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# On the Modified Supplementary Variable Technique for a Discrete-Time GI/G/1 Queue with Multiple Vacations

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# 복수휴가형 이산시간 GI/G/1 대기체계에 대한 수정부가변수법

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This work suggests a new analysis approach for a discrete-time GI/G/1 queue with multiple vacations. The method used is called a modified supplementary variable technique and our result is an exact transform-free expression for the steady state queue length distribution. Utilizing this result, we propose a simple two-moment approximation for the queue length distribution. From this, approximations for the mean queue length and the probabilities of the number of customers in the system are also obtained. To evaluate the approximations, we conduct numerical experiments which show that our approximations are remarkably simple yet provide fairly good performance, especially for a Bernoulli arrival process.

***Keywords:*** Modified Supplementary Variable Technique, Discrete-Time Queue, Queue Length Distribution, Multiple Vacations, Two-Moment Approximation

## 1. Introduction

Most real queueing situations arising in banks, computer networks, telecommunication systems, manufacturing systems, etc. can be modeled as queueing systems with general interarrival times and general service times. These queueing systems present an interest subject for which to devise a practical analysis method. However, due to the limited information on their distributions, the analysis of such a queue is notoriously difficult. While several approximations have been proposed, they are often computationally demanding. Moreover, most of the approximation methods have been applied to continuous- time queueing systems.

Thanks to recent advances in computer and telecommunication technology, the importance of discrete-time queueing systems has been increased. That is why continuous-time queueing systems can not accurately give the performance measures of computer and telecommunication systems where basic operational units are bits, packets and cells, although they have been used in the past to approximately evaluate some performance measures of these systems. Hence, discrete-time queueing systems are potentially more suitable for application to the digital computer and communication networks.

In discrete-time queueing systems, the time axis is segmented into a sequence of equal intervals of unit duration, called slots. It is always assumed that interarrival times, service times, and vacation times are integral multiples of a slot duration. Also, it is assumed that the state of the system changes only at a slot boundary $t = 0, 1, 2, \cdots$. Under these assumptions, note that an arrival and a departure can occur simultaneously at a slot boundary. Considering the order of

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these simultaneous events, there have been two typical assumptions: late arrival system (LAS) and early arrival system (EAS). According to the LAS model, a potential arrival takes place in the interval $(t^-, t)$ and a service completion occurs in the interval $(t, t^+)$, where $t^+$ and $t^-$ represent $\lim_{\Delta t \to 0}(t+|\Delta t|)$ and $\lim_{\Delta t \to 0}(t-|\Delta t|)$, respectively. On the other hand, a potential arrival takes place in the interval $(t, t^+)$ and a service completion occurs in the interval $(t^-, t)$ under the EAS model. For more details, see Hunter (1983) and Bruneel and Kim (1993)

The conventional supplementary variable technique (SVT) is known to be originated by Cox (1955), and has become one of the most frequently used approaches for both the continuous and discrete-time queuing systems. The method we use in the present work is a modified SVT, where the last step of the modified SVT is different from that of the conventional SVT. The first step is to define a Markov chain by including appropriate supplementary variables into the state vector. The second step is to construct the steady state system equations. The last step is to solve these equations. The last step of the conventional SVT is to obtain the probability generating functions (PGFs) of the number of customers in the system/queue by solving the system equations in the transform domain. On the other hand, the last step of the modified SVT is to directly sum each equation after multiplying a supplement variate. The result thus obtained is the steady state queue length distribution not expressed as the form of transformation but in terms of conditional expectations. In other words, we derive an exact transform-free expression for the steady state queue length distribution. This method is illustrated by the GI/G/1 queue with multiple vacations (for more definitions, see the following sections).

There have been several studies on the discrete-time queue with general interarrival times and general service times. For the standard GI/G/1 queue, see Murata and Miyahara (1991), who obtain the waiting time distribution under the assumption that PFGs of the sojourn time distributions are represented as rational polynomials. Chaudhry (1993) also obtains closed-form expressions of waiting time distributions via a root-finding method. For the batch arrival $\mathrm{GI}^X$/G/1 queue, Chaudhry and Gupta (2001) present a procedure for computing waiting time probabilities and its PGF by analyzing the PGF of the unfinished work. For the finite waiting spaces queue, Haβlinger (1995) and Linwong *et al.* (2004) analyze the queue length distributions for the GI/G/1/K and $\mathrm{GI}^X$/G/1/K queue, respectively. They use a polynomial factorization approach. The restriction of their approach is that the interarrival time distribution, service time distribution, and batch size distribution should all be of finite support. For the infinite server GI/G/$\infty$ queue, Eliazar (2008) analyzes the output process and the queue process making use of the statistical properties of the stochastic maps. The studies noted above are based on the transformation technique. As a consequence, all their results are expressed as the transformed terms.

Addressing a continuous-time queue, Chae *et al.* (2004) present the transform-free queue length distribution for the GI/G/1/K queue with multiple vacations. They propose obtaining the queue length distribution by using the modified SVT. However, to the best of the author's knowledge, there have been only one report on utilizing the modified SVT for the discrete-time queue. Chae *et al.* (2008) first apply the modified supplementary variable technique to the discrete-time GI/G/1/K queue without vacations. The purpose of this paper is to extend the work of Chae *et al.* (2008) considering not finite buffer but multiple vacations. In other words, this work shows the modified SVT for discrete-time queues with general interarrival times, general service times, and general vacation times.

The remainder of this paper is organized as follows. In Section 2, we analyze the queue length distribution of the discrete-time GI/G/1 queue with multiple vacations. In Section 3, we propose a simple approximation, called a two-moment approximation, for the queue length distribution. The two-moment approximation for the continuous-time queue has been reported in the literatures Kim and Chae (2003) and Choi *et al.* (2005), but there is no precedent for the discrete-time queue. Numerical experiments are conducted to demonstrate that our approximations are remarkably simple yet provide fairly good performance, especially for a Bernoulli arrival process.

