Honam Mathematical J. **39** (2017), No. 4, pp. 475–483
https://doi.org/10.5831/HMJ.2017.39.4.475

# TIGHT CLOSURE OF IDEALS RELATIVE TO SOME MODULES

F. Dorostkar * and R. Khosravi

**Abstract.** In this paper we consider the tight closure of an ideal relative to a module whose its zero submodule has a primary decomposition.

## 1. Introduction

Throughout this paper $R$ denotes a commutative Noetherian ring with identity and with a positive prime characteristic $p$. Further $\mathbf{N}$ will denote the set of natural integers and throughout the remainder of this paper $R^{\circ}$ will denote the subset of $R$ consisting of all elements which are not contained in any minimal prime ideal of $R$.

The main idea of tight closure of an ideal in a commutative Noetherian ring (with a positive prime characteristic) introduced by Hochster and Huneke in [5].

Let $I$ be an ideal of $R$. The ideal $(a^{p^e} : a \in I)$ is denoted by $I^{[p^e]}$ and is called the eth Frobenius power of $I$. In particular if $I = (a_1, a_2, ..., a_n)$, then $I^{[p^e]} = (a_1^{p^e}, a_2^{p^e}, ..., a_n^{p^e})$. In the remainder of this paper, to simplify notation, we will write $q$ to stand for a power $p^e$ of p. For any ideals $I$ and $J$, $I^{[q]} + J^{[q]} = (I + J)^{[q]}$, $I^{[q]}J^{[q]} = (IJ)^{[q]}$. We recall that an element $x$ of $R$ is said to be in the tight closure $I^*$, of $I$, if there exists an element $c \in R^{\circ}$ such that for all sufficiently large $q$, $cx^q \in I^{[q]}$. More details for the tight closure of an ideal can be found in [10].

In [1], the dual notion of tight closure of ideals relative to modules was introduced and some properties of this concept which reflect results of tight closure in the classical situation were obtained. It is appropriate for us to begin by briefly summarizing some of main aspects.

Received November 8, 2016. Accepted October 9, 2017.
2010 Mathematics Subject Classification. 13A35.
Key words and phrases. tight closure, primary decomposition.
*Corresponding author

Let $M$ be an $R-$module and let $I$ and $J$ be ideals of $R$. $I$ is an $F-$reduction of the ideal $J$ relative to $M$, if $I \subseteq J$ and there exists $c \in R^{\circ}$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M cJ^{[q]}) \text{ } for \text{ } all \text{ } q \gg 0.$$

It is straightforward to see that the set of ideals of $R$ which have $I$ as an $F-$reduction relative to $M$ has a unique maximal member, denoted by $I^{*[M]}$ and called the tight closure of $I$ relative to $M$. This is in fact the largest ideal which has $I$ as $F-$reduction relative to $M$ (see [1]).

An element $x$ of $R$ is said to be tight dependent on $I$ relative to $M$, if there exists an element $c \in R^{\circ}$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M cx^q) \text{ } for \text{ } all \text{ } q \gg 0.$$

In [1], it is shown that $I \subseteq I^* \subseteq I^{*[M]}$ and

$$I^{*[M]} = \{x \in R : x \text{ } is \text{ } tight \text{ } dependent \text{ } on \text{ } I \text{ } relative \text{ } to \text{ } M\}.$$

Moreover in [3], it is shown that if S is a multiplicatively closed subset of R and $M$ is a Noetherian $R-$module then

$$S^{-1}I^{*[M]} = (S^{-1}I)^{*[S^{-1}M]}.$$

Now, let $M$ be an $R-$module not necessarily finitely generated. In this paper, we will show that the last equation is still true if the zero submodule of $M$ has a minimal primary decomposition and every associated prime ideal of $M$ is isolated.

## 2. Auxiliary Results

In this section, we provide some definitions, notations and base facts which we need for the main sections of this paper.

