CONVOLUTION THEOREMS FOR FRACTIONAL FOURIER COSINE AND SINE TRANSFORMS AND THEIR EXTENSIONS TO BOEHMIANS

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Abstract. By introducing two fractional convolutions, we obtain the convolution theorems for fractional Fourier cosine and sine transforms. Applying these convolutions, we construct two Boehmian spaces and then we extend the fractional Fourier cosine and sine transforms from these Boehmian spaces into another Boehmian space with desired properties.

1. Introduction

Let \( \mathbb{N}, \mathbb{R}, \) and \( \mathbb{C} \) denote the sets of all natural, real and complex numbers respectively. The Banach space of all Lebesgue measurable complex valued functions \( f \) on \([0, \infty)\) satisfying \( \|f\|_p = \left( \int_0^\infty |f(t)|^p \, dt \right)^{\frac{1}{p}} < \infty, \) is denoted by \( L^p_{\alpha} \), where \( p = 1, 2, \) After the introduction of fractional Fourier transform \([11]\), many integral transforms have been generalized as the corresponding fractional integral transforms. In particular, fractional Fourier cosine transform (FRFC), fractional Fourier sine transform (FRFS) and Fractional Hartley transform were defined and used extensively in signal processing. See \([2, 16]\). We now recall the definitions of FRFC and FRFS of \( f \in L^1_{\alpha} \) from \([2]\).

\[
(F_{\alpha}^C(f))(u) = c_{\alpha} \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) e^{ia_{\alpha}(a^2 + u^2)} \cos(b_{\alpha}ux) \, dx, \quad u \in [0, \infty),
\]

\[
(F_{\alpha}^S(f))(u) = c_{\alpha} \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) e^{ib_{\alpha}(a^2 + u^2)} \sin(b_{\alpha}ux) \, dx, \quad u \in [0, \infty),
\]

where \( a_{\alpha} = \cot{\frac{\alpha}{2}}, \quad b_{\alpha} = \frac{1}{\sin{\alpha}} \) and \( c_{\alpha} = \sqrt{\sin{\alpha}}. \)

Analogous to the Plancherel theorem for Fourier transform \([22, p. 186]\), we can define the fractional Fourier cosine (sine) transform of \( f \in L^2_{\alpha} \) by \( L^2_{\alpha} - \lim_{n \to \infty} F_{\alpha}^C(f_n) \) (\( L^2_{\alpha} - \lim_{n \to \infty} F_{\alpha}^S(f_n) \)), where \( (f_n) \) is a sequence from \( L^1_{\alpha} \cap L^2_{\alpha} \), such that \( f_n \to f \) in \( L^2_{\alpha} \) as \( n \to \infty \). The existence of \( (f_n) \) is possible by the
fact that \( L^1 \cap L^2 = \text{dense in } L^2 \) and the existence of \( L^2 - \lim_{n \to \infty} F^\alpha_S(f_n) \) \((L^2 - \lim_{n \to \infty} F^\alpha_S(f_n))\) follows from the identity \( \|f\|_2 = \|F^\alpha_S f\|_2 \) \((\|f\|_2 = \|F^\alpha_S f\|_2)\), \( \forall f \in L^1 \cap L^2 \). We refer the reader to [2, Eqn. (19)], for the Parseval’s identity for FRFCT and FRFST, which implies the above identities. Thus fractional Fourier cosine and sine transforms become isometries from \( L^2 \) onto itself with self inverse.

The convolution theorems for Fourier sine and cosine transforms were first studied in [23] and then they are generalized by various researchers in [4, 24, 25, 26, 27]. Motivated by the convolutions discussed in [23, 28], we introduce two convolutions, denoted by \( * \) and \( *_{nc} \), which are suitable for discussing the convolution theorems for the fractional Fourier cosine and sine transforms. Using these convolutions, we construct suitable Boehmian spaces \( B \) from [6, 8]. An abstract Boehmian space is, in general, denoted by \( B \). Using these convolutions, we construct suitable Boehmian spaces \( B \) from \( [6, 8] \). An abstract Boehmian space is, in general, denoted by \( B \), which is defined on the collection of all quotients by \( \sim \) from \( [6, 8] \). An abstract Boehmian space is, in general, denoted by \( B \), which is defined on the collection of all quotients by \( \sim \).

This generalization motivates many researchers to extend the theory of integral transforms to the context of Boehmians (see [1, 3, 7, 9, 10, 12, 13, 14, 15, 17, 18, 19, 20, 21, 28, 29]).

The concept of Boehmian space was first introduced by J. Mikusiński and P. Mikusiński [5], which is, in general, a generalization of the space of distributions. This generalization motivates many researchers to extend the theory of integral transforms to the context of Boehmians (see [1, 3, 7, 9, 10, 12, 13, 14, 15, 17, 18, 19, 20, 21, 28, 29]).

Before ending this section, we briefly recall the construction of Boehmians from [6, 8]. An abstract Boehmian space is, in general, denoted by \( B = B(G, (S, \cdot), \circ, \Delta) \), where \( G \) is a topological vector space over \( C \), \((S, \cdot)\) is a commutative semi-group, \( \circ : G \times S \to G \) satisfies the following conditions:

- \((g_1 + g_2) \circ s = g_1 \circ s + g_2 \circ s, \forall g_1, g_2 \in G \) and \( \forall s \in S \).
- \((cg) \circ s = c(g \circ s), \forall c \in C, \forall g \in G \) and \( \forall s \in S \).
- \( g \circ (s \cdot t) = (g \circ s) \cdot t, \forall g \in G \) and \( \forall s, t \in S \).
- \( g_n \to g \) as \( n \to \infty \) in \( G \) and \( s \in S \), then \( g_n \circ s \to g \circ s \) as \( n \to \infty \),

and \( \Delta \) is a collection of sequences from \( S \) with the following properties:

- \( (s_n), (t_n) \in \Delta \), then \( (s_n, t_n) \in \Delta \).
- \( (g_n) \to g \) as \( n \to \infty \) in \( G \) and \( (s_n) \in \Delta \), then \( g_n \circ s_n \to g \) as \( n \to \infty \) in \( G \).

If \( g_n \in G, \forall n \in N \) and \( (s_n) \in \Delta \) are such that \( g_n \circ s_m = g_m \circ s_n, \forall m, n \in N \), then the pair of sequences \((g_n), (s_n)\) is called a quotient and is denoted by \( \frac{g_n}{s_n} \).

The equivalence class \( \left[ \frac{g_n}{s_n} \right] \) containing \( \frac{g_n}{s_n} \) induced by the equivalence relation \( \sim \), which is defined on the collection of all quotients by

\[
\frac{g_n}{s_n} \sim \frac{h_m}{t_n} \text{ if } g_n \circ t_m = h_m \circ s_n, \forall m, n \in N
\]
is called a Boehmian and the collection $\mathcal{B}$ of all Boehmians is a vector space with respect to the following addition and scalar multiplication:
\[
\frac{g_n}{s_n} + \frac{h_n}{t_n} = \frac{g_n \circ t_n + h_n \circ s_n}{s_n \cdot t_n}, \quad c \left( \frac{g_n}{s_n} \right) = \left( \frac{cg_n}{s_n} \right).
\]

Every member $g \in G$ can be uniquely identified as a member of $\mathcal{B}$ by $\left[ \frac{g_{2n}}{s_n} \right]$, where $(s_n) \in \Delta$ is arbitrary and the operation $\circ$ is also extended to $\mathcal{B} \times S$ by $\left[ \frac{g_n}{s_n} \right] \circ t = \left[ \frac{g_{n\odot t}}{s_n} \right]$. There are two notions of convergence on $\mathcal{B}$ namely $\delta$-convergence and $\Delta$-convergence, which are defined as follows.

