedge z)$. If $A(y\wedge z, y) < A(x\vee y, y\wedge z)$, then $A(x\vee y, y) \geq \min[A(x\vee y, y\wedge z), A(y\wedge z, y)] = A(y\wedge z, y) \geq A(y, y\wedge z) = A(x\vee y, y)$, that is, $A(y\wedge z, y) = A(y, y\wedge z) > 0$, thus $y = y\wedge z$, and hence $A(y\wedge z, x\vee y) = A(y, x\vee y) \geq A(x\vee y, y) = A(x\vee y, y\wedge z)$.

From (i), (ii), and (iii), $A(y \wedge z, x \vee y) \geq A(x \vee y, y \wedge z)$. Clearly $A(y \wedge z, x \vee y) > 0$. Also $A(x, x \vee y) > 0$ and $A(x, x \vee y) \geq A(x \vee y, x)$. By (5) of Proposition 2.5,

$$A(x \vee (y \wedge z),\ x \vee y) \geq A(x \vee y, x \vee (y \wedge z)) \text{ and } A(x \vee (y \wedge z),\ x \vee y) > 0.$$

That is, $x \vee (y \wedge z)$ is a lower bound of $\{x \vee y\}$.
Since $A(y \wedge z,\ z) > 0$ and $A(z,\ x \vee z) > 0$, $A(y \wedge z,\ x \vee z) > 0$. Clearly

$$\begin{aligned} A(y \wedge z, x \vee z) &\geq \min\,[A(y \wedge z, z), A(z, x \vee z)] \\ &\geq \min\,[A(z, y \wedge z), A(x \vee z,\ z)]. \end{aligned}$$

(i)$'$ We consider the case of $A(z, y \wedge z) > A(x \vee z, z)$.
Then $A(y \wedge z, x \vee z) \geq A(x \vee z, z)$. Also

$$\begin{aligned} A(x \vee z, z) &\geq \min\,[A(x \vee z, y \wedge z), A(y \wedge z,\ z)] \\ &\geq \min\,[A(x \vee z, y \wedge z), A(z,\ y \wedge z)]. \end{aligned}$$

Since $A(z, y \wedge z) > A(x \vee z, z)$, $A(x \vee z, z) \geq A(x \vee z, y \wedge z)$. Thus $A(y \wedge z, x \vee z) \geq A(x \vee z, y \wedge z)$.
(ii)$'$ We consider the case of $A(x \vee z, z) > A(z, y \wedge z)$.
Then $A(y \wedge z, x \vee z) \geq A(z, y \wedge z)$. Also

$$\begin{aligned} A(z, y \wedge z) &\geq \min\,[A(z, x \vee z), A(x \vee z, y \wedge z)] \\ &\geq \min\,[A(x \vee z, z), A(x \vee z, y \wedge z)]. \end{aligned}$$

Since $A(x \vee z, z) > A(z, y \wedge z)$, $A(z, y \wedge z) \geq A(x \vee z, y \wedge z)$. Thus $A(y \wedge z, x \vee z) \geq A(x \vee z, y \wedge z)$.
(iii)$'$ We consider the case of $A(x \vee z, z) = A(z, y \wedge z)$.
Then $A(y \wedge z, x \vee z) \geq A(z, y \wedge z) = A(x \vee z, z)$. Also $A(x \vee z, z) \geq \min\,[A(x \vee z, y \wedge z), A(y \wedge z, z)]$. If $A(y \wedge z, z) \geq A(x \vee z, y \wedge z)$, then $A(y \wedge z, x \vee z) \geq A(x \vee z, z) \geq A(x \vee z, y \wedge z)$. If $A(y \wedge z, z) < A(x \vee z, y \wedge z)$, then $A(x \vee z, z) \geq A(y \wedge z, z) \geq A(z, y \wedge z) = A(x \vee z, z)$, that is, $A(y \wedge z, z) = A(z, y \wedge z) > 0$, thus $z = y \wedge z$, and hence $A(y \wedge z, x \vee z) = A(z, x \vee z) \geq A(x \vee z, z) = A(x \vee z, y \wedge z)$.

