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ENVELOPES OF SUBMODULES

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Abstract. We introduce and investigate envelope of submodules. We show that the proper envelopes of certain submodules is a union of prime submodules.

1. INTRODUCTION

We always assume that A is a commutative ring with nonzero identity and B is a unitary A-module. Also, $\mathfrak{a}$ denotes a proper ideal of A. We define the envelope of a submodule N of an A-module B with respect to an ideal $\mathfrak{a}$ of A to be the set of all elements m of B such that there exists an element r of $R \setminus \mathfrak{a}$ and $rm \in N$. This is a generalization of torsion subset $T(B)$ of B. We use this notion to provide some results on prime submodules. For example, we show that the envelope of some submodule respect to some ideal is a union of prime submodules.

2. MAIN RESULTS

Definition 2.1. *Let N be a submodule of B. We define the* envelope *of N with respect to $\mathfrak{a}$ as following*

$$E_{\mathfrak{a}}(N) = \{m \in B \mid \exists r \in R \setminus \mathfrak{a}; rm \in N\} = \bigcup_{r \notin \mathfrak{a}} (N :_B r).$$

We say N is $\mathfrak{a}$-closed if $E_{\mathfrak{a}}(N) = N$.

Envelope of a submodule is no longer a submodule in general. For example, let $A = B = \mathbb{Z}$ and $N = \mathfrak{a} = 6\mathbb{Z}$. Then it is easy to see that $2, 3 \in E_{\mathfrak{a}}(N)$ but $2+3 \notin E_{\mathfrak{a}}(N)$. However, if $\mathfrak{p}$ is a prime ideal of A, then it is easy to see that $E_{\mathfrak{p}}(N)$ is submodule. Moreover, if $E_{\mathfrak{p}}(N) \neq B$, then

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it is a prime submodule of B (see [5]). The next proposition provides simple conditions under which envelope $E_{\mathfrak{a}}(N)$ of N is submodule.

Proposition 2.2. *Let $E_{\mathfrak{a}}(\mathfrak{a}) = \sqrt{\mathfrak{a}}$ and let it is a principal ideal of A. Then for each submodule L of B, $E_{\mathfrak{a}}(L)$ is a submodule of B.*

Proof. Suppose that $E_{\mathfrak{a}}(\mathfrak{a}) = Ax$ for some $x \in A$ and $E_{\mathfrak{a}}(L)$ is not a submodule of B. Then there exist $m_1, m_2 \in E_{\mathfrak{a}}(L)$ such that $m_1 + m_2 \notin E_{\mathfrak{a}}(L)$. By definition, there are elements $s_1, s_2 \in A \backslash \mathfrak{a}$ such that $s_1 m_1 \in L$ and $s_2 m_2 \in L$. Hence, $s_1 s_2(m_1 + m_2) \in L$ and $m_1 + m_2 \notin E_{\mathfrak{a}}(L)$. Therefore, $s_1 s_2 \in \mathfrak{a}$. This implies that $s_1, s_2 \in E_{\mathfrak{a}}(\mathfrak{a})$. Since $x \in \sqrt{\mathfrak{a}}$, there are positive integers t_1, t_2 such that $s_1 = ax^{t_1}$ and $s_2 = bx^{t_2}$, for some $a, b \notin E_{\mathfrak{a}}(\mathfrak{a})$. Without loss of generality, we may assume that $t_1 \geq t_2$. Since $b \notin E_{\mathfrak{a}}(\mathfrak{a})$ and $s_1 \in A \setminus \mathfrak{a}$, the element bs_1 does not belong to $\mathfrak{a}$ and therefore $bs_1(m_1 + m_2) \in L$. This yields that $m_1 + m_2 \in E_{\mathfrak{a}}(L)$, a contradiction. □

Example 2.3. *Let $A = \mathbb{Z}$ be the ring of integers and $\mathfrak{a} = 4\mathbb{Z}$ be the principal ideal generated by $4 \in \mathbb{Z}$. Then $E_{\mathfrak{a}}(\mathfrak{a}) = \sqrt{4\mathbb{Z}} = 2\mathbb{Z}$. By Theorem 2.2, for each A-module M and each submodule L of M, $E_{\mathfrak{a}}(L)$ is a submodule of M.*

In the next theorem, we show that the envelope of some submodules of a flat modules is submodule. This theorem is a generalization of a useful and well-known result in the theory of prime submodules (see [3, Theorem 3]).