**Definition** ([6]). We write that $X_m \xrightarrow{\delta} X$ as $m \to \infty$ in $\mathcal{B}$, if there exist $g_{m,n}, g_n \in G$, $m, n \in \mathbb{N}$ and $(s_n) \in \Delta$ such that $X_m = \left[ \frac{2m \odot s_n}{s_n} \right]$, $X = \left[ \frac{2n}{s_n} \right]$ and for each $n \in \mathbb{N}$, $g_{m,n} \to g_n$ as $m \to \infty$ in $G$.

**Definition** ([6]). We write that $X_m \xrightarrow{\Delta} X$ as $m \to \infty$ in $\mathcal{B}$, if there exists $(s_n) \in \Delta$ such that $(X_m - X) \odot s_m \in G \forall m \in \mathbb{N}$ and $(X_m - X) \odot s_m \to 0$ as $m \to \infty$ in $G$. This means that there exist $g_m \in G$, $\forall m \in \mathbb{N}$ and $(s_n) \in \Delta$ such that $(X_m - X) \odot s_m = \left[ \frac{2m \odot s_n}{s_n} \right]$ and $g_m \to 0$ as $m \to \infty$ in $G$.

### 2. Convolution for fractional Fourier cosine and sine transforms

In this section, we introduce two special convolutions and prove all the preliminary results required for constructing the Boehmian spaces $\mathcal{B}_C^\alpha$ and $\mathcal{B}_S^\alpha$.

**Definition.** For $f, g \in L_1^\alpha$ and $x \in [0, \infty)$,

(i) The convolution $*^\alpha_c$ is defined by $$(f *^\alpha_c g)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(y) e^{\beta x y} [f(x+y)e^{\beta x y} + f(|x-y|)e^{-\beta x y}] dy.$$

(ii) The convolution $*^\alpha_s$ is defined by $$(f *^\alpha_s g)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) e^{\beta x y} [g(|x-y|)e^{-\beta x y} - g(x+y)e^{\beta x y}] dy,$$

where $\beta = 2i\alpha$.

It is easy to verify the following two inequalities:

$$\|f *^\alpha_c g\|_1 \leq c_\alpha \sqrt{\frac{2}{\pi}} \|f\|_1\|g\|_1$$

and

$$\|f *^\alpha_s g\|_1 \leq c_\alpha \sqrt{\frac{2}{\pi}} \|f\|_1\|g\|_1.$$
\[ \int_{0}^{\infty} g(y)f(x + y)e^{\beta(y^2 + xy)} \, dy + \int_{0}^{\infty} g(y)f(|x - y|)e^{\beta(y^2 - xy)} \, dy \]

\[ = \int_{x}^{\infty} g(z - x)f(z)e^{\beta[(z - x)^2 + z(z - x)]} \, dz \]

\[ \quad + \int_{-\infty}^{x} g(x - z)f(|z|)e^{\beta[(z - x)^2 - x(z - x)]} \, dz \]

\[ = \int_{x}^{\infty} g(z - x)f(z)e^{\beta[z^2 - 2xz]} \, dz + \int_{0}^{\infty} g(x + z)f(z)e^{\beta[z^2 + 2xz]} \, dz \]

\[ \quad + \int_{0}^{x} g(x - z)f(z)e^{\beta[z^2 - 2xz]} \, dz \]

\[ = \int_{0}^{\infty} f(z)e^{\beta z^2} \left| g(|x - z|)e^{-\beta xz} + g(x + z)e^{\beta xz} \right| dz = \frac{\sqrt{2\pi}}{c_\alpha} (g * _c^\alpha f)(x) \]

and hence \( f * _c^\alpha g = g * _c^\alpha f. \) \( \square \)

**Lemma 2.2.** If \( f, g \) and \( h \in L_1 \), then \( f * _c^\alpha (g * _c^\alpha h) = (f * _c^\alpha g) * _c^\alpha h. \)

**Proof.** For \( x \in [0, \infty), \)

\[ [(f * _c^\alpha g) * _c^\alpha h](x) \]

\[ = \int_{0}^{\infty} [(f * _c^\alpha g)(x + z)e^{\beta z^2} + (f * _c^\alpha g)(|x - z|)e^{-\beta xz}] h(z)e^{\beta z^2} \, dz \]

\[ = \int_{0}^{\infty} h(z)e^{\beta(z^2 + xz)}(f * _c^\alpha g)(x + z)dz + \int_{0}^{\infty} h(z)e^{\beta(z^2 - xz)}(f * _c^\alpha g)(|x - z|)dz \]

\[ = \int_{0}^{\infty} h(z)e^{\beta(z^2 + xz)}I(x, z)dz + \int_{0}^{\infty} h(z)e^{\beta(z^2 - xz)}J(x, z)dz, \]

where

\[ I(x, z) = \int_{0}^{\infty} g(u)e^{\beta u^2}[f(x + z + u)e^{\beta(z + u)} + f(|x + z - u|)e^{-\beta(z + u)}] \, du; \]

\[ J(x, z) = \int_{0}^{\infty} g(u)e^{\beta u^2}[f(|x - z| + u)e^{\beta|x - z|} + f(|x - z - u|)e^{-\beta|x - z|}] \, du. \]

Since

\[ \int_{0}^{\infty} h(z)e^{\beta(z^2 - xz)}J(x, z)dz \]

\[ = \int_{0}^{\infty} h(z)e^{\beta(z^2 - xz)} \int_{0}^{\infty} g(u)e^{\beta u^2}[f(x - z + u)e^{\beta(z - u)} + f(|x - z - u|)e^{-\beta(z - u)}] \, du \, dz, \]

the equation (1) becomes

\[ [(f * _c^\alpha g) * _c^\alpha h](x) \]
By using Jensen’s inequality and Fubini’s theorem, we obtain that
\[\int_0^\infty h(z)e^{\beta z^2} \left( \int_0^\infty g(u)e^{\beta u^2} |f(x + z + u)e^{\beta (x+z)u} + f(x + z - u)e^{-\beta (x+z)u} du \right) dz\]
\[= \int_0^\infty h(z)e^{\beta z^2} \left\{ \int_0^\infty f(|x + u - z|)g(u)e^{\beta (u^2 + xu - uz)} du + \int_0^\infty g(u)e^{\beta (u^2 + xu - uz)} du \right\} \]
\[= \int_0^\infty e^{\beta y^2} [f(x + y)e^{\beta xy} + f(|x - y|)e^{-\beta xy}] \int_0^\infty h(z)e^{\beta z^2} g(y + z)e^{\beta yz} dz dy \]
\[= |f *_c g(x)|.\]
Since \(x \in [0, \infty)\) is arbitrary, the proof follows. \(\square\)

In the following sequel, the well known inequality \((a + b)^2 \leq 2(a^2 + b^2)\), \(\forall a, b \geq 0\), will be used at many places, without quoting it, explicitly.

**Lemma 2.3.** If \(f \in L^2_+\) and \(g \in L^1_+\), then \(\|f *_c g\|_2 \leq |c_\alpha| \sqrt{\frac{2}{\pi}} \|f\|_2 \|g\|_1\), and hence \(f *_c g \in L^2_+\).

**Proof.** By using Jensens’ inequality and Fubini’s theorem, we obtain that
\[\|f *_c g\|_2^2 = \int_0^\infty |(f *_c g)(x)|^2 dx\]
\[= \int_0^\infty \left| \int_0^\infty g(y)e^{\beta y^2} [f(x + y)e^{\beta xy} + f(|x - y|)e^{-\beta xy}] dy \right|^2 dx\]
\[\leq \frac{\|g\|^2_2 |c_\alpha|^2}{2\pi} \int_0^\infty \left( \int_0^\infty |g(y)||f(x + y)e^{\beta xy} + f(|x - y|)e^{-\beta xy}|^2 dy \right) dx\]
\[\leq \frac{\|g\|_1 |c_\alpha|^2}{\pi} \int_0^\infty |g(y)| \left( \int_0^\infty (|f(x + y)|^2 + |f(|x - y|)|^2) dy \right) dy,
\]
Theorem 2.4. If $f_n \to f$ as $n \to \infty$ in $L^2_+$ and $g \in L^1_+$, then $f_n *^c g \to f *^c g$ as $n \to \infty$ in $L^2_+$.