## 2. The Steady State Queue Length Distribution of a GI/G/1 Queue with Multiple Vacations

In this section, we derive a transform-free distribution of the steady state queue length in the LAS of the GI/G/1/MV queue, where MV stands for multiple vacations. In the multiple vacations model, the server leaves for a vacation if there is no customer to serve in the system at the end of a service. If the server returns from a vacation finding the system nonempty, it begins to serve the customers and continues serving until the system becomes empty again. If the server returns from a vacation finding the system empty, it leaves for another vacation, and repeats vacations in this manner until it returns from a vacation finding the system nonempty (see Takagi (1993)). Just like a service, a vacation is assumed to end in $(t^-, t)$ and immediately after the arrival of a customer and to begin in $(t, t^+)$ and immediately after a service completion. We further assume that customers leave the system on a first-in first-out basis. Interarrival times, service times, and vacation times are independent and identically distributed random variables (R.V.s) denoted by the generic R.V.s *A*, *S*, and *V*, respectively.

Consider the Markov chain $\{N(t^-), A_R(t^-), S_R(t^-), V_R(t^-) | t = 1, 2, \cdots\}$, where $N(t^-)$ denotes the number of customers in the system at $t^-$ and the supplementary variables $A_R(t^-)$,

$S_R(t^-)$, and $V_R(t^-)$ respectively denote the remaining interarrival time, the remaining service time, and the remaining vacation time all at $t^-$. We define the probability mass functions as follows:

$$
\begin{aligned}
&a_i = \Pr\{A=i\},\ i=1,2,\cdots,\\
&v_j = \Pr\{V=j\},\ j=1,2,\cdots,\\
&s_k = \Pr\{S=k\},\ k=1,2,\cdots,\\
&P_n^V(i,j) = \lim_{t\to\infty}\Pr\{N(t^-)=n,\ A_R(t^-)=1,\ V_R(t^-)=j\},\\
&\qquad n\ge 0,\ i,j=0,1,\cdots,\\
&P_n^B(i,k) = \lim_{t\to\infty}\Pr\{N(t^-)=n,\ A_R(t^-)=i,\ S_R(t^-)=k\}\\
&\qquad n\ge 1,\ i,k=0,1,\cdots.
\end{aligned}
$$

Let $\{A_i\}_{i=1}^{\infty}$, $\{S_i\}_{i=1}^{\infty}$, and $\{V_i\}_{i=1}^{\infty}$ denote the sequences of interarrival times, service times, and vacation times and they are mutually independent. By considering mutually exclusive events that can occur during one slot, we construct the steady state system equations as follows :

$$
\left.
\begin{aligned}
&P_0^V(i,j) = P_0^V(i+1,j+1) + \left\{P_0^V(i+1,0)+P_1^B(i+1,0\right\}v_{j+1}\\
&P_n^V(i,j) = P_n^V(i+1,j+1) + P_{n-1}^V(0,j+1)a_{i+1},\ n\ge 1,
\end{aligned}
\right\} \quad (1a)
$$

$$
\left.
\begin{aligned}
&P_1^B(i,k) = P_1^B(i+1,k+1) + \left\{P_1^V(i+1,0)+P_2^B(i+1,0)\right\}s_{k+1}\\
&+\left\{P_0^V(0,0)+P_1^B(0,0)\right\}a_{i+1}s_{k+1},\\
&P_n^B(i,k) = P_n^B(i+1,k+1) + P_{n-1}^B(0,k+1)a_{i+1}\\
&+\left\{P_n^V(i+1,0)+P_{n+1}^B(i+1,0)\right\}s_{k+1}\\
&+\left\{P_{n-1}^V(0,0)+P_n^B(0,0)\right\}a_{i+1}s_{k+1},\ n\ge 2.
\end{aligned}
\right\} \quad (1b)
$$

Due to $P_0^B(i,k)=0$, one less equation is required. The left-hand sides of (1) represent the probabilities of the system state at $(t+1)^-$ in a steady state. The right-hand sides of (1) are then expressed in terms of the probabilities of the system state at $t^-$ in a steady state, together with the probabilities of all potential queueing activities that can happen during $(t^-, t^+)$. Notice that (1) is the difference equations, which correspond to the differential equations in the continuous-time queue.

**Remark 1** : $P_n^B(0,0)\ \left(P_n^V(0,0)\right)$ is the joint probability of three events when the server is busy (on vacation) in a steady state. One event is that the number of customers in the system is *n* at $t^-$. Another event is that service (vacation) completion occurs in $(t^-, t)$ due to $S_R(t^-)=0\ (V_R(t^-)=0)$. The other event is that arrival takes place in $(t, t^+)$ due to $A_R(t^-)=0$.

In the modified SVT, we first sum (1a), both over *i* and *j*, $0\le i, j\le\infty$ and sum (1b) both over *i* and *k*, $0\le i,k\le\infty$. Then, we multiply *i*+1 to both sides of (1) and sum over *i*, *j*, and *k*, $0\le i,j,k\le\infty$. Finally, we multiply *j*+1 to both sides of (1a) and sum both over *i* and *j*, $0\le i,j\le\infty$, and then multiply *k*+1 to both sides of (1b) and sum both over *i* and *k*, $0\le i,k\le\infty$.

Now, we apply above procedure to the model. We first sum (1a), both over *i* and *j*, $0\le i,j\le\infty$, and sum (1b) both over *i* and *k*, $0\le i,k\le\infty$. Simplifying the results (for more details, see Appendix A), we obtain

$$
\sum_{i=1}^{\infty}P_n^B(i,0) = \sum_{j=0}^{\infty}P_{n-1}^V(0,j) + \sum_{k=1}^{\infty}P_{n-1}^B(0,k),\ n\ge 1, \quad (2a)
$$

$$
\sum_{i=1}^{\infty}P_n^V(i,0) = \sum_{j=1}^{\infty}P_{n-1}^V(0,j) - \sum_{j=0}^{\infty}P_n^V(0,j),\ n\ge 1. \quad (2b)
$$

In order to express (2) in terms of meaningful quantities, let $\pi_n^A(V)$ and $\pi_n^A(B)$ denote the probability that an arriving customer sees *n* customers when the server is on vacation and when the server is busy, respectively. Likewise, let $\pi_n^D$ denote the probability that a departing customer leaves behind *n* customers. Note that $\pi_0^A(B)=0$. We verify that the arrival rate and departure rate are $\lambda = 1/E[A]$ and a stable system satisfies $\rho=\lambda E[S]<1$.

Based on the assumption mentioned above, expressing (2) in terms of $\pi_n^A(V)$, $\pi_n^A(B)$, and $\pi_n^D$ leads to

$$
\lambda\pi_n^A(V) + \lambda\pi_n^A(B) = \lambda\pi_n^D,\ n\ge 0, \quad (3)
$$

which is known as the Burke's theorem.

