EXISTENCE OF POSITIVE SOLUTIONS TO NONLOCAL
BOUNDARY VALUE PROBLEMS WITH BOUNDARY
PARAMETER

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Abstract. We establish the existence of positive solutions to nonlocal boundary value problems with integral boundary condition and non-negative real boundary parameter by mainly using the Schauder-Fixed point theorem.

1. Introduction

The theory of boundary-value problems with integral boundary conditions for differential equations arises in various areas of applied mathematics and physics like heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. For the integral boundary value problems and comments on their importance, we refer the reader to the papers by M. Feng, D. Ji and W. Ge [2], Gallardo [3], Karakostas and Tsamatos [5], Lomtatidze and Malaguti [6] and the references therein. In this paper, we study the existence of positive solutions to the nonhomogeneous integral boundary value problem of the following,

$$\begin{cases}
    u''(t) + a(t)f(u(t)) = 0, & t \in (0, 1), \\
    u(0) = 0, & u(1) - \int_0^1 g(s)u(s)ds = b
\end{cases} \quad (P_b)$$

satisfying $g \in L^1(0,1), g(t) \geq 0$ and $0 < \int_0^1 sg(s)ds < 1, a \in C([0,1], [0,\infty]), f \in C([0,\infty), [0,\infty))$. We consider the following assumptions:

(A1) $a(t) \equiv 0$ does not hold on any subinterval of $[0,1]$  
(A2) $f_0 = \lim_{u \to 0} \frac{f(u)}{u} = 0, f_\infty = \lim_{u \to \infty} \frac{f(u)}{u} = \infty$

The main result of the this paper is as follows.
Theorem 1.1. Assume (A1) and (A2) hold. Then there exists a positive number $\bar{b}$ such that if $0 < b \leq \bar{b}$ then $(P_b)$ has at least one positive solution and if $b > \bar{b}$ then $(P_b)$ has no positive solution.

The proof of above theorem is based upon the following well known fixed point theorem.

Theorem 1.2. (Schauder-Fixed Point Theorem) Let $S$ be a nonempty, closed, bounded and convex subset of a Banach space $X$ and $A$ be a continuous self-map on $S$ such that $cl_X(A(S))$ is compact. Then $A$ has a fixed point on $S$.

2. Preliminaries

Lemma 2.1. The problem
\[
\begin{cases}
u''(t) = 0, & t \in (0,1), \\
u(0) = 0, & u(1) - \int_0^1 g(s)u(s)ds = 1
\end{cases}
\] (1)
has a unique solution $h(t) = \frac{1}{1-\sigma}t$, when $\sigma = \int_0^1 sg(s)ds$.

Proof. Since $h''(t) = 0$ and $h(0) = 0$, we have $h(t) = at$ for some $a \in \mathbb{R}\{0\}$.
From $h(1) = 1 + \int_0^1 g(s)h(s)ds = a$,
\[
h(t) = (1 + \int_0^1 g(s)h(s)ds) t
\] (2)
From $at = (1 + \int_0^1 g(s)ds + \sigma)sds) t$, we have $a = 1 + a \int_0^1 sg(s)ds$ and thus $a = \frac{1}{1-\int_0^1 sg(s)ds} = \frac{1}{1-\sigma}$. The proof is completed. □

Lemma 2.2. If $h$ is a solution of (1) and $v$ is a solution of
\[
\begin{cases}
u''(t) + a(t)f(v(t) + bh(t)) = 0, & t \in (0,1), \\
u(0) = 0, & v(1) - \int_0^1 g(s)v(s)ds = 0
\end{cases}
\] (\tilde{P}_b)
Then $u = v + bh$ is a solution of $(P_b)$