Lemma 2.5. If $f \in L^2_+$ and if $g, h \in L^1_+$, then $f *^c (g *^c h) = (f *^c g) *^c h$.

Proof. Choose a sequence $f_n \in L^1_+ \cap L^2_+$ such that $f_n \to f$ as $n \to \infty$ in $L^2_+$.

By using Lemma 2.2 and Theorem 2.4, we obtain that

$$f *^c (g *^c h) = \lim_{n \to \infty} f_n *^c (g *^c h) = \lim_{n \to \infty} (f_n *^c g) *^c h = (f *^c g) *^c h.$$ 

Hence the theorem follows. □

The following theorem is an immediate consequence of the inequality proved in the previous theorem.

Theorem 2.6 (Convolution theorems). If $f, g \in L^1_+$ and $u \in [0, \infty)$, then

(i) $[F_C^a(f *^c g)](u) = e^{-iau^2}[F_C^a(f)](u)[F_C(g)](u).

(ii) $[F_S^a(f *^c g)](u) = e^{-iau^2}[F_S^a(f)](u)[F_C(g)](u).$

Proof. For $f, g \in L^1_+$ and $u \in [0, \infty)$,

(i) $$[F_C^a(f *^c g)](u) = c_0 \sqrt{\frac{2}{\pi}} \int_0^\infty (f *^c g)(x)e^{iau(x^2+u^2)} \cos(b_0ux)dx$$

$$= \frac{c_0}{\pi} \int_0^\infty g(y)e^{iau(2y^2+u^2)} \left\{ \int_0^\infty [f(x+y)e^{iau(x^2+xy)} + f(|x-y|e^{iau(x^2-xy)} \cos(b_0ux)dx)dyight\}dy$$

$$= \frac{c_0}{\pi} \int_0^\infty g(y)e^{iau(y^2+u^2)} \left\{ \int_0^\infty [f(x+y)e^{iau(x+y)^2} + f(|x-y|e^{iau(x-y)^2} \cos(b_0ux)dx)dyight\}dy$$

$$= \frac{c_0}{\pi} \int_0^\infty g(y)e^{iau(y^2+u^2)} \left\{ \int_y^\infty f(z)e^{iau^2} \cos(b_0u(z-y))dz + \int_{-y}^\infty f(|z|e^{iau^2} \cos(b_0u(z+y))dz)dy$$
\begin{align*}
&= \frac{c_a}{\pi} \int_0^\infty g(y) e^{ia_n(y^2 + y^2)} \{ \int_0^\infty f(z) e^{ia_n z^2} \cos(b_n u(z - y)) dz \\
&\quad + \int_0^\infty f(z) e^{ia_n z^2} \cos(b_n u(z + y)) dz \} dy \\
&= c_a \sqrt{\frac{2}{\pi}} \int_0^\infty g(y) e^{ia_n(y^2 + y^2)} \int_0^\infty f(z) e^{ia_n z^2} \cos(b_n uz) dz dy \\
&= e^{-ia_n u^2} \left[ F_S^a(f)(u) \cdot [F_C^a(g)](u) \right].
\end{align*}

Hence the theorem follows. \hfill \Box

**Theorem 2.7** (Convolution theorems on \(L^2_+\)). If \(f \in L^2_+\), \(g \in L^1_+\) and \(u \in [0, \infty)\), then

\begin{enumerate}
\item [(i)] \( [F_S^a(f \ast_s^a g)](u) = e^{-ia_n u^2} [F_S^a(f)(u)] [F_C(g)](u) \).
\item [(ii)] \( [F_S^a(f \ast_s^a g)](u) = e^{-ia_n u^2} [F_S^a(f)(u)] [F_C(g)](u) \).
\end{enumerate}

**Proof.** Using Theorem 2.6, this proof follows from the facts that both \(FCT\) and \(FRST\) are continuous from \(L^2_+\) onto \(L^1_+\) and \(L^1_+ \cap L^2_+ \) is dense in \(L^2_+\). \hfill \Box
Definition. A sequence \((\phi_n)\) in \(L^2_+\) is called a \(\delta\)-sequence if it satisfies the following conditions:

\[\begin{align*}
(\Delta 1) \ & c_0 \sqrt{\frac{2}{\pi}} \int_0^\infty e^{ia_n y^2} \phi_n(y) \, dy = 1, \forall n \in \mathbb{N}. \\
(\Delta 2) \ & \int_0^\infty |\phi_n(y)| \, dy \leq M, \forall n \in \mathbb{N}, \text{ for some } M > 0. \\
(\Delta 3) \ & \text{Given } \epsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that support of } \phi_n \subseteq (0, \epsilon), \forall n \geq N.
\end{align*}\]

The collection of all \(\delta\)-sequences is denoted by \(\Delta(\alpha)\).

Lemma 2.8. If \((\delta_n), (\psi_n) \in \Delta(\alpha)\), then \((\delta_n \ast^\alpha \psi_n) \in \Delta(\alpha)\).

Proof. Let \((\delta_n), (\psi_n) \in \Delta(\alpha)\). By a routine calculation, we obtain that

\[\int_0^\infty e^{ia_n y^2} [\delta_n \ast^\alpha \psi_n](y) \, dy = c_0 \sqrt{\frac{2}{\pi}} \int_0^\infty \delta_n(u) e^{ia_n u^2} \, du \int_0^\infty \psi_n(u) e^{ia_n u^2} \, du,
\]

which implies that \(c_0 \sqrt{\frac{2}{\pi}} \int_0^\infty e^{ia_n y^2} [\delta_n \ast^\alpha \psi_n](y) \, dy = 1\), using the property \((\Delta 1)\) of \(\Delta(\alpha)\). It is easy to verify that \(\|\delta_n \ast^\alpha \psi_n\|_1 \leq c_0 \sqrt{\frac{2}{\pi}} \|\delta_n\|_1 \|\psi_n\|_1 < P, \forall n \in \mathbb{N}\) and for some \(P > 0\). For a given \(\epsilon > 0\), we choose \(N \in \mathbb{N}\) such that supp \(\delta_n, \text{ supp } \psi_n \subseteq [0, \frac{\epsilon}{2}]\) for all \(n \geq N\). Using the fact that supp \((\delta_n \ast^\alpha \psi_n) \subseteq [\text{supp } \delta_n + \text{ supp } \psi_n] \cup \{[0, \infty] \cap [\text{supp } \delta_n - \text{ supp } \psi_n]\} \cup \{[0, \infty] \cap [\text{supp } \psi_n - \text{ supp } \delta_n]\} \), we get that supp \((\delta_n \ast^\alpha \psi_n) \subseteq [0, \frac{\epsilon}{2}] + [0, \frac{\epsilon}{2}] = [0, \epsilon)\) for all \(n \geq N\). Hence it follows that \((\delta_n \ast^\alpha \psi_n) \in \Delta(\alpha)\). \(\square\)

Lemma 2.9. If \(f \in L^2_+\) and \((\phi_n) \in \Delta(\alpha)\), then \(f \ast \phi_n \rightarrow f\) as \(n \rightarrow \infty\) in \(L^2_+\).