From (i)$'$, (ii)$'$, and (iii)$'$, $A(y \wedge z, x \vee z) \geq A(x \vee z, y \wedge z)$. Clearly $A(y \wedge z, x \vee z) > 0$. Also $A(x, x \vee z) > 0$ and $A(x, x \vee z) \geq A(x \vee z, x)$. By (5) of Proposition 2.5,

$$A(x \vee (y \wedge z), x \vee z) \geq A(x \vee z,\ x \vee (y \wedge z)) \text{ and } A(x \vee (y \wedge z),\ x \vee z)) > 0.$$

Thus $x \vee (y \wedge z)$ is a lower bound of $\{x \vee y,\ x \vee z\}$. Since $(x \vee y) \wedge (x \vee z)$ is the greatest lower bound of $\{x \vee y,\ x \vee z\}$, $A(x \vee (y \wedge z), (x \vee y) \wedge (x \vee z)) \geq A((x \vee y) \wedge (x \vee z),\ x \vee (y \wedge z))$ and $A(x \vee (y \wedge z),\ (x \vee y) \wedge (x \vee z)) > 0$. $\square$

Definition 3.2. Let (X, A) be a fuzzy lattice. (X, A) is *distributive* iff $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$.

From the distributive inequalities, (X, A) is distributive iff $A(x \wedge (y \vee z),\ (x \wedge y) \vee (x \wedge z)) > 0$, $A(x \wedge (y \vee z), (x \wedge y) \vee (x \wedge z)) \geq A((x \wedge y) \vee (x \wedge z),\ x \wedge (y \vee z))$, $A((x \vee y) \wedge (x \vee z),\ x \vee (y \wedge z)) > 0$, and $A((x \vee y) \wedge (x \vee z), x \vee (y \wedge z)) \geq A(x \vee (y \wedge z),\ (x \vee y) \wedge (x \vee z))$.

Proposition 3.3. *Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \ \text{ iff } \ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Proof. ($\Rightarrow$) From the hypothesis, $(x \vee y) \wedge (x \vee z) = [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z]$. Since $(x \vee y) \wedge x = x$ by (3) of Proposition 2.6, $(x \vee y) \wedge (x \vee z) = x \vee [(x \vee y) \wedge z] = x \vee [z \wedge (x \vee y)] = x \vee [(z \wedge x) \vee (z \wedge y)] = x \vee (z \wedge x) \vee (z \wedge y)$. Since $x \vee (z \wedge x) = x$ by (3) of Proposition 2.6, $(x \vee y) \wedge (x \vee z) = x \vee (z \wedge y) = x \vee (y \wedge z)$.
($\Leftarrow$) From the hypothesis, $(x \wedge y) \vee (x \wedge z) = [(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z]$. Since $(x \wedge y) \vee x = x$ and $x \wedge (z \vee x) = x$ by (3) of Proposition 2.6, $(x \wedge y) \vee (x \wedge z) = x \wedge [z \vee (x \wedge y)] = x \wedge [(z \vee x) \wedge (z \vee y)] = [x \wedge (z \vee x)] \wedge (z \vee y) = x \wedge (z \vee y) = x \wedge (y \vee z)$. Thus $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$. □

Theorem 3.4. *Let (X, A) be a fuzzy totally ordered set. Then (X, A) is a distributive fuzzy lattice.*