Theorem 2.4. *Let $\mathfrak{q}$ be an $\mathfrak{a}$-closed ideal of A for some ideal $\mathfrak{a}$ of A and let B be a flat A-module. Then $E_{\mathfrak{a}}(\mathfrak{q}B) = \mathfrak{q}B$.*

Proof. We prove the theorem in three steps.
Step 1: We claim that $\mathfrak{q} = (\mathfrak{q} :_A x)$ for each element $x \in A \setminus \mathfrak{a}$. It is obvious that $\mathfrak{q} \subseteq (\mathfrak{q} :_A x)$. So, let $y \in (\mathfrak{q} :_A x)$. Then $xy \in \mathfrak{q}$. Thus $y \in E_{\mathfrak{a}}(\mathfrak{q}) = \mathfrak{q}$, since $x \in A \setminus \mathfrak{a}$ and $\mathfrak{q}$ is $\mathfrak{a}$-closed.
Step 2: Let $x \in A \setminus \mathfrak{a}$ and consider the exact sequence of A-modules

$$0 \longrightarrow \frac{(\mathfrak{q} :_A x)}{\mathfrak{q}} \xrightarrow{f} \frac{A}{\mathfrak{q}} \xrightarrow{g} \frac{A}{\mathfrak{q}}$$

where f is the canonical injection mapping and g is the mapping obtained by taking quotients under multiplication by x. Since, B is flat we have the following exact sequence

$$0 \longrightarrow \frac{(\mathfrak{q} :_A x)}{\mathfrak{q}} \otimes B \xrightarrow{f \otimes 1_B} \frac{A}{\mathfrak{q}} \otimes B \xrightarrow{g \otimes 1_B} \frac{A}{\mathfrak{q}} \otimes B.$$

Since $\frac{A}{\mathfrak{q}} \otimes B \cong \frac{B}{\mathfrak{q}B}$ and $Img(f \otimes 1_B) = Ker(g \otimes 1_B)$ we have

$$(\mathfrak{q} :_A x)B = (\mathfrak{q}B :_B x).$$

Step 3: Suppose that $m \in E_{\mathfrak{a}}(\mathfrak{q}B)$. Then there exists $r \in A \setminus \mathfrak{a}$ such that $m \in (\mathfrak{q}B :_B r)$. On account of Step 2, we have $m \in (\mathfrak{q} :_A r)B$. From Step 1 we conclude that $m \in \mathfrak{q}B$. This completes the proof. □

Let L be a submodule of B. Then L is called *prime* if L is proper and if $tl \in L$ (where $(t, l) \in A \times B$), then $t \in \mathrm{Ann}(B/L)$ or $l \in L$. If L is prime, then ideal $\mathfrak{p} := \mathrm{Ann}(B/L)$ is prime and L is said to be $\mathfrak{p}$-*prime* (see [3, 6]). If B has no prime submodule, then B is said to be *primeless.*

Lemma 2.5. *Let N be a submodule of B. Then the following statements hold.*

1. *If N is a prime submodule and $\mathfrak{a} \supseteq (N :_A B)$, then $E_{\mathfrak{a}}(N) = N$. In particular, $E_{(N:_A B)}(N) = N$ if and only if N is a prime submodule.*
2. *If N is a primary submodule and $\mathfrak{a} \supseteq \sqrt{(N :_A B)}$, then $E_{\mathfrak{a}}(N) = N$. In particular, $E_{\sqrt{(N:_A B)}}(N) = N$ if and only if N is a primary submodule.*

Proof. It is easy. □

Lemma 2.5 shows that every $\mathfrak{p}$-primary (also $\mathfrak{p}$-prime) submodule is $\mathfrak{p}$-closed. As we mentioned, Theorem 2.4 is a generalization of [3, Theorem 3]. Using the notion of envelope of submodule we provide a new proof of [3, Theorem 3].