Proof. Let \(\epsilon > 0\) be given. Since \(C_c([0, \infty))\) is dense in \(L^2_+\), we can find \(g \in C_c([0, \infty))\) such that \(\|g - f\|_2 < \epsilon\). If \(g_y(x) = g(x - y), \forall x \in [0, \infty),\) then the mapping \(y \mapsto g_y\) is continuous on \([0, \infty)\) and hence for \(0 < \delta < \min\{1, \epsilon^2\}\), we have \(\|g_y - g_0\|_2 < \epsilon, \forall y \in [0, \delta]\). Therefore, for each \(y \in (0, \delta)\), we have

\[\int_0^\infty |g(x + y) - g(x)|^2 \, dx < \epsilon^2.
\]

Indeed,

\[\int_0^\infty |g(x + y) - g(x)|^2 \, dx = \int_y^\infty |g(z) - g(z - y)|^2 \, dz \leq \int_0^y |g(z) - g(y - z)|^2 \, dz + \int_0^\infty |g(z) - g(y - z)|^2 \, dz = \int_0^\infty |g(z) - g(|z - y|)|^2 \, dz = \|g_y - g_0\|_2^2 < \epsilon^2.
\]

We choose \(N \in \mathbb{N}\) such that supp \(\phi_n \subseteq [0, \delta)\) \(\forall n \geq N\). Applying Jensen’s inequality and Fubini’s theorem, for \(n \geq N\), we get that

\[\|(g \ast^\alpha \phi_n) - g\|_2^2 \]
Now for \( 0 \leq y < \delta \), we have

\[
\int_{0}^{\infty} |g(x + y)e^{ia_{n}(2xy+y^{2})} - g(x)|^{2} \, dx < C, \\
\leq 2\epsilon^{2} + 2\int_{0}^{\infty} |g(x)| e^{ia_{n}(2xy+y^{2})} - 1|^{2} \, dx, \text{ (using (2))}
\leq 2\epsilon^{2} + 2\int_{0}^{\infty} |g(x)| g_{2}\left(|a_{n}|(2xy+y^{2})\right)^{2} \, dx
\leq 2\epsilon^{2}C_{1}, \text{ since } y^{2} < y < \delta < \epsilon^{2},
\]
where \( C_{1} = 1 + 2|a_{n}|\int_{0}^{\infty} (2x+\delta)^{2}|g(x)| \, dx < \infty \). Similarly, we can prove that

\[
\int_{0}^{\infty} |g(x - y)e^{-\beta xy} - g(x)|^{2} \, dx < \epsilon^{2}C_{2}
\]
for some \( 0 < C_{2} < \infty \).

Using these estimates in (3), we get that

\[
\|\left(g \ast e^{\alpha}\phi_{n}\right) - g\|_{2}^{2} < \frac{1}{M^{2}}M^{2}|c_{\alpha}|^{2}\epsilon^{2}(C_{1} + C_{2}),
\]
where \( M > 0 \) is such that \( \int_{0}^{\infty} |\phi_{n}(x)| \, dx \leq M, \forall n \in \mathbb{N} \). Thus, using (4), Lemma 2.3 and property (\( \Delta 2 \)) of \( (\phi_{n}) \), we have

\[
\|f \ast e^{\alpha}\phi_{n} - f\|_{2} \leq \|f \ast e^{\alpha}\phi_{n} - g \ast e^{\alpha}\phi_{n}\|_{2} + \|g \ast e^{\alpha}\phi_{n} - g\|_{2} + \|g - f\|_{2} < K\epsilon
\]
for some \( K > 0 \). Hence the lemma follows.

**Theorem 2.10.** If \( f_{n} \to f \text{ as } n \to \infty \text{ in } L_{2}^{+} \text{ and } (\delta_{n}) \in \Delta(\alpha) \), then \( f_{n} \ast e^{\alpha}\delta_{n} \to f \text{ as } n \to \infty \text{ in } L_{2}^{+} \).

**Proof.** Let \( f_{n}, f \in L_{2}^{+} \) be such that \( f_{n} \to f \text{ as } n \to \infty \text{ in } L_{2}^{+} \) and let \( (\delta_{n}) \in \Delta(\alpha) \). Using Lemma 2.9 and the property (\( \Delta 2 \)) of \( (\delta_{n}) \), we get that

\[
\|f_{n} \ast e^{\alpha}\delta_{n} - f\|_{2} = \|f_{n} \ast e^{\alpha}\delta_{n} - f \ast e^{\alpha}\delta_{n} + f \ast e^{\alpha}\delta_{n} - f\|_{2}
\leq M|c_{\alpha}|\sqrt{\frac{2}{\pi}}\|f_{n} - f\|_{2} + \|f \ast e^{\alpha}\delta_{n} - f\|_{2} \to 0
\]
as \( n \to \infty \). \( \square \)
Thus, we have proved all auxiliary results required to construct the Boehmian space $\mathcal{B}_C^\alpha = \mathcal{B}_C^\alpha(L^2_{\alpha}, (L^2_{\alpha}, \ast_{\alpha}^C, \ast_{\alpha}^C, \Delta(\alpha)))$. We shall denote a typical element of $\mathcal{B}_C^\alpha$ by $X = [(f_n), (\delta_n)]$.

In the following sequel, we obtain some lemmas which are required to construct the Boehmian space $\mathcal{B}_C^\alpha = \mathcal{B}_C^\alpha(L^2_{\alpha}, (L^2_{\alpha}, \ast_{\alpha}^C, \ast_{\alpha}^C, \Delta(\alpha))$.

**Lemma 2.11.** If $f, g$ and $h \in L^1_{\alpha}$, then $(f \ast_{sc}^\alpha g) \ast_{sc}^\alpha h = f \ast_{sc}^\alpha (g \ast_{sc}^\alpha h)$.

**Proof.** For arbitrary $x \in [0, \infty)$,

$$
\frac{2\pi}{c^2} [f \ast_{sc}^\alpha (g \ast_{sc}^\alpha h)](x)
= \int_0^\infty f(y) e^{\beta(y^2 - xy)} \int_0^\infty g(z) e^{\beta z^2} |h(x - y + z) e^{\beta y^2 - y^2 z}| dy
+ h(||x - y|| - z) e^{\beta z^2} |(x + y + z) e^{\beta(x + y)^2 - (x + y)^2 z}| dy
- \int_0^\infty f(y) e^{\beta(y^2 + xy)} \int_0^\infty g(z) e^{\beta z^2} |h(x + y + z) e^{\beta(x + y)^2 - (x + y)^2 z}| dy
+ h(||x + y - z|| - z) e^{\beta(x + y)^2 - (x + y)^2 z} dy
= \int_0^\infty f(y) e^{\beta(y^2 - xy)} \int_0^\infty g(z) e^{\beta z^2} |h(x - y + z) e^{\beta(x - y)^2 - (x - y)^2 z}| dz
+ h(||x - y - z|| - z) e^{\beta(x - y)^2 - (x - y)^2 z} dz)
- \int_0^\infty f(y) e^{\beta(y^2 + xy)} \int_0^\infty g(z) e^{\beta z^2} |h(x + y + z) e^{\beta(x + y)^2 - (x + y)^2 z}| dz
+ h(||x + y - z|| - z) e^{\beta(x + y)^2 - (x + y)^2 z} dz
= \int_0^\infty f(y) e^{\beta(y^2 - xy)} \int_0^\infty g(z) e^{\beta z^2} h(||x - y|| - z) e^{\beta x^2 - x^2 z} dz
+ g(z) e^{\beta z^2} h(||x - y|| - z) e^{\beta x^2 - x^2 z} dy
- \int_0^\infty f(y) e^{\beta(y^2 + xy)} \int_0^\infty g(z) e^{\beta z^2} h(x + y + z) e^{\beta(x + y)^2 - (x + y)^2 z} dz
+ g(z) e^{\beta z^2} h(||x + y|| - z) e^{\beta(x + y)^2 - (x + y)^2 z} dz
= \int_0^\infty f(y) e^{\beta y^2} \int_0^\infty g(u + y) e^{\beta u(y + y^2)} e^{\beta x^2 u} h(x + u) du
+ g(||u - y||) e^{\beta u(y - y^2)} e^{\beta x^2 u} h(||x - u||) du}
Proof.\[ \int_0^\infty f(y)e^{\beta y^2} \int_0^\infty g(|y-u|e^{\beta(u-y)})e^{\beta ux}h(x+u)du \\
+ \int_0^\infty g(u+y)e^{\beta u+y}e^{-\beta ux}h(|x-u|)du dy \]