Proof. Let (X, A) be a fuzzy totally ordered set and let $x, y \in X$. Then $A(x, y) > 0$ and $A(y, x) \leq A(x, y)$ or $A(y, x) > 0$ and $A(x, y) \leq A(y, x)$.
(i) We consider the case of $A(x, y) > 0$ and $A(y, x) \leq A(x, y)$.
Since $A(y, y) = 1 > 0$, y is an upper bound of $\{x, y\}$. Let u be an upper bound of $\{x, y\}$. Then $A(y, u) > 0$ and $A(u, y) \leq A(y, u)$. Thus y is the least upper bound of $\{x, y\}$. Since $A(x, y) > 0$, $A(y, x) \leq A(x, y)$, and $A(x, x) = 1 > 0$, x is a lower bound of $\{x, y\}$. Let v be a lower bound of $\{x, y\}$. Then $A(v, x) > 0$ and $A(x, v) \leq A(v, x)$. Thus x is the greatest lower bound of $\{x, y\}$. Thus (X, A) is a fuzzy lattice. By (8) of Proposition 2.5, $x \wedge y = x$. Since $A(x \wedge (y \vee z),\ x) > 0$, $A(x \wedge (y \vee z),\ x \wedge y) > 0$. By (1) of Proposition 2.5, $A(x \wedge y,\ (x \wedge y) \vee (x \wedge z)) > 0$. Thus

$$\begin{aligned} &A(x \wedge (y \vee z), (x \wedge y) \vee (x \wedge z)) \\ &\geq \min\,[A(x \wedge (y \vee z),\ x \wedge y),\ A(x \wedge y, (x \wedge y) \vee (x \wedge z))] > 0. \end{aligned}$$

Since $x \vee (x \wedge z) = x$ and $x \wedge y = x$,

$$\begin{aligned} A(x \wedge (y \vee z), (x \wedge y) \vee (x \wedge z)) &= A(x \wedge (y \vee z), x \vee (x \wedge z)) \\ &= A(x \wedge (y \vee z), x) \\ &\geq A(x, x \wedge (y \vee z)) \\ &= A(x \vee (x \wedge z), x \wedge (y \vee z)) \\ &= A((x \wedge y) \vee (x \wedge z), x \wedge (y \vee z)). \end{aligned}$$

By the distributive inequalities, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. By Proposition 3.3, $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$. Hence (X, A) is a distributive fuzzy lattice.

(ii) We consider the case of $A(y, x) > 0$ and $A(x, y) \leq A(y, x)$. Since $A(x, x) = 1 > 0$, x is an upper bound of $\{x, y\}$. Let u be an upper bound of $\{x, y\}$. Then $A(x, u) > 0$ and $A(u, x) \leq A(x, u)$. Thus x is the least upper bound of $\{x, y\}$. Since $A(y, x) > 0$, $A(x, y) \leq A(y, x)$, and $A(y, y) = 1 > 0$, y is a lower bound of $\{x, y\}$. Let v be a lower bound of $\{x, y\}$. Then $A(v, y) > 0$ and $A(y, v) \leq A(v, y)$. Thus y is the greatest lower bound of $\{x, y\}$. Thus (X, A) is a fuzzy lattice. By (7) of Proposition 2.5, $x \vee y = x$. Thus $A((x \vee y) \wedge (x \vee z),\ x) = A(x \wedge (x \vee z),\ x) > 0$. By (1) of Proposition 2.5, $A(x,\ x \vee (y \wedge z)) > 0$. Thus $A((x \vee y) \wedge (x \vee z),\ x \vee (y \wedge z)) \geq \min\,[A((x \vee y) \wedge (x \vee z),\ x),\ A(x,\ x \vee (y \wedge z))] > 0$. Since $x \vee y = x$ and $(x \vee z) \wedge x = x$,

$$\begin{aligned} A((x \vee y) \wedge (x \vee z),\ x \vee (y \wedge z)) &= A(x \wedge (x \vee z), x \vee (y \wedge z)) \\ &= A(x, x \vee (y \wedge z)) \\ &\geq A(x \vee (y \wedge z), x) \\ &= A(x \vee (y \wedge z), x \wedge (x \vee z)) \\ &= A(x \vee (y \wedge z), (x \vee y) \wedge (x \vee z)). \end{aligned}$$