Corollary 2.6. *Let B be a flat A-module and $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p}B \neq B$. If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal of A, then $\mathfrak{q}B$ is a $\mathfrak{p}$-primary submodule of B. In particular, $\mathfrak{p}B$ is a $\mathfrak{p}$-prime submodule.*

Proof. The ideal $\mathfrak{q}$ is a $\mathfrak{p}$-closed ideal of A by Lemma 2.5. According to Theorem 2.4 we have $\mathfrak{q}B = E_{\mathfrak{p}}(\mathfrak{q}B)$. It is easy to verify that $E_{\mathfrak{p}}(\mathfrak{q}B)$ (resp. $E_{\mathfrak{p}}(\mathfrak{p}B)$) is a $\mathfrak{p}$-primary (resp. $\mathfrak{p}$-prime) submodule of B. □

Since each projective module is flat, the next corollary is a generalization of [1, Theorem 2.2].

Corollary 2.7. *Let B be a projective A-module. Then either $\mathfrak{q}B = B$ or $\mathfrak{q}B$ is a $\mathfrak{p}$-primary submodule of B for every $\mathfrak{p}$-primary ideal $\mathfrak{q}$ of A.*

Proof. Let $\mathfrak{q}B \neq B$. By Lemma 2.5, $\mathfrak{q}$ is a $\mathfrak{p}$-closed ideal of A. Since each projective module is flat, it follows from Theorem 2.4 that $E_{\mathfrak{p}}(\mathfrak{q}B) = \mathfrak{q}B$. Now, the result follows from Lemma 2.5(2). □

It is shown in [3, Theorem 2] that if B is a faithful Noetherian A-module, then for every prime ideal $\mathfrak{p}$ of A there is a prime submodule N of B such that $(N :_A B) = \mathfrak{p}$. McCasland and Moore proved this result for finitely generated (not necessarily Noetherian) modules in [6, Theorem 3.3]. Also, it is proved in [4, Lemma, p.3746] using a different method. We are going to present a generalization of this result.

Theorem 2.8. *Let B be a finitely generated A-module and $\mathfrak{a}$ be a prime ideal of A. Then for every $\mathfrak{a}$-closed ideal $\mathfrak{q}$ of A containing* $\operatorname{Ann}(B)$, *there is an $\mathfrak{a}$-closed submodule Q of B such that $(Q :_A B) = \sqrt{\mathfrak{q}}$.*

Proof. Suppose that $\mathfrak{q} \supseteq \operatorname{Ann}(B)$ is an $\mathfrak{a}$-closed ideal of A. We claim that $\sqrt{\mathfrak{q}}$ is $\mathfrak{a}$-closed. Let $x \in E_{\mathfrak{a}}(\sqrt{\mathfrak{q}})$. Then there exists $y \in A \setminus \mathfrak{a}$ such that

$$x \in \sqrt{(\sqrt{\mathfrak{q}} :_A Ay)} = (\sqrt{\mathfrak{q}} :_A \sqrt{Ay}) = \sqrt{(\mathfrak{q} :_A Ay)}.$$

Hence, $x^n \in (\mathfrak{q} :_A Ay)$ for some integer n. This yields that $x \in \sqrt{\mathfrak{q}}$, since $\mathfrak{q}$ is an $\mathfrak{a}$-closed ideal of A. Therefore, $\sqrt{\mathfrak{q}}$ is $\mathfrak{a}$-closed.

From [3, Proposition 8], we have

$$(\sqrt{\mathfrak{q}}B :_A B) = \sqrt{\mathfrak{q}}. \tag{1}$$

We claim that $E_{\mathfrak{a}}(\sqrt{\mathfrak{q}}B)$ is the desired $\mathfrak{a}$-closed submodule of B such that $(E_{\mathfrak{a}}(\sqrt{\mathfrak{q}}B) :_A B) = \sqrt{\mathfrak{q}}$. Obviously, we have $E_{\mathfrak{a}}(\sqrt{\mathfrak{q}}B :_A B) \subseteq (E_{\mathfrak{a}}(\sqrt{\mathfrak{q}}B) :_A B)$. Now, let $g \in (E_{\mathfrak{a}}(\sqrt{\mathfrak{q}}B) :_A B)$ and $B = (b_1, \ldots, b_n)$. Then $gb_i \in E_{\mathfrak{a}}(\sqrt{\mathfrak{q}}B)$ for each $1 \le i \le n$. So, there are $r_1, \ldots, r_n \in R \setminus \mathfrak{a}$ such that $r_i g b_i \in \sqrt{\mathfrak{q}}B$ for each $1 \le i \le n$. Since $r := r_1 \cdots r_n \in R \setminus \mathfrak{a}$ and $rgb_i \in \sqrt{\mathfrak{q}}B$ for each $1 \le i \le n$, we infer that $rg \in (\sqrt{\mathfrak{q}}B :_A B)$. This implies that $g \in E_{\mathfrak{a}}(\sqrt{\mathfrak{q}}B :_A B)$. Consequently, By (1) we have