\[ = \int_0^\infty \int_0^\infty f(y)e^{\beta y^2}e^{\beta u}g(|u-y|)e^{-\beta uy} - g(u+y)e^{\beta uy}du dy \\
\times \{h(|x-u|)e^{-\beta ux} - h(x+u)e^{\beta xu}\}dy \\
= \int_0^\infty e^{\beta u^2} \{h(|x-u|)e^{-\beta ux} - h(x+u)e^{\beta xu}\}du \\
\times e^{\beta u} \{g(|u-y|)e^{-\beta uy} - g(u+y)e^{\beta uy}\}dy \\
= \frac{2\pi}{c^2}(f \ast_{sc} g) \ast_{sc} h(x).\]

Hence the theorem follows. \[\square\]

Remark 2.12. For \( f \in L^2_+ \), \( g \in L^1_+ \) and \( x \in [0, \infty) \), we have \((f \ast_{sc} g)(x) = (f \ast_{sc} g)(x) - c_\alpha \sqrt{2 \pi} \int_0^\infty f(u-x)g(u)e^{\beta(u-x)^2}du.\]

Lemma 2.13. For \( f \in L^2_+ \) and \( g \in L^1_+ \), \( \|f \ast_{sc} g\|_2 \leq 2|c_\alpha|\sqrt{2 \pi}\|f\|_2\|g\|_1.\)

Proof. By previous remark, we have \( \|f \ast_{sc} g\|_2 \leq \|f \ast_{sc} g\|_2 + |c_\alpha|\sqrt{2 \pi} \left( \int_0^\infty \left| \int_0^\infty f(u-x)g(u)e^{\beta(u-x)^2}du \right|^2 dx \right)^{\frac{1}{2}}.\)

Using Jensen’s inequality and Fubini’s theorem, we obtain that \( \int_0^\infty \int_0^\infty f(u-x)g(u)e^{\beta(u-x)^2}du \right| dx \leq \|g\|_1 \int_0^\infty \int_0^u |f(u-x)|^2g(u)|du| \)

\[ \leq \|g\|_1 \int_0^\infty |g(u)| \int_0^u |f(u-x)|^2dx du \leq \|g\|_1^2 \|f\|_1^2.\]

Therefore, using Lemma 2.3 and the above estimate, we obtain that \( \|f \ast_{sc} g\|_2 \leq 2|c_\alpha|\sqrt{2 \pi}\|f\|_2\|g\|_1 \), and hence \( f \ast_{sc} g \in L^2_+.\) \[\square\]

Lemma 2.14. If \( f \in C_\infty([0, \infty)) \) and \( (\delta_n) \in \Delta(\alpha) \), then \( f \ast_{sc} \delta_n \to f \) as \( n \to \infty \) in \( L^2_+.\)

Proof. In view of Remark 2.12 and Lemma 2.9, we have \( f \ast_{sc} \delta_n = (f \ast_{sc} \delta_n - f) + c_\alpha \sqrt{2 \pi} \int_0^\infty f(u-x)\delta_n(u)e^{\beta(u-x)^2}du.\)
Lemma 2.17. If \( n \to \infty \) as \( n \to \infty \). Therefore, to conclude this proof, we shall show that \( \int_x^\infty (u-x)\delta_n(u)e^{\beta(u-x)}du \to 0 \) in \( L^2_+ \), as \( n \to \infty \). Let \( \epsilon > 0 \) be given. We choose \( N \in \mathbb{N} \) such that \( \text{supp} \delta_n \subset [0, \epsilon), \forall n \geq N \). For any \( n \geq N \), we have

\[
\int_0^\infty \left| \int_x^\infty (u-x)\delta_n(u)e^{\beta(u-x)}du \right|^2 \, dx \\
\leq \int_0^\infty \left( \int_x^\infty |(u-x)|\delta_n(u)du \right)^2 \, dx \\
\leq \int_0^\infty M \int_x^\infty |(u-x)|^2 \delta_n(u)du \, dx, \text{ (by Jensen’s inequality)}
\]

(Here \( M > 0 \) is as in the property \((\Delta 2)\) of \((\delta_n)\))

\[
\leq M \int_0^\infty |\delta_n(u)| \int_0^u |(u-x)|^2 \, dx \, du, \quad \text{(by Fubini’s theorem)}
\]

\[
\leq M^2 \|f\|_{L^2}^2 \epsilon, \quad \text{where} \quad \|f\|_{L^2} = \sup_{t \geq 0} |f(t)|.
\]

Since \( \epsilon > 0 \) is arbitrary, the proof follows. \( \square \)

Lemma 2.15. If \( f \in L^2_+ \) and \((\delta_n) \in \Delta(\alpha)\), then \( f \ast\ast \delta_n \to f \) as \( n \to \infty \) in \( L^2_+ \).

Proof. Let \( f \in L^2_+ \) and \((\delta_n) \in \Delta(\alpha)\). For \( \epsilon > 0 \), choose \( g \in C_c([0, \infty)) \) such that \( \|f-g\|_2 < \epsilon \). By Lemma 2.14, there is a positive integer \( N \) with \( \|g \ast\ast \delta_n - g\|_2 < \epsilon \) for all \( n \geq N \). For any \( n \geq N \), we get that

\[
\|f \ast\ast \delta_n - f\|_2 \\
\leq |c_a| \sqrt{\frac{2}{\pi}} \|f - g\|_2 \|\delta_n\|_1 + \|g \ast\ast \delta_n - g\|_2 + \|g - f\|_2, \quad \text{(by Lemma 13)}
\]

\[
\leq M |c_a| \sqrt{\frac{2}{\pi}} \|f - g\|_2 + \|g \ast\ast \delta_n - g\|_2 + \|g - f\|_2, \quad \text{(by property \((\Delta 2)\) of \((\delta_n)\))}
\]

\[
< \epsilon M |c_a| \sqrt{\frac{2}{\pi}} + \epsilon + \epsilon = \epsilon \left( M |c_a| \sqrt{\frac{2}{\pi}} + 2 \right).
\]

Since \( \epsilon > 0 \) is arbitrary, it follows that \( f \ast\ast \delta_n \to f \) as \( n \to \infty \) in \( L^2_+ \). \( \square \)

Theorem 2.16. If \( f_n \to f \) as \( n \to \infty \) in \( L^2_+ \) and \((\delta_n) \in \Delta(\alpha)\), then \( f_n \ast\ast \delta_n \to f \) as \( n \to \infty \) in \( L^2_+ \).

Proof. As a consequence of Lemma 2.15 and Lemma 2.13, it follows that

\[
\|f_n \ast\ast \delta_n - f\|_2 \leq M |c_a| \sqrt{\frac{2}{\pi}} \|f_n - f\|_2 + \|f \ast\ast \delta_n - f\|_2 \to 0
\]

as \( n \to \infty \), which completes the proof. \( \square \)

Lemma 2.17. If \( f \in L^2_+ \) and if \( g, h \in L^1_+ \), then \( f \ast\ast (g \ast\ast h) = (f \ast\ast g) \ast\ast h \).
Proof. It follows immediately, by using the same technique applied in the proof of Lemma 2.5. □

Now, let $\mathcal{B}_3^n = \mathcal{B}(L_2^+, (L_1^+, \ast_0^p), \ast_{sc}^n, \Delta(\alpha))$ and denote a typical element of $\mathcal{B}_3^n$ by $U = ([f_n]/(\delta_n))$.