In order to express the results of the next procedure, we define the following probabilities and conditional expectations for $n\ge 0$: $\pi_n(V)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P_n^V(i,j)$, $\pi_n(B)=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}P_n^B(i,k)$, $\pi_n=\pi_n(V)+\pi_n(B)$, $\sigma_n=E[S_R^{n,A}]$, $v_n=E[V_R^{n,A}]$, $\alpha_n^D=E[A_R^{n,D}]$, and $\alpha_n^V=E[A_R^{n,V}]$, where $S_R^{n,A}$ $(V_R^{n,A})$ is the remaining service time (vacation time) at the arrival epoch of a customer who sees *n* customers in the system and $A_R^{n,D}(A_R^{n,V})$ is the remaining interarrival time at the departure epoch of a customer who leaves behind *n* customers in the system (at the epoch when the server completes its vacations, finding *n* customers in the system). Here, $\pi_0(B)=0$ and $\sigma_0=E[S]$.

Next, we multiply *i*+1 to both sides of (1a) and sum over *i* and *j*, $0\le i,j\le\infty$. Simplifying the results (for more details, see Appendix B), we obtain

$$
\left.
\begin{aligned}
\pi_0(V) &= \sum_{i=1}^{\infty} iP_1^B(i,0),\\
\pi_n(V) &= \sum_{j=0}^{\infty}P_{n-1}^V(0,j)E[A]\\
&\quad -\left\{\sum_{i=1}^{\infty} iP_n^V(i,0) + P_{n-1}^V(0,0)E[A]\right\},\ n\ge 1.
\end{aligned}
\right\} \quad (4)
$$

On the other hand, we multiply *j*+1 to both sides of (1a) and sum over *i* and *j*, $0\le i,j\le\infty$. Simplifying the results, we obtain

$$
\left.
\begin{aligned}
&\pi_0(V) + \sum_{j=1}^{\infty} jP_0^V(0,j)\\
&= \left\{\sum_{i=1}^{\infty}P_0^V(i,0) + \sum_{i=1}^{\infty}P_1^B(i,0)\right\}E[V] = 1-\rho,\\
&\pi_n(V) = \sum_{j=1}^{\infty} jP_{n-1}^V(0,j) - \sum_{j=1}^{\infty} jP_n^V(0,j),\ n\ge 1.
\end{aligned}
\right\} \quad (5)
$$

**Remark 2** : $\sum_{i=1}^{\infty}P_0^V(i,0)+\sum_{i=1}^{\infty}P_1^B(i,0)$ in (5) means the rate that the server takes vacations. By multiplying the expected length of a vacation time, $\{\sum_{i=1}^{\infty}P_0^V(i,0)+\sum_{i=1}^{\infty}P_1^B(i,0)\}E[V]$ becomes the long-run proportion of time the server is on vacation. Since $\rho$ is not only the traffic intensity but also the long-run proportion of time that the server is busy, $1-\rho$ becomes the long-run proportion of time that the server is on vacation.

Now, we want to express (4) and (5) in terms of identities $v_n$, $\alpha_0^D$, and $\alpha_n^V$. For this, we introduce the next lemma :

**Lemma 1** : In a steady state, the following relations hold :

$$\lambda\pi_n^A(V)v_n=\sum_{j=1}^{\infty}jP_n^V(0,j),\ n\geq 0,$$
$$\lambda\pi_n^D\alpha_n^D=\sum_{i=1}^{\infty}iP_{n+1}^B(i,0)+P_n^B(0,0)E[A],\ n\geq 0,$$
$$\lambda\left(\pi_{n-1}^A(V)-\pi_n^A(V)\right)\alpha_n^V=\sum_{i=1}^{\infty}iP_n^V(i,0)+P_{n-1}^V(0,0)E[A],\ n\geq 0,$$
and $\lambda\pi_n^A(B)\sigma_n=\sum_{k=1}^{\infty}kP_n^B(0,k),\ n\geq 0.$

Proof : **See Appendix C.** □

By utilizing the results of Lemma 1, we obtain the queue length equations of the GI/G/1/MV when the server is on vacation as follows :

**Theorem 1** : The queue length distribution for GI/G/1/MV when the server is on vacation satisfies the following simultaneous equations:

$$\left.\begin{aligned}&\pi_0(V)=\lambda\pi_0^D\alpha_0^D,\\&\pi_n(V)=\lambda\pi_{n-1}^A(V)E[A]\\&\quad-\lambda\{\pi_{n-1}^A(V)-\pi_n^A(V)\}\alpha_n^V,\ n\geq 1,\end{aligned}\right\}\tag{6}$$

$$\left.\begin{aligned}&\pi_0(V)=1-\rho-\lambda\pi_0^A(V)v_0,\\&\pi_n(V)=\lambda\pi_{n-1}^A(V)v_{n-1}-\lambda\pi_n^A(V)v_n,\ n\geq 1\end{aligned}\right\}\tag{7}$$

Proof. $\sum_{j=0}^{\infty}P_n^V(0,j)$ is the rate (or the expected frequency per unit time) that an arriving customer sees *n* customers when the server is on vacation. Since $\lambda$ is the expected number of arrivals per unit time and $\pi_n^A(V)$ is the probability that an arriving customer sees *n* customers when the server is on vacation, we have the concrete result : $\sum_{j=0}^{\infty}P_n^V(0,j)=\lambda\pi_n^A(V)$. In addition, $P_n^B(0,0)+\sum_{i=1}^{\infty}P_{n+1}^B(i,0)$ is the rate that a departing customer leaves behind *n* customers in the system. Since $\lambda$ is also the expected number of departures per unit time and $\pi_n^D$ is the probability that a departing customer leaves behind *n* customers in the system, $P_n^B(0,0)+\sum_{i=1}^{\infty}P_{n+1}^B(i,0)=\lambda\pi_n^D$ should hold. Using Lemma 1, we can rewrite the results of (4) and (5) as (6) and (7). □

We then have the following transform-free expressions for the queue length distribution just before an arrival and at an arbitrary epoch when the server is on vacation, all in product forms.