$$(E_{\mathfrak{a}}(\sqrt{\mathfrak{q}}B) :_A B) = E_{\mathfrak{a}}(\sqrt{\mathfrak{q}}B :_A B) = E_{\mathfrak{a}}(\sqrt{\mathfrak{q}}) = \sqrt{\mathfrak{q}}.$$

This completes the proof. □

Corollary 2.9. *Let B be a nonzero finitely generated A-module. Then for each prime ideal $\mathfrak{p}$ of A containing* $\operatorname{Ann}(B)$, *there exists a $\mathfrak{p}$-prime submodule of B.*

Proof. Let $\mathfrak{p} \supseteq \operatorname{Ann}(B)$. By Lemma 2.5, $\mathfrak{p}$ is $\mathfrak{p}$-closed. There exists a $\mathfrak{p}$-closed submodule Q of B by Theorem 2.8 such that $(Q :_A B) = \sqrt{\mathfrak{p}} = \mathfrak{p}$. Lemma 2.5 implies that Q is prime. □

It is shown in [2, Theorem 3.3] that if the torsion set $T(B)$ of a module B is a proper set, then it is a union of prime submodules. We are going to use the notion of envelope to provide a similar result.

Lemma 2.10. *Let L be a submodule of B such that $E_{\mathfrak{a}}(L) \neq B$, where $\mathfrak{a} := (L :_A B)$ and*

$$S = \{N \leq B \mid N \subseteq E_{\mathfrak{a}}(L) \text{ and } N = \bigcup_{a \in D} (L :_B a) \text{ for some } D \subseteq A\}.$$

Then each maximal element of S is a prime submodule of B.

Proof. Let P be a maximal element of S. Then there exists a subset D of A such that

$$P = \bigcup_{b \in D} (L :_B b).$$

We are going to show that P is a prime submodule of B. Suppose that $(r, m) \in A \times B$ such that $m \notin P$ and $rm \in P$. We must show that $r \in (P :_A B)$.

First, suppose that $rb \notin \mathfrak{a}$ for each element $b \in D$. Let $Q := \bigcup_{b \in D} (L :_B rb)$. Then, by definition, $P \subseteq Q \subseteq E_{\mathfrak{a}}(L)$. We claim that Q is a submodule of B. If $s \in A$ and $q \in Q$, then it is easy to see that $sq \in Q$. Hence, suppose that $m_1, m_2 \in Q$. Then there are $b_1, b_2 \in D$ such that $m_i \in (L :_B rb_i)$ for each $i = 1, 2$. Therefore, $rm_i \in (L :_B b_i) \subseteq P$ for each $i = 1, 2$, because $b_i \notin \mathfrak{a}$. This implies that

$$rm_1 + rm_2 \in P.$$

So, there exists $b_3 \in D$ such that $rm_1 + rm_2 \in (L :_B b_3)$. Thus,

$$m_1 + m_2 \in (L :_B rb_3) \subseteq Q.$$

This implies that Q is a submodule of B and by maximality of P we have $Q = P \subseteq E_{\mathfrak{a}}(L) \neq B$.

Since $rm \in P$, there is $c \in D$ such that $rm \in (L :_B c)$. Thus,

$$m \in (L :_B rc) \subseteq Q \subseteq P,$$

a contradiction. Hence, we assume that $rb \in \mathfrak{a}$ for some $b \in D$. This yields that

$$rbB \subseteq \mathfrak{a}B \subseteq L.$$

Therefore, $rB \subseteq (L :_B b) \subseteq P$. So, P is a prime submodule of B. □

Theorem 2.11. *Let L be a submodule of an A-module B and let $\mathfrak{a} := (L :_A B)$. If $E_{\mathfrak{a}}(L) \neq B$, then $E_{\mathfrak{a}}(L)$ is a union of prime submodules of B.*