3. Fractional Fourier cosine transform on Boehmians

In this section, first we extend the FRFT as a map from $\mathcal{B}_3^n$ onto the Bohemian space $\mathcal{B}_3^n = \mathcal{B}(L_2^+, (C_0^+ \cap L_1^+, \cdot), \Delta_2^n)$, where $C_0^+$ is the Banach space of complex-valued continuous functions on $[0, \infty)$ vanishing at infinity, with the norm $\|\psi\|_\infty = \sup_{x \geq 0} |\psi(x)|$, ‘′ denotes the usual point-wise multiplication of functions and $\Delta_2^n = \left\{ \langle e^{-i\alpha u^2} F_0^\alpha(\phi_n) \rangle : (\phi_n) \in \Delta(\alpha) \right\}$.

Lemma 3.1. Let $f \in L_2^+$ and $\psi \in C_0^+$. Then $f \cdot \psi \in L_2^+$ and $f_n \cdot \psi \to f \cdot \psi$ as $n \to \infty$ in $L_2^+$, whenever $f_n \to f$ as $n \to \infty$ in $L_2^+$.

Proof. Since $\|f \cdot \psi\|_2 \leq \|f\|_2\|\psi\|_\infty$, the proof of this lemma follows immediately. □

Lemma 3.2. If $(\delta_n) \in \Delta(\alpha)$, then $e^{-i\alpha u^2} F_0^\alpha(\delta_n) \to 1$ as $n \to \infty$ uniformly on each compact subset of $[0, \infty)$.

Proof. Let $K$ be a compact subset of $[0, \infty)$ and let $\epsilon > 0$ be given. Choose a positive integer $N$ such that $\text{supp} \delta_n \subset [0, \epsilon)$ for all $n \geq N$. Then for $u \in K$ and $n \geq N$, we have

\[
|e^{-i\alpha u^2} [F_0^\alpha(\delta_n)](u) - 1| \\
\leq |c_n| \sqrt{\frac{2}{\pi}} \int_0^\epsilon |x| \cos(b_nux) \, dx - 1, \forall n \geq N \\
\leq |c_n| \sqrt{\frac{2}{\pi}} \int_0^\epsilon |x| |b_nux| |\sin z| \, dx, \\
\text{by mean-value theorem, such a } z \text{ exists in } (0, |b_nux|) \\
\leq \epsilon M |b_n c_n| \sqrt{\frac{2}{\pi}} \sup_{t \in K} |t|.
\]

Since $\epsilon > 0$ is arbitrary, the proof follows. □

Lemma 3.3. If $f \in L_2^+$ and $(\delta_n) \in \Delta(\alpha)$, then $f \cdot e^{-i\alpha u^2} F_0^\alpha(\delta_n) \to f$ as $n \to \infty$ in $L_2^+$.

Proof. Let $\epsilon > 0$ be arbitrary. Since $C_0([0, \infty))$ is dense in $L_2^+$, choose $g \in C_0([0, \infty))$ such that $\|f - g\|_2 < \frac{\epsilon}{2}$. By the property $(\Delta 2)$ of $(\delta_n)$, we get that $\|F_0^\alpha(\delta_n)(u)\| = |e^{-i\alpha u^2} [F_0^\alpha(\delta_n)](u)| \leq \int_0^\infty |\delta_n(x)| \, dx \leq M, \forall n \in N$ for some $M > 0$. If $K = \text{supp} g$, then $K$ is compact. Then, using Lemma 3.2, we find
Lemma 3.4. Let \( f_n \to f \) as \( n \to \infty \) in \( L^2_+ \) and \( (\delta_n) \in \Delta(\alpha) \). Then \( f_n \cdot e^{-i\alpha y^2}F_C^\alpha(\delta_n) \to f \) as \( n \to \infty \) in \( L^2_+ \).

Proof. This lemma is similar to that of Lemma 2.10, which can be obtained by using Lemma 3.1 and 3.3.

Lemma 3.5. If \( (\delta_n), (\phi_n) \in \Delta(\alpha) \), then \( (e^{-i\alpha y^2}F_C^\alpha(\delta_n) \cdot e^{-i\alpha y^2}F_C^\alpha(\phi_n)) \in \Delta_C^\alpha \).

Proof. It follows immediately from Lemma 2.8 and Theorem 2.6.

Thus the Bohemian space \( \hat{B}_C^\alpha \) is constructed and we denote a typical element of \( \hat{B}_C^\alpha \) by \( \mathcal{V} = [(g_n)/(e^{-i\alpha y^2}F_C^\alpha(\delta_n))] \).

Definition. We define the extended FRCT \( \mathcal{F}_C^\alpha : \hat{B}_C^\alpha \to \hat{B}_C^\alpha \) by \( \mathcal{F}_C^\alpha (X) = [(F_C^\alpha f_n)/(e^{-i\alpha y^2}F_C^\alpha(\delta_n))] \), where \( X = [(f_n), (\delta_n)] \in \mathcal{P}_C^\alpha \).

Suppose \( [(f_n), (\delta_n)] \in \mathcal{P}_C^\alpha \), then for all \( n, m \in \mathbb{N} \), we have \( f_n *_c \delta_m = f_m *_c \delta_n \), which implies that \( F_C^\alpha(f_n *_c \delta_m)(u) = F_C^\alpha(f_m *_c \delta_n)(u) \). Applying the convolution theorem, we get that

\[
e^{-i\alpha u^2}[F_C^\alpha f_n](u) \cdot [F_C^\alpha \delta_m](u) = e^{-i\alpha u^2}[F_C^\alpha f_m](u) \cdot [F_C^\alpha \delta_n](u)
\]

and hence \( [F_C^\alpha f_n](u) \cdot e^{-i\alpha u^2}[F_C^\alpha \delta_m](u) = [F_C^\alpha f_m](u) \cdot e^{-i\alpha u^2}[F_C^\alpha \delta_n](u) \). By a similar argument, it is easy to prove that \( \mathcal{F}_C^\alpha (X) \) is independent of the choice of the representative of \( X \). Thus, FRCT is well-defined.

Theorem 3.6. The FRCT \( \mathcal{F}_C^\alpha : \hat{B}_C^\alpha \to \hat{B}_C^\alpha \) is consistent with \( F_C^\alpha : L^2_+ \to L^2_+ \).

Proof. If \( f \in L^2_+ \) then \( \mathcal{F} = [(f *_c \delta_n), (\delta_n)] \) is the Bohemian representing \( f \) in \( \hat{B}_C^\alpha \). By definition, we have \( \mathcal{F}_C^\alpha (\mathcal{F}) = [(F_C^\alpha f *_c \delta_n), (e^{-i\alpha u^2}F_C^\alpha \delta_n)] = [(F_C^\alpha f)(e^{-i\alpha u^2}F_C^\alpha \delta_n)/(e^{-i\alpha u^2}F_C^\alpha \delta_n)] \), which is the Bohemian representing \( F_C^\alpha f \) in \( \hat{B}_C^\alpha \).