Let $M$ be an $R-$module. A proper submodule $N$ of $M$ is said to be primary submodule if for every $r \in R$ and $m \in M$, $rm \in N$ implies that $m \in N$ or $r \in \sqrt{(N :_R M)}$. If $N$ is a primary submodule of $M$ then $P = \sqrt{(N :_R M)}$ is a prime ideal of $R$ and $N$ is said a $P-$primary submodule of $M$. A non zero module is said to be $P-$coprimary if its zero submodule is a $P-$primary submodule. If $N$ is a $P-$primary submodule of $M$ then the quotient $M/N$ is $P-$coprimary.

For a submodule $N$ of $M$ the intersection $N = N_1 \cap N_2 \cap .... \cap N_k$ where $N_i$ is a $P_i-$primary submodule of $M$ for every $1 \leq i \leq k$, is called a primary decomposition of $N$ and if

(a) the prime ideals $P_1, P_2, ..., P_k$ are all distinct and (b)$N \neq \bigcap_{\substack{i=1 \\ i \neq j}}^{k} N_i$ for every $1 \leq j \leq k$

then it is called a minimal primary decomposition of $N$. We know that every primary decomposition of $N$ can be refined to a minimal primary decomposition.

Let the zero submodule of $M$ have a minimal primary decomposition $0 = N_1 \cap N_2 \cap .... \cap N_k$ where $N_i$ is a $P_i-$primary submodule of $M$ for every $1 \leq i \leq n$. Every minimal prime ideal of $\{P_1, ..., P_k\}$ is called an isolated prime ideal. We know that if $P_i$ is a minimal prime ideal of $\{P_1, ..., P_k\}$ then the corresponding $P_i-$primary component is the same in any minimal primary decomposition of 0 (see [7, page 55]).

Now we recall that, a prime ideal $P$ of $R$ is called an associated prime ideal of $M$ if there exists $x \in M$ such that $P = Ann(x)$. The set of all associated prime ideals of $M$, is denoted by $Ass(M)$.

**Remark 2.1.** *(See* [11, 1.1].*) Let $M$ be an $R-module$. Then*

(a) *$Ass(M) \neq \emptyset$ if and only if $M \neq \emptyset$;*
(b) *If $N$ is a submodule of $M$ then $Ass(N) \subseteq Ass(M)$;*
(c) *$Ass(M) \subseteq Supp(M)$ (where $Supp(M)$ denotes the support of the module $M$);*
(d) *If $S$ is a multiplicatively closed subset of $R$ then*

$$Ass_{S^{-1}R}(S^{-1}M) = \{S^{-1}P | P \in Ass(M) \text{ with } P \cap S = \emptyset\}.$$

**Remark 2.2.** *(See* [11, 1.3].*) Let the zero submodule of $M$ have a minimal primary decomposition $0 = N_1 \cap N_2 \cap .... \cap N_k$ where $N_i$ is a $P_i-$primary submodule of $M$ for every $1 \leq i \leq n$. Then $Ass(M) = \{P_1, ..., P_k\}$ and so the prime ideals $P_1, ..., P_k$ are independent of any minimal primary decomposition of* 0.

**Lemma 2.3.** *Let $I$ be an ideal of $R$ and let $M$ be an $R-$module. Let $N$ be a $P-$primary submodule of $M$. Further assume that $P'$ is a prime ideal of $R$ such that $P$ is not contained in $P'$. Then*

$$(N :_M I)R_{P'} = NR_{P'} = MR_{P'}.$$

*Proof.* We first show that $NR_{P'} = MR_{P'}$. To do this let $\frac{m}{1} \in MR_{P'}$. By assumption we can choose an element $t \in P$ such that $t \notin P'$. So there exists a positive integer $n$ such that

$$\frac{m}{1} = \frac{t^n m}{t^n} \in NR_{P'}$$

and this shows that $NR_{P'} = MR_{P'}$. Now, since $N \subseteq (N :_M I) \subseteq M$, the claim is clear. □

