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ON A TOTALLY UMBILIC HYPERSURFACE OF FIRST ORDER

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Abstract. In this paper, we define a totally umbilic hypersurface of first order and show that a totally umbilic hypersurface of first order in an Einstein manifold has a parallel second fundamental form. Furthermore we prove that a complete, simply connected and totally umbilic hypersurface of first order in a space of constant curvature is a Riemannian product of Einstein manifolds. Finally we show a proper example which is a totally umbilic hypersurface of first order but not a totally umbilic hypersurface.

1. Introduction

A totally umbilic hypersurface of a Riemannian manifold has received a great deal of attention and has been studied in considerable detail by many authors. For instance, a totally umbilic hypersurface in a space of constant curvature was investigated by Cheng and Nakagawa [5]; Hasanis [8]; Okumura [11]; Kim and Park [9]. Also a totally umbilic hypersurface in a conformally flat symmetric space (resp. a homogeneous space) was studied by Calvaruso, Kowalczyk and Van der Veken [3]; Van der Veken and Vrancken [14] (resp. Souam and Toubiana [12]; Van der Veken [13]). In this paper, as a natural generalization of the notion of a totally umbilic hypersurface, we introduce the notion of a totally umbilic hypersurface of first order, and investigate some properties of such a manifold in a space of constant curvature or in an Einstein manifold.

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2. Preliminaries

Let $(\bar{M}^{n+1}, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; y^\alpha\}$ and (M^n, g) a hypersurface of $(\bar{M}^{n+1}, \bar{g})$ covered by a system of coordinate neighborhoods $\{V; x^i\}$. Let $y^\alpha = y^\alpha(x^i)$ be the parametric representation of the hypersurface M^n in $\bar{M}^{n+1}$, where Greek indices take the values $1, 2, ..., n+1$ and Latin indices take the values $1, 2, ..., n$. Then we have

$$g_{ij} = \bar{g}_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}. \tag{1}$$

Here we adopt the Einstein convention, that is, when an index variable appears once in an upper and once in a lower position in a term, it implies summation of that term over all the values of the index. Let N^α be a local unit normal to (M^n, g). Then we have the relations

$$\bar{g}_{\alpha\beta} N^\alpha \frac{\partial y^\beta}{\partial x^j} = 0,$$

$$\bar{g}_{\alpha\beta} N^\alpha N^\beta = 1,$$

$$\bar{g}^{\alpha\beta} = g^{ij} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} + N^\alpha N^\beta. \tag{2}$$

We also have the following equations of Gauss and of Weingarten:

$$\left(\frac{\partial y^\alpha}{\partial x^i}\right)_{;j} = \omega_{ij} N^\alpha,$$

$$N^\alpha_{;i} = -\omega_{ij} g^{jk} \left(\frac{\partial y^\alpha}{\partial x^k}\right), \tag{3}$$

where ω_{ij} is the second fundamental form of (M^n, g) and semicolon ";" indicates covariant differentiation. The structure equations of Gauss and Codazzi for a hypersurface (M^n, g) of $(\bar{M}^{n+1}, \bar{g})$ can be respectively written as

$$R_{ijkl} = \bar{R}_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + \omega_{il}\omega_{jk} - \omega_{ik}\omega_{jl}, \tag{4}$$

$$\bar{R}_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta = \omega_{jk;i} - \omega_{ik;j}, \tag{5}$$

where R_{ijkl} and $\bar{R}_{\alpha\beta\gamma\delta}$ are the curvature tensors of (M^n, g) and $(\bar{M}^{n+1}, \bar{g})$, respectively. The hypersurface (M^n, g) is said to be a totally umbilic hypersurface of $(\bar{M}^{n+1}, \bar{g})$ [4] if its second fundamental form ω_{ij} satisfies

$$\omega_{ij} = H g_{ij},$$

$$(\frac{\partial y^\alpha}{\partial x^i})_{;j} = g_{ij} H N^\alpha, \tag{6}$$

where H denotes the mean curvature of (M^n, g) defined by $H = \frac{1}{n} g^{ij}\omega_{ij}$. In particular, if H=0, then the totally umbilic hypersurface (M^n, g) is called a totally geodesic hypersurface of $(\bar{M}^{n+1}, \bar{g})$ [4]. The equations of Weingarten, Gauss and Codazzi for a totally umbilic hypersurface (M^n, g) of $(\bar{M}^{n+1}, \bar{g})$ are respectively obtained as

$$N^\alpha_{;i} = -H \frac{\partial y^\alpha}{\partial x^i}, \tag{7}$$

$$R_{ijkl} = \bar{R}_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} + H^2 (g_{il} g_{jk} - g_{ik} g_{jl}), \tag{8}$$

$$\bar{R}_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} N^\delta = H_{;i} g_{jk} - H_{;j} g_{ik}. \tag{9}$$

3. Totally umbilic hypersurfaces of first order

A hypersurface (M^n, g) of $(\bar{M}^{n+1}, \bar{g})$ is said to be a totally umbilic hypersurface of first order if its second fundamental form ω_{ij} satisfies

$$\omega_{ij;k} = A_k g_{ij}, \tag{10}$$

where A_k are the components of 1-form A.
It is easy to see that every totally umbilic hypersurface is a totally umbilic hypersurface of first order.

