ADDITIVE $\rho$-FUNCTIONAL EQUATIONS IN $\beta$-HOMOGENEOUS $F$-SPACES

EunHwa Shim

Abstract. In this paper, we solve the additive $\rho$-functional equations

\begin{equation}
(0.1) \quad f(x+y) + f(x-y) - 2f(x) = \rho \left( 2f \left( \frac{x+y}{2} \right) + f(x-y) - 2f(x) \right),
\end{equation}

and

\begin{equation}
(0.2) \quad 2f \left( \frac{x+y}{2} \right) + f(x-y) - 2f(x) = \rho (f(x+y) + f(x-y) - 2f(x)),
\end{equation}

where $\rho$ is a fixed (complex) number with $\rho \neq 1$.

Using the direct method, we prove the Hyers-Ulam stability of the additive $\rho$-functional equations (0.1) and (0.2) in $\beta$-homogeneous (complex) $F$-spaces.

1. Introduction and Preliminaries


The functional equation $f(x+y) = f(x) + f(y)$ is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers’ Theorem was generalized by Aoki [2] for additive mappings and by Rassias [14] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruţa [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach. The stability of quadratic functional equation was proved by Skof [22] for mappings $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group. The stability problems of various functional
equations have been extensively investigated by a number of authors (see [1, 3, 4, 6, 9, 10, 11, 12, 13, 15, 17, 18, 19, 20, 21, 24, 25]).

**Definition 1.1.** Let $X$ be a linear space. A nonnegative valued function $\| \cdot \|$ is an $F$-norm if it satisfies the following conditions:

1. \((FN_1)\) $\| x \| = 0$ if and only if $x = 0$;
2. \((FN_2)\) $\| \lambda x \| = \| x \|$ for all $x \in X$ and all $\lambda$ with $|\lambda| = 1$;
3. \((FN_3)\) $\| x + y \| \leq \| x \| + \| y \|$ for all $x, y \in X$;
4. \((FN_4)\) $\| \lambda_n x \| \to 0$ provided $\lambda_n \to 0$;
5. \((FN_5)\) $\| \lambda x_n \| \to 0$ provided $\| x_n \| \to 0$.

Then $(X, \| \cdot \|)$ is called an $F^*$-space.

A sequence $\{x_n\}$ is called a Cauchy sequence if, for a given $\epsilon > 0$, there is a natural number $N$ such that $\| x_n - x_m \| \leq \epsilon$ for all $n, m \geq N$. A sequence $\{x_n\}$ is called a convergent sequence if, for a given $\epsilon > 0$, there are a natural number $N$ and $x_0 \in X$ such that $\| x_n - x_0 \| \leq \epsilon$ for all $n \geq N$. If every Cauchy sequence converges, then the space is called *complete*. An $F$-space is a complete $F^*$-space.

An $F$-norm is called $\beta$-homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \| x \|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [16]).

In Section 2, we solve the additive $\rho$-functional equation (0.1) and prove the Hyers-Ulam stability of the additive $\rho$-functional equation (0.1) in $\beta_2$-homogeneous (complex) $F$-spaces.

In Section 3, we solve the additive $\rho$-functional equation (0.2) and prove the Hyers-Ulam stability of the additive $\rho$-functional equation (0.2) in $\beta_2$-homogeneous (complex) $F$-spaces.

Throughout this paper, let $\beta_1, \beta_2$ be positive real numbers with $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Assume that $X$ is a $\beta_1$-homogeneous (complex) normed space with norm $\| \cdot \|$ and that $Y$ is a $\beta_2$-homogeneous (complex) $F$-space with norm $\| \cdot \|$. Assume that $\rho$ is a (complex) number with $\rho \neq 1$.

**2. ADDITIVE $\rho$-FUNCTIONAL EQUATION (0.1) IN $\beta$-HOMOGENEOUS (COMPLEX) $F$-SPACES**

We solve and investigate the additive $\rho$-functional equation (0.1) in (complex) normed spaces.
Lemma 2.1. If a mapping \( f : X \to Y \) satisfies \( f(0) = 0 \) and
\[
(2.1) \quad f(x + y) + f(x - y) - 2f(x) = \rho \left( 2f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) \right)
\]
for all \( x, y \in X \), then \( f : X \to Y \) is additive.

Proof. Assume that \( f : X \to Y \) satisfies (2.1).

Letting \( y = x \) in (2.1), we get
\[
\| f(2x) - 2f(x) \| = 0
\]
and so
\[
f \left( \frac{x}{2} \right) = \frac{1}{2} f(x)
\]
for all \( x \in X \).