Proof. Let $m \in E_{\mathfrak{a}}(L)$ and

$$S_m := \{N \leq B | m \in N \subseteq E_{\mathfrak{a}}(L) \text{ and } N = \bigcup_{a \in D} (L :_B a) \text{ for some } D \subseteq A\}.$$

By definition, there exists $r \in A \setminus \mathfrak{a}$ such that $rm \in L$. Thus,

$$m \in (L :_B r) \subseteq E_{\mathfrak{a}}(L).$$

Hence, $S_m \neq \emptyset$. Now, an easy application of Zorn's Lemma shows that S_m has a maximal element P_m. By Lemma 2.10, P_m is a prime submodule of B. This completes the proof. □

Example 2.12. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$ and let $N = 6\mathbb{Z} \times 12\mathbb{Z}$ and $\mathfrak{a} = 12\mathbb{Z}$. Then $E_{\mathfrak{a}}(N)$ is a proper subset of M, since $(5, 7) \in M \setminus E_{\mathfrak{a}}(N)$. Also, note that, $E_{\mathfrak{a}}(N)$ is not a submodule of M, since $(2, 0), (0, 3) \in E_{\mathfrak{a}}(N)$ but $(2, 3) \notin E_{\mathfrak{a}}(N)$. By Theorem 2.11, $E_{\mathfrak{a}}(N)$ is a union of prime submodules. Indeed, it is easy to see that $2M$ and $3M$ are prime submodules of M and $E_{\mathfrak{a}}(N) = 2M \cup 3M$.*

We Conclude the paper with some results on primeless modules. It is shown in [7, Lemma 1.3], if A is an integral domain and B is a primeless A-module, then B is torsion. Using Theorem 2.11, we can extend this result.

Corollary 2.13. *B is primeless if and only if $E_{\mathfrak{a}}(\mathfrak{a}B) = B$ for each ideal $\mathfrak{a}$ of A. In particular, if B is primeless, then it is torsion.*

Proof. Let B be primeless. Then by Theorem 2.11, $E_{\mathfrak{a}}(\mathfrak{a}B) = B$ for each ideal $\mathfrak{a}$ of A. Now, suppose that $E_{\mathfrak{a}}(\mathfrak{a}B) = B$ for each ideal $\mathfrak{a}$ of A and P be a $\mathfrak{p}$-prime submodule of B. Then

$$B = E_{\mathfrak{p}}(\mathfrak{p}B) \subseteq E_{\mathfrak{p}}(P) = P \subsetneqq B,$$

a contradiction. □

Corollary 2.14. *Let A be a one-dimensional Noetherian integral domain. Then B is torsion divisible if and only if $E_{\mathfrak{a}}(\mathfrak{a}B) = B$ for each ideal $\mathfrak{a}$ of A.*

Proof. Let $E_{\mathfrak{a}}(\mathfrak{a}B) = B$ for each ideal $\mathfrak{a}$ of A. Then it is primeless by Corollary 2.13. It follows from [7, Proposition 1.4] that B is torsion divisible.

On the other hand, if B is torsion divisible, then B is primeless by [7, Proposition 1.4]. So, Corollary 2.13 implies that $E_{\mathfrak{a}}(\mathfrak{a}B) = B$ for each ideal $\mathfrak{a}$ of A. □

References

[1] M. Alkan and Y. Tiras, *Projective modules and prime submodules*, Czechoslovak Math. J. **56** (2006), no. 2, 601–611.

[2] D. D. Anderson and S. Chun, *The set of torsion elements of a module*, Comm. Algebra **42** (2014), 1835–1843.
[3] Chin-Pi Lu, *Prime submodules of modules*, Comment. Math. Univ. St. Pauli **33** (1984), no. 1, 61–69.
[4] ______, *Spectra of modules*, Comm. Algebra **23** (1995), no. 10, 3741–3752.
[5] ______, *Saturations of submodules*, Comm. Algebra **31** (2003), no. 6, 2655–2673.
[6] R. L. McCasland and M. E. Moore, *Prime submodules*, Comm. Algebra **20** (1992), no. 6, 1803–1817.
[7] R. L. McCasland, M. E. Moore, and P. F. Smith, *On the spectrum of a module over a commutative ring*, Comm. Algebra **25** (1997), no. 1, 79–103.

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