Theorem 3.7. The FRCT \( \mathcal{F}_C^\alpha : \hat{B}_C^\alpha \to \hat{B}_C^\alpha \) is a bijective linear map.
Proof. Let \( X = [(f_n), (\delta_n)] \), and \( Y = [(g_n), (\phi_n)] \) be such that \( \mathcal{F}_C(X) = \mathcal{F}_C(Y) \). Therefore, it follows that

\[
[(F^0_C f_n) / (e^{-i\alpha u^2} F^0_C \delta_n)] = [(F^0_C g_n) / (e^{-i\alpha u^2} F^0_C \phi_n)]
\]

and hence for any \( n, m \in \mathbb{N} \),

\[ e^{-i\alpha u^2} [F^0_C f_n](u) \cdot [F^0_C \phi_m](u) = e^{-i\alpha u^2} [F^0_C g_m](u) \cdot [F^0_C \delta_n](u). \]

Using the convolution Theorem 2.7, we have \( F^0_C (f_n * e \phi_m) = F^0_C (g_m * e \delta_n) \), \( \forall m, n \in \mathbb{N} \), which implies that \( f_n * e \phi_m = g_m * e \delta_n \), \( \forall n, m \in \mathbb{N} \) and hence \( X = Y \).

Let \( X = [(g_n) / (e^{-i\alpha u^2} F^0_C \delta_n)] \in \mathcal{S}_C \). Since \( F^0_C : L^2_+ \rightarrow L^1_+ \) is onto, choose \( f_n \in L^2_+ \) such that \( g_n = F^0_C f_n \) for each \( n \in \mathbb{N} \). For any \( n, m \in \mathbb{N} \), we get that \( [F^0_C f_n](u) \cdot e^{-i\alpha u^2} [F^0_C \delta_n](u) = [F^0_C f_m](u) \cdot e^{-i\alpha u^2} [F^0_C \delta_n](u) \). Therefore, by convolution theorem, we have \( [F^0_C (f_n * e \delta_n)](u) = [F^0_C (f_m * e \delta_n)](u) \), and hence \( f_n * e \delta_n = f_m * e \delta_n \). Thus \( X = [(f_n), (\delta_n)] \in \mathcal{F}_C(X) \) and \( \mathcal{F}_C(X) = [(F^0_C f_n) / (e^{-i\alpha u^2} F^0_C \delta_n)] = X \), which implies that \( \mathcal{F}_C(X) \) is a surjective map.

The linearity of \( \mathcal{F}_C \) follows from the linearity of \( F^0_C \) and Theorem 2.7. \( \square \)

**Theorem 3.8** (Convolution theorem for FRFCT on Boehmians). If \( X \in \mathcal{S}_C \) and \( h \in L^1_+ \), then \( \mathcal{F}_C(X * e \hbar) = \mathcal{F}_C(X) \cdot e^{-i\alpha u^2} F^0_C \hbar \).

Proof. Let \( X = [(f_n), (\delta_n)] \) and \( h \in L^1_+ \). By using Theorem 2.7,

\[
\mathcal{F}_C(X * e \hbar) = \mathcal{F}_C([(f_n * e \hbar), (\delta_n)] = [F^0_C (f_n * e \hbar) / (e^{-i\alpha u^2} F^0_C \delta_n)]
\]

\[
= [(e^{-i\alpha u^2} F^0_C f_n \cdot e \hbar) / (e^{-i\alpha u^2} F^0_C \delta_n)]
\]

\[
= [(F^0_C f_n) / (e^{-i\alpha u^2} F^0_C \delta_n)] \cdot e^{-i\alpha u^2} F^0_C \hbar
\]

\[
= \mathcal{F}_C(X) \cdot e^{-i\alpha u^2} F^0_C \hbar.
\]

\( \square \)

**Theorem 3.9.** The FRFCT on \( \mathcal{S}_C \) is continuous with respect to \( \delta \)-convergence and \( \Delta \)-convergence.

Proof. Let \( X_n \xrightarrow{\delta} X \) as \( n \to \infty \) in \( \mathcal{S}_C \). By the definition of \( \delta \)-convergence, \( X_n * e \delta_k, X_n * e \delta_k \in L^2_+ \) and \( X_n * e \delta_k \to X * e \delta_k \) as \( n \to \infty \) in \( L^2_+ \) for each fixed \( k \in \mathbb{N} \) and for some \( (\delta_n) \in \Delta(\alpha) \).

In view of Theorems 3.8 and 3.6, we get that

\[
\mathcal{F}_C(X_n) \cdot e^{-i\alpha u^2} F^0_C \delta_k = \mathcal{F}_C(X_n * e \delta_k) = F^0_C(X_n * e \delta_k) \in L^2_+
\]

and

\[
\mathcal{F}_C(X) \cdot e^{-i\alpha u^2} F^0_C \delta_k = \mathcal{F}_C(X * e \delta_k) = F^0_C(X * e \delta_k) \in L^2_+
\]

for all \( n, k \in \mathbb{N} \). Further, using the continuity of the FRFCT, we obtain

\[
\mathcal{F}_C(X_n) \cdot e^{-i\alpha u^2} F^0_C \delta_k = F^0_C(X_n * e \delta_k) \to F^0_C(X * e \delta_k) = \mathcal{F}_C(X) \cdot e^{-i\alpha u^2} F^0_C \delta_k
\]

as \( n \to \infty \) in \( L^2_+ \), for each fixed \( k \in \mathbb{N} \). Hence \( \mathcal{F}_C(X_n) \xrightarrow{\delta} \mathcal{F}_C(X) \) as \( n \to \infty \).
Let $X_n \overset{\Delta}{\to} X$ as $n \to \infty$. Then there exists $(\delta_n) \in \Delta(\alpha)$ such that

$$(X_n - X) \ast_n \delta_n \in L^2_+, \forall n \in \mathbb{N} \text{ and } (X_n - X) \ast_c \delta_n \to 0 \text{ as } n \to \infty \text{ in } L^2_+.$$ 

Now using Theorems 3.7 and 3.6, we have for each $n \in \mathbb{N},$

$$\begin{align*}
[\mathcal{F}_S^\alpha(X_n) - \mathcal{F}_S^\alpha(X)] \cdot e^{-ia_nu^2F_C^\alpha\delta_n} &= \mathcal{F}_S^\alpha((X_n - X) \ast_c \delta_n) \\
&= F_C^\alpha((X_n - X) \ast_c \delta_n),
\end{align*}$$

which belongs to $L^2_+$ and the continuity of $F_C^\alpha$ on $L^2_+$ yields that

$$[\mathcal{F}_S^\alpha(X_n) - \mathcal{F}_S^\alpha(X)] \cdot e^{-ia_nu^2F_C^\alpha\delta_n} = F_C^\alpha((X_n - X) \ast_c \delta_n) \to 0$$

as $n \to \infty$ in $L^2_+$. This shows that $\mathcal{F}_S^\alpha(X_n) \overset{\Delta}{\to} \mathcal{F}_S^\alpha(X)$ as $n \to \infty$. \hfill \Box

4. Fractional Fourier sine transform on Boehmians

In this section, we extend the FRST as a mapping from the Boehmian space $\mathcal{B}_C^\alpha$ onto the Boehmian space $\mathcal{B}_C$. 

**Definition.** For each $U = \{(f_n), (\delta_n)\} \in \mathcal{B}_C^\alpha$, we define the extended FRST of $U$ by $\mathcal{F}_S^\alpha(U) = [(F_S^\alpha f_n)/(e^{-ia_nu^2F_C^\alpha\delta_n})]$. 