Proof. If $u = v + bh$, then we have
\[
u''(t) = v''(t) + bh''(t) = -a(t)f(v(t) + bh(t)) = -a(t)f(u(t)),
\]
u(0) = v(0) + bh(0) = 0 and
\[
u(1) = v(1) + bh(1) = \int_0^1 g(s)v(s)ds + b(1 + \int_0^1 g(s)h(s)ds) = b + \int_0^1 g(s)[v(s) + bh(s)]ds = b + \int_0^1 g(s)u(s)ds.
\] □
Let \( P = \{ u \in C[0,1] | u(t) \geq 0 \text{ for all } t \in [0,1] \} \). Define \( A_b : P \to C[0,1] \) by
\[
A_b w(t) = \int_0^1 H(t, s) a(s) f(w(s) + bh(s)) ds,
\]
where
\[
H(t, s) = G(t, s) + \frac{t}{1 - \sigma} \int_0^1 G(s, \tau) g(\tau) d\tau
\]
\[
G(t, s) = \begin{cases} 
  t(1-s), & 0 \leq t \leq s \leq 1 \\
  s(1-t), & 0 \leq s \leq t \leq 1.
\end{cases}
\]
Then \( v = A_b w \) be the solution of
\[
\begin{cases}
  v''(t) + a(t) f(w(t) + bh(t)) = 0, & t \in (0,1), \\
  v(0) = 0, & v(1) - \int_0^1 g(s)v(s) ds = 0,
\end{cases}
\]
and we know that the fixed point of \( A_b \) is the solution of \((\tilde{P}_b)\).

**Lemma 2.3.** If \( A : [0, \infty) \times P \to C[0,1] \) is defined by \( A(b,u) = A_b(u) \), then \( A \) is compact and continuous.

**Proof.** For bounded subset \( M \) of \([0, \infty) \times P\), we claim that \( A(M) \) is relatively compact. Firstly, since \( M \) is bounded in \([0, \infty) \times P\), there exists \( B > 0 \) such that \(|b| + \|w\|_{\infty} \leq B\), for all \((b, w) \in M\). If we take \( M_1 > 0 \) by \( M_1 = \max_{s \in [0, B + B \|h\|_{\infty}} f(s) \), then for \((b, w) \in M\),
\[
A(b, w)(t) = A_b w(t) = \int_0^1 G(t, s) a(s) f(w(s) + bh(s)) ds
\]
\[
+ \frac{t}{1 - \sigma} \int_0^1 g(s) \left( \int_0^1 G(\tau, s) a(\tau) f(w(\tau) + bh(\tau)) d\tau \right) ds
\]
\[
\leq \int_0^1 G(t, s) a(s) M_1 ds + \frac{t}{1 - \sigma} \left[ \int_0^1 g(s) \left( \int_0^1 G(\tau, s) a(\tau) M_1 d\tau \right) ds \right]
\]
\[
\leq \int_0^1 G(s, s) a(s) M_1 ds + \frac{1}{1 - \sigma} \left[ \int_0^1 g(s) \left( \int_0^1 G(\tau, s) a(\tau) M_1 d\tau \right) ds \right]
\]
\[
= M_1 \left( \int_0^1 G(s, s) a(s) ds \right) (1 + \frac{1}{1 - \sigma} \int_0^1 g(s) ds).
\]
That is, \( \|A(b, w)\|_{\infty} \leq M_1 \int_0^1 (1-s) a(s) ds (1 + \frac{1}{1 - \sigma} \int_0^1 g(s) ds) \) for all \((b, w) \in M\) and thus \( A(M) \) is bounded. Secondly, for \((b, w) \in M\), we have
\[
|A(b, w)'(t)| = |(A_b w)'(t)|
\]
\[
= | - \int_0^t sa(s) f(w(s) + bh(s)) ds + \int_t^1 (1-s)a(s)f(w(s) + bh(s)) ds
\]
\[
+ \frac{1}{1 - \sigma} \left[ \int_0^1 g(s) \left( \int_0^1 G(\tau, s) a(\tau) f(w(\tau) + bh(\tau)) d\tau \right) ds \right] |
\]
\[
= M_1 (\gamma(t) + M_2),
\]
(4)
where \( M_2 = \frac{1}{1-\sigma} \int_0^1 g(s)[\int_0^1 G(\tau, s)a(\tau)d\tau]ds \) and
\[
\gamma_1(t) = \int_0^t sa(s)ds + \int_t^1 (1-s)a(s)ds.
\]
Now, we show that
\[
\int_0^1 \gamma(t)dt = \int_0^1 \int_0^t sa(s)dsdt + \int_0^1 \int_t^1 (1-s)a(s)dsdt
\]
\[
= \int_0^1 s(1-s)a(s)ds + \int_0^1 s(1-s)a(s)ds
\]
\[
= 2 \int_0^1 s(1-s)a(s)ds < \infty.
\]
This means
\[
\int_0^1 M_1(\gamma(t) + M_2)dt < \infty.
\] (5)
From (4) and (5), for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( |u - v| < \delta \) then
\[
|A(b, w)(u) - A(b, w)(v)| = \left| \int_u^v A(b, w)'(t)dt \right|
\]
\[
\leq \int_u^v |A(b, w)'(t)|dt
\]
\[
\leq \int_u^v M_1(\gamma(t) + M_2)dt < \varepsilon,
\]
for any \((b, w) \in M\). Consequently, \( A(M) \) is equicontinuous on \([0, 1]\). By Arzela-Ascoli theorem, \( A(M) \) is relatively compact and thus \( A \) is compact operator.