In particular if the associated 1-form A in (10) vanishes, then the second fundamental form is parallel [7,15]. In general if the 1-form A is closed, then the second fundamental form is semiparallel [6,10].

A Riemannian manifold (M^n, g) is said to be Einstein if its Ricci tensor r is proportional to the metric tensor g, i.e.

$$r_{ij} = \frac{s}{n} g_{ij}. \tag{11}$$

Note that the scalar curvature s of an Einstein manifold is constant when the dimension is greater or equal to 3 [1]. Concerning totally umbilic hypersurfaces of first order in an Einstein manifold, we obtain the following results:

Theorem 3.1. *Let (M^n, g) be a totally umbilic hypersurface of first order in an Einstein manifold $(\bar{M}^{n+1}, \bar{g})$. Then its associated 1-form A vanishes.*

Proof. By virtue of (5) and (10), we have

$$\bar{R}_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta = A_i g_{jk} - A_j g_{ik}. \tag{12}$$

By transvecting (12) by g^{jk}, we obtain from (2)

$$\bar{R}_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}(\bar{g}^{\beta\gamma} - N^\beta N^\gamma)N^\delta = (n-1)A_i,$$

which yields

$$\bar{r}_{\alpha\delta}\frac{\partial y^\alpha}{\partial x^i}N^\delta - \bar{R}_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}N^\beta N^\gamma N^\delta = (n-1)A_i.$$

By taking account of the Einstein condition (11), the last relation reduces to

$$\bar{R}_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}N^\beta N^\gamma N^\delta = -(n-1)A_i. \tag{13}$$

On the other hand, by considering the skew symmetric property of curvature tensor and the hypersurface condition , we have

$$\bar{R}_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}N^\beta N^\gamma N^\delta = 0. \tag{14}$$

Therefore it follows from (13) and (14) that the associated 1-form A vanishes. This completes the proof. □

Corollary 3.2. *Let (M^n, g) be a totally umbilic hypersurface of first order in an Einstein manifold $(\bar{M}^{n+1}, \bar{g})$. Then its mean curvature H is constant.*

Proof. It follows from (10) and Theorem 3.1 that the second fundamental form ω_{ij} is parallel. Hence by definition of the mean curvature H, that is, $H = \frac{1}{n}g^{ij}\omega_{ij}$, we have $H_{;p} = 0$. □

Theorem 3.3. *Let (M^n, g) be a totally umbilic hypersurface of first order in an Einstein manifold $(\bar{M}^{n+1}, \bar{g})$. Then its scalar curvature s is constant.*

Proof. By differentiating (4) covariantly, we obtain from (3)

$$
\begin{aligned}
R_{ijkl;p} &= \bar{R}_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p} + \bar{R}_{\alpha\beta\gamma\delta}(\omega_{ip}N^\alpha)\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l} \\
&+\bar{R}_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}(\omega_{jp}N^\beta)\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l} + \bar{R}_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}(\omega_{kp}N^\gamma)\frac{\partial y^\delta}{\partial x^l} \\
&+\bar{R}_{\alpha\beta\gamma\delta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}(\omega_{lp}N^\delta) + \omega_{il;p}\omega_{jk} + \omega_{il}\omega_{jk;p} - \omega_{ik;p}\omega_{jl} - \omega_{ik}\omega_{jl;p}.
\end{aligned} \tag{15}
$$

From (5), (10), (15) and Theorem 3.1 it follows that

$$
R_{ijkl;p} = \bar{R}_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l}\frac{\partial y^\mu}{\partial x^p}. \tag{16}
$$

By transvecting (16) by g^{il} we have from (2)

$$
\begin{aligned}
r_{jk;p} &= (\bar{g}^{\alpha\delta} - N^\alpha N^\delta)\bar{R}_{\alpha\beta\gamma\delta;\mu}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\mu}{\partial x^p} \\
&= \bar{r}_{\beta\gamma;\mu}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\mu}{\partial x^p} - \bar{R}_{\alpha\beta\gamma\delta;\mu}N^\alpha\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\frac{\partial y^\mu}{\partial x^p}.
\end{aligned} \tag{17}
$$

By taking account of the Einstein manifold $(\bar{M}^{n+1}, \bar{g})$, we get from (17)

$$
r_{jk;p} = -\bar{R}_{\alpha\beta\gamma\delta;\mu}N^\alpha\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}N^\delta\frac{\partial y^\mu}{\partial x^p}. \tag{18}
$$