**Theorem 2** : The steady state queue length distribution for GI/G/1/MV when the server is on vacation is given by

$$\left.\begin{aligned}&\pi_0^A(V)=\frac{(1-\rho)E[A]}{v_0+\alpha_0^D}\\&\pi_n^A(V)=\pi_0^A(V)\prod_{i=1}^{n}\frac{\lambda_i^V}{\mu_i^V},\ n\geq 1,\end{aligned}\right\}\tag{8}$$

$$\left.\begin{aligned}&\pi_0(V)=\frac{(1-\rho)\alpha_0^D}{v_0+\alpha_0^D}\\&\pi_n(V)=\pi_n^A(V)\gamma_n^V,\ n\geq 1,\end{aligned}\right\}\tag{9}$$

where $\lambda_n^V=v_{n-1}+\alpha_n^V-E[A]$, $\mu_n^V=v_n+\alpha_n^V$,
and $\gamma_n^V=\dfrac{v_{n-1}\alpha_n^V-v_n\alpha_n^V+v_nE[A]}{(v_{n-1}+\alpha_n^V-E[A])E[A]}$.

**Proof** : For each *n*, solving (6) and (7) simultaneously leads to (8). Finally, (9) is derived by combining (8) with either (6) or (7). □

Note that (8) is a similar form of the steady state queue length distribution of the birth and death process (see Wolff (1989)). The procedure of obtaining $\pi_n^A(B)$ and $\pi_n(B)$ is the same. Multiplying *i*+1 to both sides of (1b) and summing over *i* and *k*, $0\leq i,k\leq\infty$, we have

$$\left.\begin{aligned}\pi_1(B)&=\left\{\sum_{i=1}^{\infty}iP_2^B(i,0)+P_1^B(0,0)E[A]\right\}\\&\quad-\sum_{i=1}^{\infty}iP_1^B(i,0)+\left\{\sum_{i=1}^{\infty}iP_1^V(i,0)+P_0^V(0,0)E[A]\right\}\\\pi_n(B)&=\left\{\sum_{i=1}^{\infty}iP_{n+1}^B(i,0)+P_n^B(0,0)E[A]\right\}\\&\quad-\left\{\sum_{i=1}^{\infty}iP_n^B(i,0)+P_{n-1}^B(0,0)E[A]\right\}\\&\quad+\sum_{k=0}^{\infty}P_{n-1}^B(0,k)E[A]\\&\quad+\left\{\sum_{i=1}^{\infty}iP_n^V(i,0)+P_{n-1}^B(0,0)E[A]\right\},\ n\geq 2.\end{aligned}\right\}\tag{10}$$

Multiplying *k*+1 to both sides of (1b) and summing over *i* and *k*, $0\leq i,\ k\leq\infty$, we have

$$\left.\begin{aligned}\pi_1(B)&=\left\{\sum_{j=0}^{\infty}P_0^V(0,j)+\sum_{k=0}^{\infty}P_1^B(0,k)\right\}E[S]\\&\quad-\sum_{k=1}^{\infty}kP_1^B(0,k),\\\pi_n(B)&=\left\{\sum_{j=0}^{\infty}P_{n-1}^V(0,j)+\sum_{k=0}^{\infty}P_n^B(0,k)\right\}E[S]\\&\quad+\sum_{k=1}^{\infty}kP_{n-1}^B(0,k)-\sum_{k=1}^{\infty}kP_n^B(0,k),\quad n\geq 2.\end{aligned}\right\}\tag{11}$$

Expressing (10) and (11) by using Lemma 1, we obtain the queue length equations of the GI/G/1/MV when the server is busy as follows :

**Theorem 3** : The queue length distribution for GI/G/1/MV when the server is busy satisfies the following simultaneous equations :

$$\pi_n(B) = \lambda\pi_{n-1}^A(B)E[A] + \lambda\pi_n^D\alpha_n^D - \lambda\pi_{n-1}^D\alpha_{n-1}^D + \lambda\pi_{n-1}^A(V)\alpha_n^V - \lambda\pi_n^A(V)\alpha_n^V,\ n \geq 1, \tag{12}$$

$$\pi_n(B) = \lambda\pi_{n-1}^A(V)E[S] + \lambda\pi_n^A(B)E[S] + \lambda\pi_{n-1}^A(B)\sigma_{n-1} - \lambda\pi_n^A(B)\sigma_n,\ n \geq 1. \tag{13}$$

**Proof** : To simplify (10) and (11) meaningfully, we use the relations $\sum_{k=0}^{\infty}P_n^B(0, k) = \lambda\pi_n^A(B)$ and then, we obtain (12) and (13). $\sum_{k=0}^{\infty}P_n^B(0, k)$ is the rate that an arriving customer sees *n* customers when the server is busy servicing. Since $\lambda$ is the expected number of arrivals per unit time and $\pi_n^A(B)$ is the probability that an arriving customer sees *n* customers when the server is busy, we have the concrete result : $\sum_{k=0}^{\infty} P_n^B(0, k) = \lambda\pi_n^A(B)$. Using Lemma 1 and the fact that $\sum_{k=0}^{\infty} P_n^B(0, k) = \lambda\pi_n^A(B)$, we can rewrite the results of (10) and (11) as (12) and (13), respectively. □

We finally have the following transform-free expressions for the queue length distribution just before an arrival and at an arbitrary epoch when the server is busy, all in product forms.

**Theorem 4** : The steady state queue length distribution for GI/G/1/MV when the server is busy is given by

$$\pi_n^A(B) = \pi_{n-1}^A(B)\frac{\lambda_n^D}{\mu_n^B} + \pi_{n-1}^A(V)\epsilon_n,\ n \geq 1, \tag{14}$$

$$\pi_n(B) = \pi_n^A(B)\gamma_n^B + \pi_{n-1}^A(V)\delta_n,\ n \geq 1, \tag{15}$$

where $\lambda_n^B = \alpha_{n-1}^D + \sigma_{n-1} - E[A]$, $\mu_n^B = \alpha_n^D + \sigma_n - E[S]$,

$$\epsilon_n = \frac{\alpha_n^V - \alpha_n^D}{\alpha_n^D + \sigma_n - E[S]} \times \frac{\lambda_n^V}{\mu_n^V} + \frac{E[S] + \alpha_{n-1}^D - \alpha_n^V}{\alpha_n^D + \sigma_n - E[S]},$$

$$\gamma_n^B = \frac{\alpha_n^D\sigma_{n-1} + \left(E[S] - \sigma_n\right)\left(\alpha_{n-1}^D - E[A]\right)}{\left(\alpha_{n-1}^D + \sigma_{n-1} - E[A]\right)E[A]},$$

and $\delta_n = \dfrac{1}{E[A]}\left(E[S] - \dfrac{\sigma_{n-1}\epsilon_n\mu_n^B}{\lambda_n^B}\right)$.