Now, we claim \( A : [0, \infty) \times P \to C[0, 1] \) is continuous. We note that
\[
H(t, s) = G(t, s) + \frac{t}{1-\sigma} \int_0^1 G(s, \tau)g(\tau)d\tau
\]
\[
\leq s(1-s) + \frac{t}{1-\sigma} \int_0^1 s(1-s)g(\tau)d\tau
\]
\[
= s(1-s)[1 + \frac{1}{1-\sigma} \int_0^1 g(\tau)d\tau].
\]
If \((b_n, w_n)\) is the sequence of \([0, \infty) \times P\) such that \((b_n, w_n) \to (b, w)\) in \([0, \infty) \times P\), then \((b_n, w_n)\) is bounded sequence so that there exist \( \bar{B} > 0 \) such that
\[
sup\{\|w_n\|_\infty + b_n\|h\|_\infty, \|w\|_\infty + b\|h\|_\infty\} < \bar{B}.
\]
Since \( f \) is uniformly continuous on \([0, \bar{B}]\), for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( u, v \in [0, \bar{B}] \) and \( |u - v| < \delta \), then
\[
|f(u) - f(v)| < \frac{\varepsilon}{L},
\] (6)
where \( L = \left( \int_0^1 s(1-s)a(s)ds \right) \left( 1 + \frac{1}{1-\sigma} \int_0^1 g(\tau)d\tau \right) \). Since \( \|w_n - w\|_\infty \to 0 \) and \( |b_n - b| \to 0 \) as \( n \to \infty \), there exists \( N \in \mathbb{N} \) such that \( \|w_n - w\|_\infty < \frac{\delta}{2} \) and \( |b_n - b| < \frac{\delta}{2\|h\|_\infty} \) for \( n \geq N \). Then by (6), \( \|f(w_n(t) + b_n h(t)) - f(w(t) + bh(t))\| < \frac{\varepsilon}{L} \) for all \( t \in [0, 1] \). Then \( \|f(w_n + b_n h) - f(w + bh)\|_\infty < \varepsilon \). Thus,

\[
|A(b_n, w_n)(t) - A(b, w)(t)| = \left| \int_0^1 H(t, s)a(s)(f(w_n(s) + b_n h(s)) - f(w(s) + bh(s)))ds \right|
\leq \left( \int_0^1 H(t, s)a(s)ds \right) \|f(w_n + b_n h) - f(w + bh)\|_\infty
\leq \left( \int_0^1 s(1-s)a(s)ds \right) (1 + \frac{1}{1-\sigma} \int_0^1 g(\tau)d\tau) \|f(w_n + b_n h) - f(w + bh)\|_\infty
= L\|f(w_n + b_n h) - f(w + bh)\|_\infty < \varepsilon,
\]

for all \( t \in [0, 1] \). Thus, \( \|A(b_n, w_n) - A(b, w)\|_\infty < \varepsilon \). So, we verified that \( A \) is continuous. \( \square \)

### 3. Main Result

For the proof of theorem 1.1, we need the following theorems.

**Theorem 3.1.** Assume (A2) holds. Then there exists a positive number \( b_1 > 0 \) such that \((P_b)\) has a positive solution for \( 0 < b < b_1 \).