**Lemma 2.4.** *Let $I$ be an ideal of $R$ and $M$ be an $R-$module. Then for every multiplicatively closed subset $S$ of $R$ we have*

$$S^{-1}(I^{*[M]}) \subseteq (S^{-1}I)^{*[S^{-1}M]}.$$

*Proof.* The proof is straightforward. □

**Remark 2.5.** *Let $M$ be an $R-$module and let $S$ be a multiplicatively closed subset of $R$. Let $\frac{u}{1} \in (S^{-1}I)^{*[S^{-1}M]}$. Then there exists $\frac{c}{1} \in (S^{-1}R)^{\circ}$ such that*

$$(0 :_{S^{-1}M} (S^{-1}I)^{[q]}) \subseteq (0 :_{S^{-1}M} \frac{c}{1}(\frac{u}{1})^{q}) \text{ for all } q \gg 0.$$

*By using a similar method which is used in* [5, Prop. 4.14], *without loss of generality, we can assume $c \in R^{\circ}$.*

**Proposition 2.6.** *Let $I$ be an ideal of $R$ and $M$ be an $R-$module. Let $M$ be a $P-$coprimary $R-$module and let $S$ be a multiplicatively closed subset of $R$. Then we have*

$$S^{-1}(I^{*[M]}) = (S^{-1}I)^{*[S^{-1}M]}.$$

*Proof.* This is proved in two cases. First let $S \cap P \neq \emptyset$. Then we can choose an element $t \in S \cap P$ and a positive integer $k$ such that $t^k \in S \cap (0 : M)$. Let $y \in R$. Since $t^k M = 0$, we can see $t^k y \in I^{*[M]}$ and this shows that $\frac{y}{1} = \frac{t^k y}{t^k} \in S^{-1}I^{*[M]}$. Then we have $S^{-1}R \subseteq S^{-1}(I^{*[M]})$. Now by using 2.4, we have

$$S^{-1}R \subseteq S^{-1}(I^{*[M]}) \subseteq (S^{-1}I)^{*[S^{-1}M]} \subseteq S^{-1}R.$$

So in this case we have $S^{-1}(I^{*[M]}) = (S^{-1}I)^{*[S^{-1}M]} = S^{-1}R$.

Now let $S \cap P = \emptyset$. Let $\frac{u}{1} \in (S^{-1}I)^{*[S^{-1}M]}$. Then there exists a $\frac{c}{1} \in (S^{-1}R)^{\circ}$ such that

$$(0 :_{S^{-1}M} (S^{-1}I)^{[q]}) \subseteq (0 :_{S^{-1}M} \frac{c}{1}(\frac{u}{1})^{q}) \text{ for all } q \gg 0.$$

By 2.5, we can take $c \in R^{\circ}$. Let $m \in (0 :_M I^{[q]})$ for $q \gg 0$. This implies that

$$\frac{m}{1} \in (0 :_{S^{-1}M} (S^{-1}I)^{[q]}) \subseteq (0 :_{S^{-1}M} \frac{c}{1}(\frac{u}{1})^{q}).$$

Thus there exists $t_q \in S$ such that $t_q c u^q m = 0$. Since $M$ is a $P-$coprimary $R-$module and $S \cap P = \emptyset$, we can see that $cu^q m = 0$. Then

$$(0 :_M I^{[q]}) \subseteq (0 :_M cu^q) \text{ for all } q \gg 0.$$

So $u \in I^{*[M]}$. Hence $(S^{-1}I)^{*[S^{-1}M]} \subseteq S^{-1}(I^{*[M]})$. The reverse inclusion always holds by 2.4. Then $S^{-1}(I^{*[M]}) = (S^{-1}I)^{*[S^{-1}M]}$.

□

## 3. Main Results

In this section we want to study the localization of tight closure of an ideal relative to a module whose its zero submodule has a primary decomposition.