**Proof** : For each *n*, solving (12) and (13) simultaneously leads to (14). Finally, (15) is derived by combining (14) with either (12) or (13). □

Our results above are expressed in terms of $\alpha_n^D$, $\alpha_n^V$, $\sigma_n$, and $v_n$, which are all conditional expectations of supplementary variables. In general, they are not easy to compute, except for some special cases such as Bernoulli arrival, geometric service times, or geometric vacation times. However, the availability of such expressions provides a basic idea for developing approximations for various performance measures of practical interest, which will be discussed in the following sections.

**Remark 3** : These transform-free expressions for the discrete-time queues are not available in previous studies, although our method is intuitive and easy to follow. It is interesting to state that (9) and (15) in this work take the same forms as (12) and (18) in Chae *et al*. (2004) when $K \to \infty$. Specifically, the definitions of the conditional expectations $\alpha_n^D$ and $\alpha_n^V$ in this paper are slightly different from those in Chae *et al*. (2004), because $P_n^B(0, 0)$ and $P_n^V(0, 0)$ are not defined in the continuous-time queue but have positive probabilities in the discrete-time queue. Nonetheless, (9) and (15) in this work take exactly the same forms as (12) and (18) in Chae *et al*. (2004).

**Remark 4** : Heuristic interpretations for Eqs. (6), (7), (12) and (13) can be found in the Chae *et al*. (2002). Here, we briefly explain the first equation in (6). $\pi_0(V)$ is the time-average probability that there is no customer in the system when the server is on vacation. $\lambda\pi_0^D$ is the expected frequency that a departing customer leaves behind the empty system. And, $\alpha_0^D$ is the expected remaining interarrival time when the system becomes empty. If the state of the server changes from "busy" to "on vacation", there is no customer and the server leaves for vacations at the same time. Therefore, $\lambda\pi_0^D\alpha_0^D$ is the long-run proportion of time that there is no customer in the system when the server is on vacation. Chae *et al*. (2002) explains the time-average probability of the queue length distribution for the continuous-time GI/G/c/K queue by using the tax collection examples. Nevertheless, we can apply their same interpretations for the queue length distribution to our results.

# 3. The Two-Moment Approximation for the Queue Length Distribution and Its Performance

Based on the expressions given in Section 2, we now propose a simple two-moment approximation for the steady state queue length distribution through approximation of $\alpha_n^D$, $\alpha_n^V$, $\sigma_n$, and $v_n$. From this, approximations for several performance measures can also be obtained. We employ the following approximation scheme for $n \geq 0$ :

$$\left.\begin{aligned} \alpha_n^V,\ \alpha_n^D &\approx \alpha = \frac{E[A^2]+E[A]}{2E[A]} \\ \sigma_n &\approx \sigma = \frac{E[S^2]-E[S]}{2E[S]} \\ v_n &\approx v = \frac{E[V^2]+E[V]}{2E[V]} \end{aligned}\right\} \quad (16)$$

where $E[Y^2]$ is the second moment of any discrete R.V. $Y$ with a distribution function $F$, and $(E[Y^2]\pm E[Y])/2E[Y]$ is the mean of the equilibrium distribution of $F$.

**Remark 5** : In our setting, the remaining interarrival time of a customer both at a service completion epoch and at a vacation termination epoch does not contain 0. In contrast, both the remaining service time and the remaining vacation time at a customer arrival epoch contain 0. Therefore, in terms of the discrete-time inspection paradox, $\alpha_n^D$, $\alpha_n^V$, $\sigma_n$, and $v_n$ can approximate $\alpha$, $\sigma$, and $v$, respectively.

Applying (16) to (8), (9), (14), and (15), we obtain two-moment approximations for the steady state queue length distribution as follows :

$$\left.\begin{aligned} &\pi_0^A(V) \approx \frac{(1-\rho)E[A]}{v+\alpha}, \\ &\pi_n^A(V) \approx P_0^A(V)\left(\frac{\lambda^V}{\mu^V}\right)^n,\ n\ge 1, \\ &\pi_0(V) \approx \frac{(1-\rho)\alpha}{v+\alpha}, \\ &\pi_n(V) \approx P_n^A(V)\gamma^V,\ n\ge 1, \\ &\pi_n^A(B) \approx \pi_{n-1}^A(B)\frac{\lambda^B}{\mu^B}+\pi_{n-1}^A(V)\epsilon,\ n\ge 1, \\ &\pi_n(B) \approx \pi_n^A(B)\gamma^B+\pi_{n-1}^A(V)\delta,\ n\ge 1, \end{aligned}\right\} \quad (17)$$

where $\lambda^V = v+\alpha-E[A]$, $\mu^V = v+\alpha$, $\gamma^V = \dfrac{v}{v+\alpha-E[A]}$, $\lambda^B = \alpha+\sigma-E[A]$, $\mu^B = \alpha+\sigma-E[S]$, $\epsilon = \dfrac{E[S]}{\alpha+\sigma-E[S]}$, $\gamma^B = \dfrac{\alpha\rho+\sigma-E[S]}{\alpha+\sigma-E[A]}$, and $\delta_n = \rho-\dfrac{\sigma\epsilon\mu^B}{\lambda^B E[A]}$. Note that (16) is exact for the Bernoulli arrival process, geometric service times, and geometric vacation times in the LAS, respectively, due to the memoryless property of the geometric distribution. Therefore, our approximations in (17) lead to the exact queue length distribution for the Geo/Geo/1/GMV queue, where GMV stands for geometric multiple vacations. Similar approximations to those proposed in (17) have been used to approximate a continuous-time queue by Kim and Chae (2003) and Choi *et al*. (2005). For some range of parameter values, (17) may result in negative probabilities. In such a case, one can set those approximate values to zero.