**Proof.** Let \( \tilde{f}(x) = \sup_{0 \leq s \leq x} f(s) \), then \( \tilde{f} \) is monotone increasing and \( f(s) \leq \tilde{f}(s) \) for all \( s \geq 0 \). From (A2), we know \( \lim_{u \to 0^+} \frac{\tilde{f}(u)}{u} = 0 \). Then there exists \( b_1 > 0 \) such that

\[
\tilde{f}(b_1 + b_1 \|h\|_\infty)\|p\|_\infty \leq b_1,
\]

where \( p(t) = \int_0^1 H(t, s)a(s)ds \) is the solution of

\[
\begin{cases}
  u'' + a(t) = 0, & t \in (0, 1), \\
  u(0) = 0, & u(1) = \int_0^1 g(s)u(s)ds = 0.
\end{cases}
\]

Define a closed bounded convex subset in \( C[0, 1] \) by

\[
D_{b_1} = \{ u \in C[0, 1] \mid 0 \leq u(t) \leq b_1, \quad t \in [0, 1] \}.
\]
For $0 < b < b_1$, we claim that $A_b(D_{b_1}) \subset D_{b_1}$. Indeed, for $w \in D_{b_1}$, let $v = A_bw$, then $v$ is the solution of (3). From the choice of $b_1$, we have

$$0 \leq v(t) = \int_0^1 G(t, s)a(s)f(w(s) + bh(s))ds$$

$$+ \frac{t}{1 - \int_0^1 sg(s)ds} \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(w(\tau) + bh(\tau))d\tau)ds$$

$$\leq \int_0^1 G(t, s)a(s)f(w(s) + bh(s))ds$$

$$+ \frac{t}{1 - \int_0^1 sg(s)ds} \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(w(\tau) + bh(\tau))d\tau)ds$$

$$\leq \int_0^1 G(t, s)a(s)f(b_1 + b_1\|h\|_\infty)ds$$

$$+ \frac{t}{1 - \int_0^1 sg(s)ds} \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(b_1 + b_1\|h\|_\infty)d\tau)ds$$

$$\leq \tilde{f}(b_1 + b_1\|h\|_\infty)[\int_0^1 G(t, s)a(s)ds]$$

$$+ \frac{t}{1 - \int_0^1 sg(s)ds} \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)d\tau)ds$$

$$\leq \tilde{f}(b_1 + b_1\|h\|_\infty)\|p\|_\infty \leq b_1.$$  

Thus $A_bw = v \in D_{b_1}$. By Lemma 2.3, $\overline{A_b(D_{b_1})}$ is compact. Thus by Theorem 1.2, $A_b$ has a fixed point $v$ in $D_{b_1}$ and by Lemma 2.2 $u = v + bh$ is a positive solution of $(P_b)$. 

**Theorem 3.2.** Assume (A1) and (A2). There exists $B > 0$ such that $\|u\|_\infty < B$ for all positive possible solutions $u$ of $(P_b)$.

**Proof.** Let $u$ be a positive solution to $(P_b)$. Then $v = u - bh$ satisfies $(\tilde{P}_b)$. Let $J$ be a set of positive measure $J \subset (\delta, 1 - \delta)$ for some positive $\delta < \frac{1}{2}$ such that $a(t) > 0$ for $t \in J$. This $J$ can be taken by condition (A1). Let $\gamma = \min\{a^*, 1 - b^*\}$ where $a^* = \inf J$ and $b^* = \sup J$. Firstly, we will show that

$$\inf_{t \in J} v(t) \geq \gamma\|v\|_\infty. \quad (7)$$

Infact, since $v$ is the solution of the equation $(\tilde{P}_b)$, we have

$$v(t) \leq \int_0^1 G(s, s)a(s)f(v(s) + bh(s))ds$$

$$+ \frac{1}{1 - \sigma} \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(v(\tau) + bh(\tau))d\tau)ds$$
for all $t \in [0, 1]$. Thus,

$$
\|v\|_\infty \leq \int_0^1 G(s, s)a(s)f(v(s) + bh(s))ds
$$

(8)

$$
+ \frac{1}{1 - \sigma} \left[ \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(v(\tau) + bh(\tau))d\tau)ds \right].
$$