We note that sometimes every associated prime ideal of a module can be isolated. For example if $M$ is a finite length $R-$module or $M$ is a Cohen-Maculay $R-$module then $Ass(M)$ has no embedded prime ideals (see [4, 2.17] and [7, Theo. 30]).

**Theorem 3.1.** *Let $I$ be an ideal of $R$ and $M$ be an $R-$module. Assume that the zero submodule of $M$ has a minimal primary decomposition $0 = N_1 \cap N_2 \cap ... \cap N_k$, where $N_i$ is $P_i-$primary submodule of $M$ and every prime ideal in the set $\{P_1, ..., P_k\}$is isolated. Then*

$$I^{*[M]} = \bigcap_{i=1}^{k} I^{*[M/N_i]}.$$

*Proof.* Let $u \in \bigcap_{i=1}^{k} I^{*[M/N_i]}$. Then for every $1 \leq i \leq k$, there exists $c_i \in R^{\circ}$ such that

$$(0 :_{M/N_i} I^{[q]}) \subseteq (0 :_{M/N_i} c_i u^q) \ for\ all\ q \gg 0.$$

Let $c = c_1 c_2 ... c_k$. Then for every $1 \leq i \leq k$, we have

$$(N_i :_M I^{[q]}) \subseteq (N_i :_M cu^q) \ for\ all\ q \gg 0.$$

This shows that

$$\bigcap_{i=1}^{k} (N_i :_M I^{[q]}) \subseteq \bigcap_{i=1}^{k} (N_i :_M cu^q) \ for\ all\ q \gg 0.$$

Therefore

$$(\bigcap_{i=1}^{k} N_i :_M I^{[q]}) \subseteq (\bigcap_{i=1}^{k} N_i :_M cu^q) \ for\ all\ q \gg 0.$$

This implies that $u \in I^{*[M]}$ and so $I^{*[M]} \supseteq \bigcap_{i=1}^{k} I^{*[M/N_i]}$.

For the reverse inclusion, let $z \in I^{*[M]}$. Then there exists $c \in R^{\circ}$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M cz^q) \ for\ all\ q \gg 0.$$

This shows that

$$\bigcap_{i=1}^{k}(N_i :_M I^{[q]}) \subseteq \bigcap_{i=1}^{k}(N_i :_M cz^q) \ for\ all\ q \gg 0.$$

By localization and using 2.3, we can see that

$$(N_i :_M I^{[q]})R_{P_i} \subseteq (N_i :_M cz^q)R_{P_i} \ for\ all\ q \gg 0$$

for every $1 \leq i \leq k$. This implies that

$$(N_i :_M I^{[q]}) \subseteq (N_i :_M cz^q) \ for\ all\ q \gg 0$$

for every $1 \leq i \leq k$. Hence

$$(0 :_{M/N_i} I^{[q]}) \subseteq (0 :_{M/N_i} cz^q) \ for\ all\ q \gg 0$$

for every $1 \leq i \leq k$. This shows that $z \in I^{*[M/N_i]}$ for every $1 \leq i \leq k$. So $z \in \bigcap_{i=1}^{k} I^{*[M/N_i]}$. Then $I^{*[M]} \subseteq \bigcap_{i=1}^{k} I^{*[M/N_i]}$ and so the proof is completed. □

**Remark 3.2.** *The proof of 3.1 shows that the inclusion $\supset$ holds true without the isolated assumption on the set of associated primes.*

**Corollary 3.3.** *Let $I$ be an ideal of $R$ and $M$ be an $R-$module. Assume that the zero submodule of $M$ has a minimal primary decomposition $0 = N_1 \cap N_2 \cap ... \cap N_k$, where $N_i$ is $P_i-$primary submodule of $M$ and every prime ideal in the set $\{P_1, ..., P_k\}$is isolated. Then*