From (17), approximations of various mean performance measures can be obtained, such as approximations for the mean numbers of customers in service, in the queue, and in the entire system. Subsequently, approximations for the mean time a customer spends in the queue and in the system also follow from Little's formula. Among others, we present the approximate value for the mean of customers in the entire system, denoted by $E[N]$. From (6), (7), (12), and (13), we have

$$\begin{aligned} E[N] &= 1+\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\left(\pi_n^A(V)+\pi_n^A(B)\right), && (18) \\ &\quad -\sum_{n=0}^{\infty}\lambda\left(\pi_n^A(V)+\pi_n^A(B)\right)\alpha_n^D, \\ E[N] &= \rho\sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\left(\pi_n^A(V)+\pi_n^A(B)\right) && (19) \\ &\quad +\sum_{n=0}^{\infty}\lambda\left(\pi_n^A(V)E[S]+\pi_n^A(V)v_n+\pi_n^A(B)\sigma_n\right). \end{aligned}$$

Combining (18) and (19) with (16) yields

$$E[N] \approx \frac{\lambda(E[S]+v+\rho\alpha)\sum_{n=0}^{\infty}\widetilde{\pi_n^A}(V) + \lambda(\sigma+\rho\alpha)\sum_{n=1}^{\infty}\widetilde{\pi_n^A}(B)}{1-\rho} - \frac{\rho}{1-\rho}, \quad (20)$$

where $\widetilde{\pi_n^A}(V)$ and $\widetilde{\pi_n^A}(B)$ are approximate values for $\pi_n^A(V)$ and $\pi_n^A(B)$, respectively.

To evaluate the performance of our approximation, extensive numerical experiments have been carried out for a variety of interarrival times, service times, and vacation times, but only a few that exhibit representative information are presented in <Table 1> through <Table 3>. In all cases, exact values are calculated by differentiating the PGFs of each queue length distribution or carrying out simulation experiments. $NB_2$ denotes a 2-Negative binomial distribution and $MGeo_2$ denotes a mixed-geometric distribution of order 2.

<Table 1> presents results for $\pi_n$ and $E[N]$ of several Bernoulli arrival queues with GMV in low ($\rho = 0.25$) traffic, in moderate ($\rho = 0.50$) traffic, and high ($\rho = 0.75$) traffic, respectively. Through our numerical investigations, we have observed that our results closely match the exact results regardless of the traffic intensities. <Table 2> shows results of the case of Bernoulli arrival queues with general multiple vacations. Interestingly, our approximation functions well even though vacation times do not have geometric distributions.

**Table 1.** $\pi_n$ and $E[N]$ for $Geo/G/1/GMV$ queues

$Geo/Geo/1/GMV$

| $\rho$ | 0.25 | | 0.50 | | 0.75 | |
|---|---|---|---|---|---|---|
| | Ours | Exact | Ours | Exact | Ours | Exact |
| $\pi_0$ | .6818 | .6818 | .4167 | .4167 | .1923 | .1923 |
| $\pi_1$ | .2514 | .2514 | .2604 | .2604 | .2504 | .2504 |
| $\pi_2$ | .0544 | .0544 | .1411 | .1411 | .1901 | .1901 |
| $\pi_3$ | .0102 | .0102 | .0601 | .0601 | .1291 | .1291 |
| $\pi_4$ | .0018 | .0018 | .0238 | .0238 | .0845 | .0845 |
| $\pi_5$ | .0003 | .0003 | .0091 | .0091 | .0547 | .0547 |
| $E[N]$ | .4000 | .4000 | .9000 | .9000 | 2.400 | 2.400 |

$Geo/MGeo_2/1/GMV$

| $\rho$ | 0.25 | | 0.50 | | 0.75 | |
|---|---|---|---|---|---|---|
| | Ours | Exact | Ours | Exact | Ours | Exact |
| $\pi_0$ | .6818 | .6818 | .4167 | .4617 | .1923 | .1923 |
| $\pi_1$ | .2483 | .2490 | .3204 | .3241 | .2380 | .2443 |
| $\pi_2$ | .0562 | .0552 | .1532 | .1502 | .1836 | .1826 |
| $\pi_3$ | .0112 | .0112 | .0652 | .0635 | .1278 | .1248 |
| $\pi_4$ | .0021 | .0023 | .0266 | .0264 | .0863 | .0838 |
| $\pi_5$ | .0004 | .0005 | .0107 | .0110 | .0576 | .0563 |
| $E[N]$ | .4050 | .4050 | 1.030 | 1.030 | 2.535 | 2.535 |

$Geo/NB_2/1/GMV$

| $\rho$ | 0.25 | | 0.50 | | 0.75 | |
|---|---|---|---|---|---|---|
| | Ours | Exact | Ours | Exact | Ours | Exact |
| $\pi_0$ | .6818 | .6818 | .4167 | .4617 | .1923 | .1923 |
| $\pi_1$ | .2655 | .2645 | .3781 | .3706 | .3258 | .3056 |
| $\pi_2$ | .0454 | .0471 | .1430 | .1532 | .2193 | .2346 |
| $\pi_3$ | .0063 | .0059 | .0446 | .0447 | .1244 | .1334 |
| $\pi_4$ | .0008 | .0006 | .0128 | .0114 | .0665 | .0686 |
| $\pi_5$ | .0001 | .0001 | .0035 | .0027 | .0347 | .0339 |
| $E[N]$ | .3792 | .3792 | 0.875 | 0.875 | 1.838 | 1.838 |

**Table 2.** $\pi_n$ and $E[N]$ for $Geo/G/1/MV$ queues

$Geo/MGeo_2/1/GMV$

| $\rho$ | 0.25 | | 0.50 | | 0.75 | |
|---|---|---|---|---|---|---|
| | Ours | Exact | Ours | Exact | Ours | Exact |
| $\pi_0$ | .5357 | .5357 | .2778 | .2778 | .1136 | .1136 |
| $\pi_1$ | .2994 | .3000 | .2908 | .2932 | .1764 | .1801 |
| $\pi_2$ | .1119 | .1113 | .1958 | .1945 | .1722 | .1728 |
| $\pi_3$ | .0367 | .0367 | .1135 | .1121 | .1444 | .1431 |
| $\pi_4$ | .0114 | .0114 | .0609 | .0604 | .1123 | .1106 |
| $\pi_5$ | .0034 | .0035 | .0313 | .0313 | .0836 | .0821 |
| $E[N]$ | .7050 | .7050 | 1.630 | 1.630 | 3.435 | 3.435 |