For $t \in J$,

$$
v(t) = \int_0^1 G(t, s)f(v(s) + bh(s))ds + \frac{t}{1 - \sigma}\left[ \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(v(\tau) + bh(\tau))d\tau)ds \right]
$$

$$
= \int_0^t s(1-t)a(s)f(v(s) + bh(s))ds + \int_t^1 t(1-s)a(s)f(v(s) + bh(s))ds
$$

$$
+ \frac{1}{1 - \sigma}\left[ \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(v(\tau) + bh(\tau))d\tau)ds \right]
$$

$$
\geq \int_0^t s(1-b^*)a(s)f(v(s) + bh(s))ds + \int_t^1 a^*(1-s)a(s)f(v(s) + bh(s))ds
$$

$$
+ \frac{1}{1 - \sigma}\left[ \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(v(\tau) + bh(\tau))d\tau)ds \right]
$$

$$
\geq \min\{a^*, 1-b^*\}\left[ \int_0^t sa(s)f(v(s) + bh(s))ds + \int_t^1 (1-s)a(s)f(v(s) + bh(s))ds
$$

$$
+ \frac{1}{1 - \sigma}\left[ \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(v(\tau) + bh(\tau))d\tau)ds \right]\right]
$$

$$
\geq \min\{a^*, 1-b^*\}\left[ \int_0^t s(1-s)a(s)f(v(s) + bh(s))ds + \int_t^1 s(1-s)a(s)f(v(s) + bh(s))ds
$$

$$
+ \frac{1}{1 - \sigma}\left[ \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(v(\tau) + bh(\tau))d\tau)ds \right]\right]
$$

$$
= \min\{a^*, 1-b^*\}\left[ \int_0^1 G(s, s)a(s)f(v(s) + bh(s))ds
$$

$$
+ \frac{1}{1 - \sigma}\left[ \int_0^1 g(s)(\int_0^1 G(\tau, s)a(\tau)f(v(\tau) + bh(\tau))d\tau)ds \right]\right]
$$

$$
\geq \min\{a^*, 1-b^*\}\|v\|_\infty = \gamma \|v\|_\infty.
$$

For the last inequality in the above, we used (8). Now, we need to prove

$$
\inf_{t \in J}(v(t) + bh(t)) \geq \gamma \|v + bh\|_\infty.
$$

(9)

In fact, since $h(t) = \frac{t}{1 - \sigma}$, we have

$$
\inf_{t \in J} h(t) = a^* \times \frac{1}{1 - \sigma} = a^* \times \|h\|_\infty \geq \gamma \|h\|_\infty.
$$

(10)
By using (7) and (10),
\[
\inf_{t \in J} (v(t) + bh(t)) \geq \inf_{t \in J} v(t) + \inf_{t \in J} bh(t) \\
\geq \gamma \|v\|_{\infty} + b\gamma \|h\|_{\infty} \\
= \gamma (\|v\|_{\infty} + \|bh\|_{\infty}) \\
\geq \gamma \|v + bh\|_{\infty}.
\]

Let \( \overline{f}(t) = \inf_{t \leq s} f(s) \), then \( \overline{f} \) is monotone increasing, \( \overline{f}(s) \leq f(s) \) for all \( s \in [0, \infty) \) and from \( \overline{f}_{\infty} = \infty \),
\[
\overline{f}_{\infty} = \lim_{u \to \infty} \frac{\overline{f}(u)}{u} = \infty. \tag{11}
\]