(a) $Ass_R(R/I^{*[M]}) \subseteq \{\sqrt{I^{*[M/N_i]}} :\ 1 \leq i \leq k\}$.
(b) *If $P_1, P_2, ..., P_k \in Max(R)$ then $Ass_R(R/I^{*[M]}) \subseteq \{P_1, P_2, ..., P_k\}$ and so the sequence $(Ass_R(R/(I^n)^{*[M]}))_{n\in\mathbf{N}}$ is eventually constant.*

*Proof.* (a) By 3.1, we have

$$I^{*[M]} = \bigcap_{i=1}^{n} I^{*[M/N_i]}.$$

So it suffices to prove that if $I^{*[M/N_i]} \neq R$ then $I^{*[M/N_i]}$ is a primary ideal of $R$. For $x, y$ in $R$, let $xy \in I^{*[M/N_i]}$ and $y \notin \sqrt{I^{*[M/N_i]}}$.

We know $Ann_R(M/N_i) \subseteq I^{*[M/N_i]}$. Then $y \notin \sqrt{I^{*[M/N_i]}}$ implies that $y \notin \sqrt{Ann_R(M/N_i)}$. Since $xy \in I^{*[M/N_i]}$, there exists a $c \in R^{\circ}$ such that

$$(0 :_{M/N_i} I^{[q]}) \subseteq (0 :_{M/N_i} c(xy)^q) \ for\ all\ q \gg 0.$$

Now since $N_i$ is $P_i-$primary submodule of $M$ and $y \notin \sqrt{Ann_R(M/N_i)}$, it is easy to see that

$$(0 :_{M/N_i} I^{[q]}) \subseteq (0 :_{M/N_i} cx^q) \ for \ all \ q \gg 0.$$

Then $x \in I^{*[M/N_i]}$ and this shows that if $I^{*[M/N_i]} \neq R$ then $I^{*[M/N_i]}$ is a primary ideal of $R$.

(b) Let $P_1, P_2, ..., P_k \in Max(R)$ and $I^{*[M/N_i]} \neq R$. Since $P_i \in Max(R)$ and $(N_i :_R M) \subseteq I^{*[M/N_i]}$, we can see

$$\sqrt{I^{*[M/N_i]}} = \sqrt{(N_i :_R M)} = P_i.$$

By [1, 2.17(a)], the sequence $(Ass_R(R/(I^n)^{*[M]}))_{n\in\mathbf{N}}$ is an increasing sequence and so the proof is completed. □

**Theorem 3.4.** *Let $I$ be an ideal of $R$ and $M$ be an $R-$module such that its zero submodule has a primary decomposition. Moreover assume that, every prime ideal of $Ass(M)$ is isolated. Also, let $S$ be a multiplicatively closed subset of $R$. Then*

$$S^{-1}(I^{*[M]}) = (S^{-1}I)^{*[S^{-1}M]}.$$

*Proof.* Let $0 = N_1 \cap N_2 \cap ... \cap N_k$ be a minimal primary decomposition of 0 where $N_i$ is $P_i-$primary submodule of $M$ for every $1 \leq i \leq k$. Now let $S \cap P_i = \emptyset$ for every $1 \leq i \leq t$ and $S \cap P_i \neq \emptyset$ for every $t+1 \leq i \leq k$. Since every prime ideal of $Ass(M)$ is isolated, we can conclude that

$$S^{-1}0 = S^{-1}N_1 \cap S^{-1}N_2 \cap ... \cap S^{-1}N_t$$

is a minimal primary decomposition of $S^{-1}0$. By 3.1, we have

$$(S^{-1}I)^{*[S^{-1}M]} = \bigcap_{i=1}^{t} (S^{-1}I)^{*[S^{-1}M/S^{-1}N_i]}.$$

Also, by 3.1, we have

$$S^{-1}(I^{*[M]}) = (\bigcap_{i=1}^{t} S^{-1}I^{*[M/N_i]}) \cap (\bigcap_{i=t+1}^{k} S^{-1}I^{*[M/N_i]})$$

As we saw in the proof of 2.6, since $S \cap P_i \neq \emptyset$ for every $t+1 \leq i \leq k$, we have $S^{-1}I^{*[M/N_i]} = S^{-1}R$ for every $t+1 \leq i \leq k$. This shows that

$$S^{-1}(I^{*[M]}) = (\bigcap_{i=1}^{t} S^{-1}I^{*[M/N_i]}).$$

Now, by using 2.6, we have $S^{-1}(I^{*[M]}) = (S^{-1}I)^{*[S^{-1}M]}$.