It follows from (2.1) and (2.2) that
\[
f(x + y) + f(x - y) - 2f(x) = \rho \left( 2f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) \right)
\]
and so \( f(x + y) + f(x - y) = 2f(x) \) for all \( x, y \in X \). It is easy to show that \( f \) is additive. \( \square \)

We prove the Hyers-Ulam stability of the additive \( \rho \)-functional equation (2.1) in \( \beta \)-homogeneous (complex) \( F \)-spaces.

Theorem 2.2. Let \( r > \frac{\beta_2}{\beta_1} \) and \( \theta \) be nonnegative real numbers and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and
\[
\| f(x + y) + f(x - y) - 2f(x) - \rho \left( 2f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) \right) \| \leq \theta (\| x \|^r + \| y \|^r)
\]
(2.3)
for all \( x, y \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that
\[
\| f(x) - A(x) \| \leq \frac{2\theta}{2^{\beta_1}r - 2^{\beta_2}} \| x \|^r
\]
(2.4)
for all \( x \in X \).

Proof. Letting \( y = x \) in (2.3), we get
\[
\| f(2x) - 2f(x) \| \leq 2\theta \| x \|^r
\]
(2.5)
for all \( x \in X \). So
\[
\| f(x) - 2f \left( \frac{x}{2} \right) \| \leq \frac{2}{2^{\beta_1}r} \theta \| x \|^r
\]
Let \( \beta \) be a mapping satisfying sequence \((2.6)\) that the sequence for all nonnegative integers \( k \) converges. So one can define the mapping \( A : X \to Y \) by

\[
A(x) := \lim_{k \to \infty} 2^k f \left( \frac{x}{2^k} \right)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.6), we get (2.4).

It follows from (2.3) that

\[
\left\| A(x + y) + A(x - y) - 2A(x) - \rho \left( 2A \left( \frac{x + y}{2} \right) + A(x - y) - 2A(x) \right) \right\| = \lim_{n \to \infty} 2^n \left( f \left( \frac{x + y}{2} \right) + f \left( \frac{x - y}{2} \right) - f \left( \frac{x}{2} \right)
- \rho \left( 2f \left( \frac{x + y}{2n+1} \right) + f \left( \frac{x - y}{2n} \right) - 2f \left( \frac{x}{2n} \right) \right) \right) \leq \lim_{n \to \infty} \frac{2^{\beta n}}{2^{\beta n}} \rho(\|x\|^r + \|y\|^r) = 0
\]

for all \( x, y \in X \). So

\[
A(x + y) + A(x - y) - 2A(x) = \rho \left( 2A \left( \frac{x + y}{2} \right) + A(x - y) - 2A(x) \right)
\]

for all \( x, y \in X \). By Lemma 2.1, the mapping \( A : X \to Y \) is additive.

Now, let \( T : X \to Y \) be another additive mapping satisfying (2.4). Then we have

\[
\|A(x) - T(x)\| = \left\| 2^q A \left( \frac{x}{2^q} \right) - 2^q T \left( \frac{x}{2^q} \right) \right\|
\leq \left\| 2^q A \left( \frac{x}{2^q} \right) - 2^q f \left( \frac{x}{2^q} \right) \right\| + \left\| 2^q T \left( \frac{x}{2^q} \right) - 2^q f \left( \frac{x}{2^q} \right) \right\|
\leq \frac{4\theta}{2^{\beta q}} 2^q \left( \frac{x}{2^q} \right),
\]

which tends to zero as \( q \to \infty \) for all \( x \in X \). So we can conclude that \( A(x) = T(x) \) for all \( x \in X \). This proves the uniqueness of \( A \), as desired.

**Theorem 2.3.** Let \( r < \frac{\beta}{\theta} \) and \( \theta \) be nonnegative real numbers and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and (2.3). Then there exists a unique additive
mapping \( A : X \to Y \) such that

\[
\| f(x) - A(x) \| \leq \frac{2\theta}{2^{\beta_2} - 2^{\beta_1}r} \| x \|^r
\]

for all \( x \in X \).

**Proof.** It follows from (2.5) that

\[
\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{2}{2^{\beta_2}} \theta \| x \|^r
\]

for all \( x \in X \). Hence

\[
\left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|
\]

\[
\leq \frac{2}{2^{\beta_2}} \sum_{j=l}^{m-1} \frac{2^{\beta_3} \theta}{2^{\beta_4}} \| x \|^r
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.8) that the sequence \( \{ \frac{1}{2^m} f(2^m x) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ \frac{1}{2^m} f(2^m x) \} \) converges. So one can define the mapping \( A : X \to Y \) by

\[
A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.8), we get (2.7).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

**3. Additive \( \rho \)-functional Equation (0.2) in \( \beta \)-homogeneous (Complex) \( F \)-spaces**

We solve and investigate the additive \( \rho \)-functional equation (0.2) in \( \beta \)-homogeneous (complex) normed spaces.