Thus we have
\[
\|v\|_{\infty} \geq v(a^*) = \int_{0}^{1} G(a^*, s)a(s)f(v(s) + bh(s))ds \\
+ \frac{a^*}{1 - \sigma} \int_{0}^{1} g(s) \left[ \int_{0}^{1} G(\tau, s)a(\tau)f(v(\tau) + bh(\tau))d\tau \right]ds \\
\geq \int_{a^*}^{b^*} G(a^*, s)a(s)f(v(s) + bh(s))ds \\
+ \frac{a^*}{1 - \sigma} \int_{a^*}^{b^*} g(s) \left[ \int_{a^*}^{b^*} G(\tau, s)a(\tau)f(v(\tau) + bh(\tau))d\tau \right]ds \\
= \int_{a^*}^{b^*} a^*(1 - s)a(s)f(v(s) + bh(s))ds \\
+ \frac{a^*}{1 - \sigma} \int_{a^*}^{b^*} g(s) \left[ \int_{a^*}^{s} \tau(1 - s)a(\tau)f(v(\tau) + bh(\tau))d\tau \right]ds \\
+ \int_{s}^{b^*} s(1 - \tau)a(\tau)f(v(\tau) + bh(\tau))d\tau \right]ds \\
\geq a^*(1 - b^*) \int_{a^*}^{b^*} a(s)f(v(s) + bh(s))ds \\
+ \frac{a^*}{1 - \sigma} \int_{a^*}^{b^*} g(s) \left[ \int_{a^*}^{s} a^*(1 - b^*)a(\tau)f(v(\tau) + bh(\tau))d\tau \right]ds \\
+ \int_{s}^{b^*} a^*(1 - b^*)a(\tau)f(v(\tau) + bh(\tau))d\tau \right]ds
Let $\Lambda = \{ b > 0 | (P_b) \text{ has a positive solution} \}$. By Theorem 3.1, $\Lambda$ is nonempty set. Let $\bar{b} = \sup \Lambda$, then by Theorem 3.3, $0 < \bar{b} < \infty$. First, we show that $(P_b)$ has a positive solution for $b = \bar{b}$. Indeed, there is a sequence $\{b_n\}$ in $\Lambda$ such that $b_n \to \bar{b}$, and let $u_n$ be a positive solution of $(P_{b_n})$. Then $v_n = u_n - b_nh$ are the solutions of $(\tilde{P}_{b_n})$ and by Theorem 3.2 and Theorem 3.3, $\{(b_n, v_n)\}$ is bounded in $[0, \infty) \times P$. By compactness of $A$, $\{v_n\}$ has a convergent subsequence converging to, say $\tilde{v}$ and by continuity of $A$, we see that $\tilde{v}$ is a solution of $(\tilde{P}_{\bar{b}})$. Let $\bar{v} = \tilde{v} + \bar{b}h$, then $\bar{v}$ is a positive solution of $(P_{\bar{b}})$. Now, we show that $(P_b)$ has a positive solution for $b \in (0, \bar{b})$. Let $b \in (0, \bar{b})$ and $\bar{v}$ be a positive solution
of \((P_b)\). If we define \(F : [0, 1] \times [0, \infty) \to [0, \infty)\) by
\[
F(t, u) = \begin{cases} 
  f(\overline{u}(t)), & \text{if } u > \overline{u}(t), \\
  f(u), & \text{if } 0 \leq u \leq \overline{u}(t), \\
  0, & \text{if } u < 0.
\end{cases}
\]