If $\{(f_n), (\delta_n)\}, \{(g_n), (\epsilon_n)\}$ in $\mathcal{B}_C^\alpha$ such that $\{(f_n), (\delta_n)\} = \{(g_n), (\epsilon_n)\}$, then we have $f_n \ast_sc \epsilon_m = g_m \ast_{sc} \delta_n, \forall m, n \in \mathbb{N}$. As in the case of FRCT on Boehmians, applying Theorem 2.7(ii), we get that $F_S^\alpha(f_n)e^{-ia_nu^2F_C^\alpha\epsilon_m} = F_S^\alpha(g_m)e^{-ia_nu^2F_C^\alpha\delta_n}, \forall m, n \in \mathbb{N}$. This implies that the images of $\{(f_n), (\delta_n)\}$ and $\{(g_n), (\epsilon_n)\}$ are same in $\mathcal{B}_C$. Thus, $\mathcal{F}_S^\alpha : \mathcal{B}_C^\alpha \to \mathcal{B}_C$ is well-defined.

**Theorem 4.1.** The FRST $\mathcal{F}_S^\alpha : \mathcal{B}_C^\alpha \to \mathcal{B}_C$ is consistent with $F_S^\alpha : L^2_+ \to L^2_+$. 

**Proof.** Let $f \in L^2_+$ be arbitrary. Then, the Boehmian representing $f$ is of the form $\{(f_n), (\delta_n)\}$, where $(\delta_n) \in \Delta(\alpha)$ is arbitrary. Then, Theorem 2.7(ii) implies that $\mathcal{F}_S^\alpha(\{(f_n) \ast_sc \delta_n, (\delta_n)\}) = [(F_S^\alpha (f \ast_sc \delta_n))/(e^{-ia_nu^2F_C^\alpha\delta_n})] = [(F_S^\alpha (f)e^{-ia_nu^2F_C^\alpha\delta_n}]/(e^{-ia_nu^2F_C^\alpha\epsilon_m}]$, which is the representation of $\mathcal{F}_S^\alpha(f)$ in $\mathcal{B}_C$. Hence, $\mathcal{F}_S^\alpha$ is consistent with $F_S^\alpha$. \hfill \Box

**Theorem 4.2.** The FRST $\mathcal{F}_S^\alpha : \mathcal{B}_C^\alpha \to \mathcal{B}_C$ is a bijective linear map. 

**Proof.** The linearity $\mathcal{F}_S^\alpha$ is a direct consequence of linearity of the $\mathcal{F}_S^\alpha$ on $L^2_+$ and the convolution theorem (Theorem 2.7(ii)). To prove the injectivity, let $\{(f_n), (\delta_n)\}$ and $\{(g_n), (\epsilon_n)\}$ in $\mathcal{B}_C^\alpha$ be such that

$$\{(F_S^\alpha f_n)/(e^{-ia_nu^2F_C^\alpha\delta_n})\} = [(F_S^\alpha g_n)/(e^{-ia_nu^2F_C^\alpha\epsilon_n})].$$

Then, it follows that $F_S^\alpha(f_n)e^{-ia_nu^2F_C^\alpha\epsilon_m} = F_S^\alpha(g_m)e^{-ia_nu^2F_C^\alpha\delta_n}, \forall m, n \in \mathbb{N}$. Applying Theorem 2.7(ii) and using the invertibility of $F_S^\alpha$ on both sides, we obtain that $f_n \ast_sc \epsilon_m = g_m \ast_{sc} \delta_n, \forall m, n \in \mathbb{N}$, which implies that $\{(f_n), (\delta_n)\} = \{(g_n), (\epsilon_n)\}$. Therefore, $\mathcal{F}_S^\alpha$ is injective on $\mathcal{B}_C^\alpha$. To prove that $\mathcal{F}_S^\alpha : \mathcal{B}_C^\alpha \to \mathcal{B}_C$ is surjective, let $\{(g_n)/(e^{-ia_nu^2F_C^\alpha\delta_n})\} \in \mathcal{B}_C$ be arbitrary. If we choose
f_n \in L^2_\uparrow$ such that $F_C^\alpha(f_n) = g_n$ for all $n \in \mathbb{N}$, then adopting the proof of Theorem 3.7, one can show that $[[f_n]/(\delta_n)] \in \mathcal{B}_C^\alpha$ and that $\mathcal{F}_C^\alpha([[f_n]/(\delta_n)]) = [(g_n)/e^{-iu_\alpha u^2} F_C^\alpha(\delta_n))]$ in $\mathcal{B}_C^\alpha$. \hfill \Box

As the proofs of the following properties of $\mathcal{F}_C^\alpha$ are much similar to that of $\mathcal{F}_C^\alpha$ in Section 3, we prefer to leave the details.

**Theorem 4.3** (Convolution theorem for extended FRFST). If $\mathcal{U} \in \mathcal{B}_C^\alpha$ and $h \in L^1_\uparrow$, then $\mathcal{F}_C^\alpha(\mathcal{U} *_{\mathcal{C}} h) = \mathcal{F}_C^\alpha(\mathcal{U}) \cdot e^{-iu_\alpha u^2} \mathcal{F}_C^\alpha h$.

**Theorem 4.4.** The FRFST on $\mathcal{B}_C^\alpha$ is continuous with respect to $\delta$-convergence and $\Delta$-convergence.

5. Concluding remarks

It is interesting to note that the Fourier sine and cosine transforms on Boehmians discussed in [21, Section 3] becomes a particular case of this paper, when $\alpha = \frac{\pi}{2}$. This is a first benefit of this paper, in pure mathematical point of view. Furthermore, we see that the every distribution $\Lambda$ with compact support in $[0, \infty)$, that since $\Lambda \mapsto [[(\Lambda \ast_\mathcal{C} \varphi_n)/(\varphi_n)]$, where $(\varphi_n) \in \Delta(\alpha)$ is such that $\varphi_n$ is infinitely smooth function on $\mathbb{R}$ having support in $[0, \infty)$, $\forall n \in \mathbb{N}$ and

$$(\Lambda \ast_\mathcal{C} \varphi)(x) = \frac{c_\alpha}{\sqrt{2\pi}} \langle \Lambda(y), [e^{\beta(y^2+xy)}\phi(x+y) + e^{\beta(y^2-xy)}\phi(|x-y|)] \rangle, \forall x \geq 0.$$ 

Therefore, the extension of the fractional Fourier cosine transform discussed in this paper properly generalizes the fractional Fourier cosine transform on $L^2_\uparrow$, which is a second benefit of this paper, in view of pure mathematics.

In signal processing, the Dirac’s delta function $\delta$ and convolution play a vital role, especially to find the system waiting function of a linear time-invariant system (LTI system) and the output of the LTI system, respectively. A filter in signal processing is an example of an LTI system. So, if an integral transform is used in signal processing, it is necessary to study the image of $\delta$ under the transform and the convolution theorem of the transform.

In particular, it is well known that the fractional Fourier sine and cosine transforms are having many applications in signal processing. We point out that since $\delta$ is identified as the Boehmian $[(\delta \ast_\mathcal{C} \varphi_n)/(\varphi_n)]$, using the extended fractional Fourier cosine transform on $\mathcal{B}_C^\alpha$, we can find the image of the same. Since, every Boehmian $[[f_n]/(\delta_n)]$ can be approximated by the sequence of functions $(f_n)$ (see [6]), the image of any Boehmian in $\mathcal{B}_C^\alpha$ under $\mathcal{F}_C^\alpha$ can be approximated by sequence of functions. In particular, we can find the fractional Fourier cosine transform of $\delta$, approximately by sequence of functions.

In addition to this, since convolution theorems of these transforms are obtained as products, one can find the fractional cosine (respectively, sine) convolution of $f$ and $g$ easily by applying the inverse fractional Fourier cosine (respectively, sine) transform on $e^{-iu_\alpha u^2} F_C(f) F_C(g)$ (respectively, $e^{-iu_\alpha u^2} F_C(f) F_S(g)$).
Since all the properties of the fractional Fourier cosine and sine transforms on function space are extended to the context of Boehmian space, we can freely apply the extended fractional Fourier cosine and sine transform on any mathematical expression which involves both functions and generalized functions.

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