By the distributive inequalities, $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$. By Proposition 3.3, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Hence (X, A) is a distributive fuzzy lattice. □

THEOREM 3.5. *(Modular inequality) Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then $A(x, z) > 0$ and $A(z, x) \leq A(x, z)$ implies $A(x \vee (y \wedge z),\ (x \vee y) \wedge z) > 0$ and $A((x \vee y) \wedge z, x \vee (y \wedge z)) \leq A(x \vee (y \wedge z),\ (x \vee y) \wedge z)$.*

Proof. Since $A(x,\ x \vee y) > 0$ and $A(x, z) > 0$, $A(x,\ (x \vee y) \wedge z) > 0$ by (6) of Proposition 2.5. Since $A(y \wedge z,\ y) > 0$ and $A(y,\ x \vee y) > 0$,

$A(y \wedge z,\ x \vee y) > 0$. Since $A(y \wedge z,\ z) > 0$, $A(y \wedge z,\ (x \vee y) \wedge z) > 0$ by (6) of Proposition 2.5. Thus $A(x \vee (y \wedge z),\ (x \vee y) \wedge z) > 0$ by (5) of Proposition 2.5. Also

$$A(y \wedge z, x \vee y) \geq \min\,[A(y \wedge z, y),\ A(y, x \vee y)] \geq \min\,[A(y, y \wedge z),\ A(x \vee y, y)].$$

(i) We consider the case of $A(y, y \wedge z) > A(x \vee y, y)$.
Clearly $A(y \wedge z, x \vee y) \geq A(x \vee y, y)$. Also $A(x \vee y, y) \geq \min\,[A(x \vee y, y \wedge z), A(y \wedge z, y)] \geq \min\,[A(x \vee y, y \wedge z),\ A(y, y \wedge z)]$. Since $A(y, y \wedge z) > A(x \vee y, y)$, $A(x \vee y, y) \geq A(x \vee y, y \wedge z)$. Thus $A(y \wedge z, x \vee y) \geq A(x \vee y, y \wedge z)$.
(ii) We consider the case of $A(y, y \wedge z) < A(x \vee y, y)$.
Clearly $A(y \wedge z, x \vee y) \geq A(y, y \wedge z)$. Also $A(y, y \wedge z) \geq \min\,[A(y, x \vee y), A(x \vee y, y \wedge z)] \geq \min\,[A(x \vee y, y),\ A(x \vee y, y \wedge z)]$. Since $A(y, y \wedge z) < A(x \vee y, y)$, $A(y, y \wedge z) \geq A(x \vee y, y \wedge z)$. Thus $A(y \wedge z, x \vee y) \geq A(x \vee y, y \wedge z)$.
(iii) We consider the case of $A(y, y \wedge z) = A(x \vee y, y)$.
Then $A(y \wedge z, x \vee y) \geq A(y, y \vee z) = A(x \vee y, y)$. Also $A(x \vee y, y) \geq \min\,[A(x \vee y, y \wedge z), A(y \wedge z, y)]$. If $A(y \wedge z, y) \geq A(x \vee y, y \wedge z)$, then $A(y \wedge z, x \vee y) \geq A(x \vee y, y \wedge z)$. If $A(y \wedge z, y) < A(x \vee y, y \wedge z)$, then $A(x \vee y, y) \geq A(y \wedge z, y) \geq A(y, y \wedge z) = A(x \vee y, y)$, and hence $A(y \wedge z, y) = A(y, y \wedge z) > 0$. That is, $y \wedge z = y$. Thus $A(y \wedge z, x \vee y) = A(y, x \vee y) \geq A(x \vee y, y) = A(x \vee y, y \wedge z)$.