$Geo/MGeo_2/1/MGeo_2MV$

| $\rho$ | 0.25 | | 0.50 | | 0.75 | |
|---|---|---|---|---|---|---|
| | Ours | Exact | Ours | Exact | Ours | Exact |
| $\pi_0$ | .4903 | .4963 | .2428 | .2513 | .0966 | .1025 |
| $\pi_1$ | .3037 | .2982 | .2712 | .2705 | .1565 | .1616 |
| $\pi_2$ | .1293 | .1266 | .1976 | .1913 | .1606 | .1587 |
| $\pi_3$ | .0491 | .0494 | .1248 | .1208 | .1415 | .1373 |
| $\pi_4$ | .0178 | .0186 | .0734 | .0721 | .1154 | .1115 |
| $\pi_5$ | .0063 | .0069 | .0414 | .0416 | .0897 | .0870 |
| $E[N]$ | .8346 | .8346 | 1.889 | 1.889 | 3.824 | 3.824 |

$Geo/MGeo_2/1/NB_2MV$

| $\rho$ | 0.25 | | 0.50 | | 0.75 | |
|---|---|---|---|---|---|---|
| | Ours | Exact | Ours | Exact | Ours | Exact |
| $\pi_0$ | .5882 | .5813 | .3226 | .3107 | .1370 | .1278 |
| $\pi_1$ | .2876 | .2969 | .3088 | .3184 | .1998 | .2033 |
| $\pi_2$ | .0910 | .0914 | .1868 | .1931 | .1820 | .1885 |
| $\pi_3$ | .0249 | .0234 | .0970 | .0975 | .1431 | .1465 |
| $\pi_4$ | .0063 | .0055 | .0466 | .0451 | .1056 | .1055 |
| $\pi_5$ | .0015 | .0012 | .0214 | .0200 | .0744 | .0734 |
| $E[N]$ | .5800 | .5800 | 1.380 | 1.380 | 3.060 | 3.060 |

The results for the non-Bernoulli arrival queues are appended in Table 3. In this case, however, approximations can deteriorate. Thus, one should use the approximations cautiously for non-Bernoulli arrival queues. Note that our approximations do not require the whole distributions of *A*, *S*, and *V*, but only the first two moments. The first two moments alone will lead to quick and simple approximate results.

**Table 3.** $E[N]$ for $GI/G/1/MV$ queues

$NB_2/Geo/1/GMV$

| | $\rho$ | 0.2 | 0.4 | 0.6 | 0.8 |
|---|---|---|---|---|---|
| $E[N]$ | Exact | .3063 | .5977 | 1.153 | 2.788 |
| | Ours | .2750 | .5667 | 1.150 | 2.900 |

$MGeo_2/Geo/1/GMV$

| | $\rho$ | 0.2 | 0.4 | 0.6 | 0.8 |
|---|---|---|---|---|---|
| $E[N]$ | Exact | .3298 | .7389 | 1.596 | 4.234 |
| | Ours | .3583 | .7889 | 1.650 | 4.233 |

$NB_2/NB_2/1/GMV$

| | $\rho$ | 0.2 | 0.4 | 0.6 | 0.8 |
|---|---|---|---|---|---|
| $E[N]$ | Exact | .3022 | .5312 | .8516 | 1.823 |
| | Ours | .2656 | .5167 | .9813 | 2.300 |

$MGeo_2/NB_2/1/GMV$

| | $\rho$ | 0.2 | 0.4 | 0.6 | 0.8 |
|---|---|---|---|---|---|
| $E[N]$ | Exact | .3143 | .6261 | 1.213 | 3.092 |
| | Ours | .3438 | .7111 | 1.388 | 3.300 |

**Remark 6** : One may consider $\sum_{n=0}^{\infty}\widetilde{\pi}_n=1$, where $\widetilde{\pi}_n$ is the approximate value for $\pi_n$. From (6) and (12), $\sum_{n=0}^{\infty}\pi_n$ can be written as $\sum_{n=0}^{\infty}\pi_n^A(V)+\sum_{n=1}^{\infty}\pi_n^A(B)$. For Bernoulli arrival queues, $\sum_{n=0}^{\infty}\widetilde{\pi}_n=1$ holds due to $\sum_{n=0}^{\infty}\widetilde{\pi}_n^A(V)=1-\rho$ and $\sum_{n=0}^{\infty}\widetilde{\pi}_n^A(B)=\rho$. For non-Bernoulli arrival queues, however, $\sum_{n=0}^{\infty}\widetilde{\pi}_n=1$ does not hold. In this case, we approximate $\widetilde{\pi}_0$ to $1-\sum_{n=0}^{\infty}\widetilde{\pi}_n$ by normalization.

# 4. Conclusions

This paper aimed at analyzing a discrete-time GI/G/1 queue with multiple vacations and making a two-moment approximation of the mean queue length and the probabilities of the number of customers in the system for that queue. To this end, we derived a transform-free steady state queue length distribution, $\pi_n(V)$ and $\pi_n(B)$. The transform-free queue length distribution for other vacation models, such as the single vacation model and set-up time model, can be derived by the modified SVT.

Note again that our approximation is especially useful when distributions of *A*, *S*, and *V* are unknown and have to be estimated, since our methods do not require fitting distributions to sample data but only require that the first two moments be obtained. On this basis, the approximation procedure is simple and quick. We anticipate that our two-moment approximation will be beneficial to those practitioners who seek simple and quick practical answers to queueing systems with multiple vacations.

Finally, it is noted that the modified SVT is basically the same as the conventional SVT except that in the last step of solving system equations. We multiply a supplementary variate *i*+1, *j*+1, and *k*+1 and then sum over both *i*, *j*, and *k*. As a result, we obtained the simultaneous equations for the queue length probabilities in terms of conditional expectations of the supplementary variables. We believe that our approach will help the readers better understand discrete-time queueing systems and gain new insight into their analyses.