**Lemma 3.1.** If a mapping \( f : X \to Y \) satisfies

\[
2f\left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) = \rho(f(x + y) + f(x - y) - 2f(x))
\]

for all \( x, y \in X \), then \( f : X \to Y \) is additive.
Proof. Assume that \( f : X \to Y \) satisfies (3.1).

Letting \( x = y = 0 \) in (3.1), we get \( f(0) = 0 \).

Letting \( y = 0 \) in (3.1), we get \( \|2f \left( \frac{x}{2} \right) - f(x) \| \leq 0 \) and so

\[
2f \left( \frac{x}{2} \right) = f(x)
\]

(3.2)

for all \( x \in X \).

It follows from (3.1) and (3.2) that

\[
f(x + y) + f(x - y) - 2f(x) = 2f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) = \rho(f(x + y) + f(x - y) - 2f(x))
\]

and so \( f(x + y) + f(x - y) = 2f(x) \) for all \( x, y \in X \). It is easy to show that \( f \) is additive. \( \square \)

We prove the Hyers-Ulam stability of the additive \( \rho \)-functional equation (3.1) in \( \beta \)-homogeneous (complex) \( F \)-spaces.

**Theorem 3.2.** Let \( r > \frac{\beta_2}{\beta_1} \) and \( \theta \) be nonnegative real numbers and let \( f : X \to Y \) be a mapping satisfying \( f(0) = 0 \) and

\[
\left\|2f \left( \frac{x + y}{2} \right) + f(x - y) - 2f(x) - \rho(f(x + y) + f(x - y) - 2f(x))\right\| \leq \theta(\|x\|^r + \|y\|^r)
\]

(3.3)

for all \( x, y \in X \). Then there exists a unique additive mapping \( A : X \to Y \) such that

\[
\|f(x) - A(x)\| \leq \frac{2^{\beta_1} \theta}{2^{\beta_1} r - 2^{\beta_2}} \|x\|^r
\]

(3.4)

for all \( x \in X \).

**Proof.** Letting \( y = 0 \) in (3.3), we get

\[
\|f(x) - 2f \left( \frac{x}{2} \right)\| = \left\|2f \left( \frac{x}{2} \right) - f(x)\right\| \leq \theta \|x\|^r
\]

(3.5)

for all \( x \in X \). So

\[
\left\|2^j f \left( \frac{x}{2^j} \right) - 2^m f \left( \frac{x}{2^m} \right)\right\| \leq \sum_{j=1}^{m-1} \left\|2^j f \left( \frac{x}{2^j} \right) - 2^{j+1} f \left( \frac{x}{2^{j+1}} \right)\right\|
\]

(3.6)

\[
\leq \sum_{j=1}^{m-1} \frac{2^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r
\]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (3.6) that the sequence \( \{2^k f(\frac{x}{2^k})\} \) is Cauchy for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{2^k f(\frac{x}{2^k})\} \) converges. So one can define the mapping \( A : X \rightarrow Y \) by

\[
A(x) := \lim_{k \to \infty} 2^k f \left( \frac{x}{2^k} \right)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.6), we get (3.4).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)

**Theorem 3.3.** Let \( r < \frac{\beta_2}{\beta_1} \) and \( \theta \) be nonnegative real numbers and let \( f : X \rightarrow Y \) be an odd mapping satisfying (3.3). Then there exists a unique additive mapping \( A : X \rightarrow Y \) such that

\[
\|f(x) - A(x)\| \leq \frac{2^{3r} \theta}{2^{3r} - 2^{3r} \theta} \|x\|^r
\]

for all \( x \in X \).

**Proof.** It follows from (3.5) that

\[
\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{2^{3r} \theta}{2^{3r} - 2^{3r} \theta} \|x\|^r
\]

for all \( x \in X \). Hence

\[
\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|
\]

(3.8)

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (3.8) that the sequence \( \{\frac{1}{2^l} f(2^m x)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{\frac{1}{2^l} f(2^m x)\} \) converges. So one can define the mapping \( A : X \rightarrow Y \) by

\[
A(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]

for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (3.8), we get (3.7).

The rest of the proof is similar to the proof of Theorem 2.2. \( \square \)


Department of Mathematics, Hanyang University, Seoul 04763, Republic of Korea

Email address: stareun01@mate.com