Then for the problem
\[
\begin{aligned}
  & v''(t) + a(t)F(t, v(t) + bh(t)) = 0, \quad t \in (0, 1), \\
  & v(0) = 0, \quad v(1) - \int_0^1 g(s)v(s)ds = 0,
\end{aligned}
\tag{13}
\]
define \(T : P \to C[0, 1]\) by \(Tv(t) = \int_0^1 H(t, s)a(s)F(s, v(s) + bh(s))ds\). From the definition of \(F\), if \(K = \sup_{u \in [0, \|\sigma\|_\infty]} f(u)\), then \(|F(t, u(t))| \leq K\) for all \(u \in C[0, 1]\) and \(t \in [0, 1]\). Thus for \(v \in C[0, 1]\),
\[
|Tv(t)| = |\int_0^1 H(t, s)a(s)F(s, v(s) + bh(s))ds| \\
= |\int_0^1 [G(t, s) + \frac{t}{1-\sigma}] \int_0^1 G(s, \tau)g(\tau)d\tau a(s)F(s, v(s) + bh(s))ds| \\
\leq \int_0^1 s(1-s)[1 + \frac{1}{1-\sigma}] \int_0^1 g(\tau)d\tau a(s)|F(s, v(s) + bh(s))|ds \\
\leq K \int_0^1 s(1-s)[1 + \frac{1}{1-\sigma}] \int_0^1 g(\tau)d\tau a(s)ds,
\]
for all \(t \in [0, 1]\). Thus \(\|Tv\|_\infty \leq M\), where
\[
M = K \int_0^1 s(1-s)[1 + \frac{1}{1-\sigma}] \int_0^1 g(\tau)d\tau a(s)ds.
\]
Thus \(T(B_M(0)) \subset B_M(0)\), when \(B_M(0) = \{v \in C[0, 1] \mid \|v\|_\infty \leq M\}\) and \(\overline{T}(B_M(0))\) is compact by the same argument in the proof of Lemma 2.3. By Theorem 1.2, \(T\) has a fixed point \(v_b\) in \(B_M(0) \cap P\). Then \(u_b = v_b + bh\) is a positive solution of
\[
\begin{aligned}
  & u''(t) + a(t)F(t, u(t)) = 0, \quad t \in (0, 1), \\
  & u(0) = 0, \quad u(1) - \int_0^1 g(s)u(s)ds = b.
\end{aligned}
\tag{14}
\]
We need to show that \(u_b(t) \leq \overline{u}(t)\) for all \(t \in [0, 1]\). Then from the definition of \(F\), we can say that \(u_b\) is a positive solution of \((P_b)\). Let \(\Omega_0 = \{t \in (0, 1) \mid u_b(t) > \overline{u}(t)\}\) and suppose that \(\Omega_0 \neq \emptyset\). Firstly, we consider the case that \(u_b(1) \leq \overline{u}(1)\), then there exists \((a_1, b_1) \subset \Omega_0\) such that \(w(a_1) = w(b_1) = 0\) and \(w(t) > 0\) for \(t \in (a_1, b_1)\), where \(w(t) = u_b(t) - \overline{u}(t)\). Then, by the definition of \(F\), for \(t \in (a_1, b_1)\)
\[
w''(t) = u_b''(t) - \overline{u}''(t) \\
= -a(t)F(t, u_b(t)) - (-a(t)f(\overline{u}(t))) = 0.
\]
Thus $w \equiv 0$ in $(a_1, b_1)$, but it’s a contradiction to the fact that $(a_1, b_1)$ is the subset of $\Omega_0$. For the second case that $u_0(1) > \overline{u}(1)$, that is $w(1) > 0$. We claim that $\int_0^1 g(s)w(s)ds > 0$. In fact, from the assumption of

$$w(1) - \int_0^1 g(s)w(s)ds = u_b(1) - \overline{u}(1) - \int_0^1 g(s)(u_b(s) - \overline{u}(s))ds = [u_b(1) - \int_0^1 g(s)u_b(s)ds] - [\overline{u}(1) - \int_0^1 g(s)\overline{u}(s)ds] = b - \overline{b} < 0.$$

From $w(1) > 0$, we have that $\int_0^1 g(s)w(s)ds > 0$. Now, we only need to deal with the following three cases. First, for the case that $w > 0$ in $(0,1)$, we have

$$w''(t) = u''_b(t) - \overline{u}''(t) = 0, \ t \in (0, 1) \quad w(0) = u_b(0) - \overline{u}(0) = 0 - 0 = 0 \quad w(1) - \int_0^1 g(s)w(s)ds = [u_b(1) - \int_0^1 g(s)u_b(s)ds] - [\overline{u}(1) - \int_0^1 g(s)\overline{u}(s)ds] = b - \overline{b} < 0.$$ 

Since $w''(t) = 0$ for $t \in (0, 1)$, $w$ is a linear function and we have $w(t) = \alpha t$ for some $\alpha \in \mathbb{R}$. From

$$w(1) = b - \overline{b} + \int_0^1 g(s)w(s)ds,$$  

(15)

by putting $w(s) = \alpha s$ into (15), $\alpha \cdot 1 = b - \overline{b} + \int_0^1 g(s)(\alpha s)ds$, then $\alpha(1 - \int_0^1 s g(s)ds) = b - \overline{b}$. It claims that $\alpha = \frac{b - \overline{b}}{1 - \sigma}$.