□

**Example 3.5.** *Let $R$ be an Artinian ring and let $M$ be an $R-$module (not necessary Noetherian $R-$module). By* [9, 2.8], *the zero submodule of $M$ has a primary decomposition. Since $R$ is an Artinian ring, every prime ideal of $Ass(M)$ is isolated.*

For secondary representation and attached primes, we can see [6].

**Corollary 3.6.** *Let $I$ be an ideal of $R$ and $M$ be an $R-$module such that $M$ has a secondary representation and $MinAtt(M) = Att(M)$. Moreover let $S$ be a multiplicatively closed subset of $R$ and let $E$ be an injective cogenerator $R-$module. If $D(M) = Hom_R(M, E)$ is the dual of $M$ relative $E$ then*

$$S^{-1}(I^{*[D(M)]}) = (S^{-1}I)^{*[S^{-1}D(M)]}.$$

*In particular, if $M$ is a finitely presented module then*

$$S^{-1}(I^{*[D(M)]}) = (S^{-1}I)^{*[D(S^{-1}M)]}.$$

*Proof.* We can conclude from [9, 2.6(1)] and [2, IV, §1, Ex. 17(g)]that $D(M)$ has a primary decomposition and $Ass(D(M)) = Att(M)$. Then by 3.4, we can see that

$$S^{-1}(I^{*[D(M)]}) = (S^{-1}I)^{*[S^{-1}D(M)]}.$$

Now the last part is clear by [7, 1.G] and [8, §18 Lem. 5]. □

## References

[1] H. Ansari-Toroghy and F. Dorostkar, *Tight closure of ideals relative to modules*, Honam Math. J., **32(4)** (2010), 675-687.

[2] N. Bourbaki, *Algebra commutative*, Hermann, Paris, 1972.

[3] F. Dorostkar and R. Khosravi, *F-Regularity Relative To Modules*, Journal of Algebra and Related Topics, **3(1)** (2015), 41-50.

[4] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate texts in mathematics 150, Springer-Verlage, 1995.

[5] M. Hochster and C. Huneke, *Tight closure, invariant theory, and Briançon-Skoda theorem*, J. of the Amer. Math. Soc., **3** (1990), 31-116.

[6] I. G. Macdonald, *Secondary representation of modules over a commutative ring*, Symp. Math. , **XI** (1973), 23-43.

[7] H. Matsumura, *Commutative Algebra*, Benjamin, 2nd edition, Mass., 1980.

[8] H. Matsumura, *Commutative ring theory*, Cambridge Univ. press, 1986.

[9] I. Nishitani, *Associated prime ideals of the dual of an artinian module relative to an injective module*, Comm. in Algebra, **22(7)** (1994), 2651-2667.

[10] I. Swanson and C. Huneke, *Integral closure of ideals, rings, and modules*, Cambridge Univ. Press, New York, 2006.

[11] S. Yassemi, *Coassociated primes of modules over a commutative ring*, Math Scand. , **80** (1997), 175-187.

F. Dorostkar
Department of Pure Mathematics, University of Guilan,
P. O. Box 1914,
Rasht, Iran.
E-mail: dorostkar@guilan.ac.ir

R. Khosravi
Department of Pure Mathematics, University of Guilan,
P. O. Box 1914,
Rasht, Iran.
E-mail: Khosravi@phd.guilan.ac.ir