From (i), (ii), and (iii), $A(y \wedge z, x \vee y) \geq A(x \vee y, y \wedge z)$ and $A(y \wedge z, x \vee y) > 0$. Since $A(y \wedge z, z) \geq A(z, y \wedge z)$ and $A(y \wedge z, z) > 0$,

$$A(y \wedge z, (x \vee y) \wedge z) \geq A((x \vee y) \wedge z, y \wedge z) \text{ and } A(y \wedge z, (x \vee y) \wedge z) > 0.$$

Since $A(x, z) \geq A(z, x)$, $A(x, z) > 0$, $A(x, x \vee y) \geq A(x \vee y, x)$, and $A(x, x \vee y) > 0$,

$$A(x, (x \vee y) \wedge z) \geq A((x \vee y) \wedge z, x) \text{ and } A(x, (x \vee y) \wedge z) > 0.$$

Thus

$$A((x \vee y) \wedge z, x \vee (y \wedge z)) \leq A(x \vee (y \wedge z),\ (x \vee y) \wedge z) \text{ and}$$
$$A(x \vee (y \wedge z),\ (x \vee y) \wedge z) > 0. \qquad \square$$

Definition 3.6. A fuzzy lattice (X, A) is *modular* iff $A(x, z) > 0$ and $A(x, z) \geq A(z, x)$ imply $x \vee (y \wedge z) = (x \vee y) \wedge z$ for $x, y, z \in X$.

By the modular inequality, a fuzzy lattice (X, A) is *modular* iff $A(x, z) > 0$ and $A(x, z) \geq A(z, x)$ imply $A((x \vee y) \wedge z,\ x \vee (y \wedge z)) > 0$ and $A((x \vee y) \wedge z,\ x \vee (y \wedge z)) \geq A(x \vee (y \wedge z),\ (x \vee y) \wedge z)$ for $x, y, z \in X$.

THEOREM 3.7. *Let (X, A) be a distributive fuzzy lattice. Then (X, A) is modular.*

Proof. Since (X, A) is distributive, $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. Thus $A((x \vee y) \wedge z, x \vee (y \wedge z)) = A((x \wedge z) \vee (y \wedge z), x \vee (y \wedge z))$. Since $A(x, z) > 0$ and $A(z, x) \leq A(x, z)$, $x \wedge z = x$ by (8) of Proposition 2.5. Thus $A((x \vee y) \wedge z, x \vee (y \wedge z)) = A((x \wedge z) \vee (y \wedge z), x \vee (y \wedge z)) = A(x \vee (y \wedge z), x \vee (y \wedge z)) > 0$. Since $A(x, z) > 0$ and $A(z, x) \leq A(x, z)$, $x \vee z = z$ by (7) of Proposition 2.5. Since (X, A) is distributive, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Thus $A((x \vee y) \wedge z, x \vee (y \wedge z)) = A((x \wedge z) \vee (y \wedge z), (x \vee y) \wedge (x \vee z)) = A(x \vee (y \wedge z), (x \vee y) \wedge z) > 0$. Hence $(x \vee y) \wedge z = x \vee (y \wedge z)$. $\square$

References

[1] N. Ajmal and K. V. Thomas, *Fuzzy lattices*, Inform. Sci. **79** (1994), 271–291.

[2] I. Chon, *Fuzzy lattices as fuzzy relations*, Korean J. Math. **23** (2015), 557–569.

[3] I. Chon, *Fuzzy partial order relations and fuzzy lattices*, Korean J. Math. **17** (2009), 361–374.

[4] J. A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl. **18** (1967), 145–174.

[5] E. Sanchez, *Resolution of composite fuzzy relation equation*, Inform. and control **30** (1976), 38–48.

[6] P. Venugopalan, *Fuzzy ordered sets*, Fuzzy sets and systems **46** (1992), 221–226.

[7] L. A. Zadeh, *Fuzzy sets*, Inform. and control **8** (1965), 338–353.

[8] L. A. Zadeh, *Similarity relations and fuzzy orderings*, Inform. Sci. **3** (1971), 177–200.

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