Thus,

$$w(t) = \frac{b - \overline{b}}{1 - \sigma} t.$$ 

Since $1 - \sigma > 0, b - \overline{b} < 0$, we have a contradiction that $w(t) < 0$ for $t \in (0,1)$. For the second case that there exists $\theta \in (0, 1)$ such that $w(\theta) = 0, w(t) > 0$ for $t \in (\theta, 1]$ and $w(t) \leq 0$ in $[0, \theta]$. Thus, $w''(t) = 0$ on $[\theta, 1]$, and $w$ is a linear function on $[\theta, 1]$. Since $w(1) = \int_\theta^1 g(s)w(s)ds + b - \overline{b} \leq \int_\theta^1 g(s)w(s)ds + b - \overline{b},$

$$w(1) - \int_\theta^1 g(s)w(s)ds \leq b - \overline{b} < 0.$$ 

So let $r = w(1) - \int_\theta^1 g(s)w(s)ds$, then $r < 0$. From the fact that $w$ is linear on $[\theta, 1]$, for any $s \in (\theta, 1],$

$$w'(s) = w'(\theta).$$  

(16)

Integrate (16) from $t$ to $1$ for $t \in [\theta, 1]$, then

$$w(1) - w(t) = w'(\theta)(1 - t).$$  

(17)
Putting by $t = \theta$, we have

$$w'(\theta) = \frac{1}{1 - \theta} w(1) = \frac{1}{1 - \theta} \int_{\theta}^{1} g(s)w(s)ds + r.$$  \hspace{1cm} (18)

Substitute (18) into (17), we have

$$w(t) = w(1) - w'(\theta)(1 - t) = \int_{\theta}^{1} g(s)w(s)ds + r - \frac{1}{1 - \theta} \int_{\theta}^{1} g(s)w(s)ds + r(1 - t) = \left(1 - \frac{1 - t}{1 - \theta}\right) \int_{\theta}^{1} g(s)w(s)ds + (1 - \frac{1 - t}{1 - \theta})r = \left(\frac{t - \theta}{1 - \theta}\right) \left[\int_{\theta}^{1} g(s)w(s)ds + r\right].$$

Multiplying $g(t)$ to both sides,

$$g(t)w(t) = \frac{t - \theta}{1 - \theta} g(t) \left[\int_{\theta}^{1} g(s)w(s)ds + r\right].$$ \hspace{1cm} (19)

Integrate (19) from $\theta$ to 1, then

$$\int_{\theta}^{1} g(t)w(t)dt = \left[\int_{\theta}^{1} t - \frac{t - \theta}{1 - \theta} g(t)dt\right] \cdot \left[\int_{\theta}^{1} g(s)w(s)ds + r\right],$$

and

$$\left[1 - \int_{\theta}^{1} \frac{t - \theta}{1 - \theta} g(t)dt\right] \int_{\theta}^{1} g(t)w(t)dt = r \int_{\theta}^{1} \frac{t - \theta}{1 - \theta} g(t)dt.$$ \hspace{1cm} (20)

Since $\frac{t - \theta}{1 - \theta} \leq t$, we have

$$0 < \int_{\theta}^{1} \frac{t - \theta}{1 - \theta} g(t)dt < \int_{\theta}^{1} tg(t)dt < \int_{0}^{1} tg(t)dt = \sigma < 1.$$

Then the left hand side of (20) is positive but the right hand side of (20) is negative and it’s contradiction. For the last case that there exists $(a_2, b_2) \not\subset (0, 1)$ such that $w(a_2) = 0 = w(b_2)$ and $w(t) > 0$ for $t \in (a_2, b_2)$, then this implies that $w''(t) = 0$ in $(a_2, b_2)$ and thus $w(t) \equiv 0$ in $(a_2, b_2)$. Then we get a desired contradiction again. From these all cases, we can conclude that $\Omega_0 = \emptyset$. Thus

$$0 \leq u_b(t) \leq u(t)$$

for all $t \in [0, 1]$. From the definition of $F$, $u_b$ is the positive solution of $(P_b)$ for $b \in [0, \bar{b})$ and we complete the proof.
References


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