By transvecting (18) by g^{jk}, we have from (2) and the Einstein manifold $(\bar{M}^{n+1}, \bar{g})$,

$$
\begin{aligned}
s_{;p} &= -(\bar{g}^{\beta\gamma} - N^\beta N^\gamma)\bar{R}_{\alpha\beta\gamma\delta;\mu}N^\alpha N^\delta\frac{\partial y^\mu}{\partial x^p} \\
&= -\bar{r}_{\alpha\delta;\mu}N^\alpha N^\delta\frac{\partial y^\mu}{\partial x^p} + \bar{R}_{\alpha\beta\gamma\delta;\mu}N^\alpha N^\beta N^\gamma N^\delta\frac{\partial y^\mu}{\partial x^p} = 0
\end{aligned}
$$

because of the skew symmetric property of curvature tensor and the hypersurface condition. Therefore we conclude that the scalar curvature s of (M^n, g) is constant. □

A Riemannian manifold (M^n, g) is said to be a space of constant curvature if its curvature tensor R satisfies the relation:

$$R_{ijkl} = \frac{s}{n(n-1)}(g_{il}g_{jk} - g_{ik}g_{jl}). \tag{19}$$

Note that a space of constant curvature is Einstein and hence its scalar curvature s is constant. In case of a space of constant curvature, we obtain the following results:

Theorem 3.4. *Let (M^n, g) be a totally umbilic hypersurface of first order in a space of constant curvature $(\bar{M}^{n+1}, \bar{g})$. Then the manifold (M^n, g) is locally symmetric.*

Proof. By considering (4) and (19), we have

(20)

$$R_{ijkl} = \frac{\bar{s}}{(n+1)n}(\bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta})\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial x^k}\frac{\partial y^\delta}{\partial x^l} + (\omega_{il}\omega_{jk} - \omega_{ik}\omega_{jl}),$$

where $\bar{s}$ is the scalar curvature of a space of constant curvature $(\bar{M}^{n+1}, \bar{g})$. By taking account of (1) and (20), we obtain

$$R_{ijkl} = \frac{\bar{s}}{(n+1)n}(g_{il}g_{jk} - g_{ik}g_{jl}) + (\omega_{il}\omega_{jk} - \omega_{ik}\omega_{jl}). \tag{21}$$

By differentiating (21) covariantly, we get from (10), $\bar{s}=$ constant and Theorem 3.1

$$R_{ijkl;p} = 0,$$

showing that (M^n, g) is locally symmetric. This completes the proof. □

Theorem 3.5. *Let (M^n, g) be a complete, simply connected and totally umbilic hypersurface of first order in a space of constant curvature $(\bar{M}^{n+1}, \bar{g})$. Then the manifold (M^n, g) is a Riemannian product of Einstein manifolds.*

Proof. By virtue of Theorem 3.4, we have

$$R_{ijkl;p} = 0. \tag{22}$$

By transvecting (22) by g^{il}, we obtain

$$r_{jk;p} = 0, \tag{23}$$

showing that the manifold (M^n, g) has a parallel Ricci tensor r. It follows from the de Rham decomposition theorem [2] that (23) implies that the complete, simply connected manifold (M^n, g) is a Riemannian product of Einstein manifolds. This completes the proof. □

Now we show a proper example which is a totally umbilic hypersurface of first order but not a totally umbilic hypersurface:

Example 3.6. *Let $(S^n \times R^m, g)$ be a hypersurface with an induced metric g of a flat manifold $(R^{n+1} \times R^m, g_o)$. Here S^n is a standard sphere of dimension n in R^{n+1}.*

Then by the basic properties of a Riemannian product manifold, we have

$$\omega(X, Y) = \omega(X_1 + X_2, Y_1 + Y_2) = \omega(X_1, Y_1) + \omega(X_2, Y_2) \tag{24}$$

and

$$g(X, Y) = g(X_1 + X_2, Y_1 + Y_2) = g(X_1, Y_1) + g(X_2, Y_2), \tag{25}$$

where X, Y are vector fields on $S^n \times R^m$, and X_1, Y_1 (resp. X_2, Y_2) are vector fields on S^n (resp. R^m).

On the other hand, it is easy to see that

$$\omega(X_1, Y_1) = 1g(X_1, Y_1) \tag{26}$$

and

$$\omega(X_2, Y_2) = 0g(X_2, Y_2). \tag{27}$$

Therefore it follows from (24), (25), (26) and (27) that

$$\omega(X, Y) \neq cg(X, Y),$$

showing that the Riemannian product manifold $(S^n \times R^m, g)$ is not a totally umbilic hypersurface in a flat manifold $(R^{n+1} \times R^m, g_o)$. However by (26), (27) and the basic properties of a Riemannian product manifold, we get

$$(\nabla_Z \omega)(X, Y) = (\nabla_{Z_1} \omega)(X_1, Y_1) + (\nabla_{Z_2} \omega)(X_2, Y_2) = 0,$$

where Z is a vector field on $S^n \times R^m$, and Z_1, Z_2 are vector fields on S^n, R^m respectively. Hence we obtain

$$(\nabla_Z \omega)(X, Y) = 0g(X, Y),$$

showing that the Riemannian product manifold $(S^n \times R^m, g)$ is a totally umbilic hypersurface of first order with vanishing 1-form A in a flat manifold $(R^{n+1} \times R^m, g_o)$.

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