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# <Appendix A> Derivation of the Results in (2)

We first sum the first equation in (1a), both over *i* and *j*, $0 \le i,\, j \le \infty$, we have

$$\sum_{i,j=0}^{\infty} P_0^V(i,\, j) = \sum_{i,j=0}^{\infty} P_0^V(i+1,\, j+1) + \sum_{i=0}^{\infty}\left\{P_0^V(i+1,\, 0) + P_1^B(i+1,\, 0)\right\}\sum_{j=0}^{\infty} v_{j+1}. \tag{A.1}$$

The left hand side of (A.1) is split into four terms as following :

$$P_0^V(0,\, 0) + \sum_{i=1}^{\infty} P_0^V(i,\, 0) + \sum_{j=1}^{\infty} P_0^V(0,\, j) + \sum_{i,j=1}^{\infty} P_0^V(i,\, j). \tag{A.2}$$

The right hand side of (A.1) is simplified as following :

$$\sum_{i,j=1}^{\infty} P_0^V(i,\, j) + \sum_{i=1}^{\infty} P_0^V(i,\, 0) + \sum_{i=1}^{\infty} P_1^B(i,\, 0). \tag{A.3}$$

Thus, we have

$$\sum_{i=1}^{\infty} P_1^B(i,\, 0) = P_0^V(0,\, 0) + \sum_{j=1}^{\infty} P_0^V(0,\, j). \tag{A.4}$$

Applying the same procedure the rest of equations of (1), we get the following relations :

$$\left.\begin{aligned}
&\sum_{j=1}^{\infty} P_{n-1}^V(0,\, j) = P_n^V(0,\, 0) + \sum_{i=1}^{\infty} P_n^V(i,\, 0) + \sum_{j=1}^{\infty} P_n^V(0,\, j)\\
&\sum_{j=1}^{\infty} P_0^V(0,\, j) = \sum_{k=1}^{\infty} P_1^B(0,\, k) = \sum_{i=1}^{\infty} P_2^B(i,\, 0) + \sum_{i=1}^{\infty} P_1^V(i,\, 0)\\
&\sum_{i=1}^{\infty} P_n^B(i,\, 0) + \sum_{k=1}^{\infty} P_n^B(0,\, k) = \sum_{i=1}^{\infty} P_{n+1}^B(i,\, 0) + \sum_{k=1}^{\infty} P_{n-1}^B(0,\, k) + \sum_{i=1}^{\infty} P_n^B(i,\, 0) + P_{n-1}^B(0,\, 0)
\end{aligned}\right\} \tag{A.5}$$

By recursively solving (A.4) and (A.5), we finally get the results in Eq. (2).

# <Appendix B> Derivation of the Results in (4)

We will show the derivation of the second equation in (4), and , nonetheless, we can prove the results of Eqs. (5), (10), and (11) by using the same procedure.

We multiply *i*+1 to the left hand side of (1a) and sum over *i* and *j*, $0 \le i,\, j \le \infty$ and we have

$$\sum_{i,j=0}^{\infty} (i+1) P_n^V(i,\, j) = \sum_{i,j=0}^{\infty} i P_n^V(i,\, j) + \sum_{i,j=0}^{\infty} P_n^V(i,\, j) = \sum_{i,j=1}^{\infty} i P_n^V(i,j) + \sum_{i=1}^{\infty} i P_n^V(1,0) + \pi_n(V),\ n \ge 1. \tag{B.1}$$

We multiply *i*+1 to the right hand side of (1b) and sum over *i* and *j*, $0 \le i,\, j \le \infty$ and we have

$$\sum_{i,j=0}^{\infty} (i+1) P_n^V(i+1,\, j+1) + \sum_{j=0}^{\infty} P_{n-1}^V(0,\, j+1) \sum_{i=0}^{\infty} (i+1)\alpha_{i+1} = \sum_{i,j=1}^{\infty} i P_n^V(i,\, j) + \sum_{j=1}^{\infty} P_{n-1}^V(0,\, j) E[A],\ n \ge 1. \tag{B.2}$$

Equalizing (B.1) and (B.2), we have

$$\pi_n(V) = \sum_{j=1}^{\infty} P_{n-1}^V(0,\, j) E[A] - \sum_{i=1}^{\infty} i P_n^V(i,\, 0) = \sum_{j=0}^{\infty} P_{n-1}^V(0,\, j) E[A] - \left\{\sum_{i=1}^{\infty} i P_n^V(i,\, 0) + P_{n-1}^V(0,\, 0) E[A]\right\},\ n \ge 1. \tag{B.3}$$

Applying the similar procedure to (1b), we finally get the results of (5), (10), and (11).

# <Appendix C> Proof of Lemma 1

By the definition of $\alpha_n^V$, it is expressed as $\alpha_n^V = \lim_{t\to\infty} E\,[A_R(t^+)|N(t^+)=n, E_V]$, where $E_V$ denotes the event that the vacation will end during $(t,\ t^-)$. Therefore, we have

$$
\begin{aligned}
\alpha_n^V &= \lim_{t\to\infty} E[A_R(t^+)|N(t^+)=n,\ E_V] \\
&= \lim_{t\to\infty}\sum_{i=1}^{\infty} i\frac{\Pr\{A_R(t^+)=i,\ N(t^+)=n,\ E_V\}}{\Pr\{N(t^+)=n,\ E_V\}} \\
&= \lim_{t\to\infty}\sum_{i=1}^{\infty} i\frac{\Pr\{A_R(t^-)=i,\ V_R(t^-)=0,\ N(t^-)=n,\ E_V\}}{\Pr\{N(t^+)=n,\ E_V\}} \\
&+ \lim_{t\to\infty}\sum_{i=1}^{\infty} i\frac{\Pr\{A_R(t^-)=0,\ V_R(t^-)=0, N(t^-)=n-1,\ E_V\}\Pr\{A=i\}}{\Pr\{N(t^+)=n,\ E_V\}} \\
&= \frac{\sum_{i=1}^{\infty} iP_n^V(i,0)+P_{n-1}^V(0,0)E[A]}{\sum_{i=1}^{\infty} P_n^V(i,0)+P_{n-1}^V(0,0)} = \frac{\sum_{i=1}^{\infty} iP_n^V(i,0)+P_{n-1}^V(0,0)E[A]}{\sum_{i=1}^{\infty} P_n^V(i,0)+P_{n-1}^V(0,0)} = \frac{\sum_{i=1}^{\infty} iP_n^V(i,0)+P_{n-1}^V(0,0)E[A]}{\lambda\left(P_{n-1}^A-P_n^A\right)}
\end{aligned}
\tag{C.1}
$$

In a similar manner in (B.1), we can prove other relations in Lemma 1 but we here omit their proofs.